Further order reduction of a singular H_∞ controller

Takao Watanabe[1]
Dept. of Electrical Engineering
Tokyo Metropolitan University
1-1 Minami-osawa Hachioji, Tokyo 192-0397
Japan
takao@eei.metro-u.ac.jp

Anton A. Stoorvogel
Dept. of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
A.A.Stoorvogel@tue.nl

Abstract

A new class of reduced-order controllers is obtained for the H_∞ problem. The reduced-order controller does not compromise the performance attained by the full-order controller. An algorithm for deriving the reduced-order H_∞ controller is presented in the case of continuous time. The reduction in order is related to unstable invariant zeros of the subsystem from disturbance inputs to measurement outputs. In the case where the subsystem has no infinite zeros, the resulting order of the H_∞ controller is lower than that of the existing reduced-order H_∞ controller designs which are based on reduced-order observer design. Furthermore, the mechanism of the controller order reduction is analyzed on the basis of the two-Riccati equation approach.

1 Introduction

Consider the linear continuous-time system

$$\Sigma : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $z \in \mathbb{R}^{p_1}$ is the controlled output, $y \in \mathbb{R}^{p_2}$ is the measurement output, $w \in \mathbb{R}^{m_1}$ is the disturbance input and $u \in \mathbb{R}^{m_2}$ is the control input. For this system we make the assumptions that (A, B_2) is stabilizable and (C_2, A) is detectable. The transfer function of Σ is denoted by

$$\Sigma(s) = \begin{pmatrix} \Sigma_{11}(s) & \Sigma_{12}(s) \\ \Sigma_{21}(s) & \Sigma_{22}(s) \end{pmatrix}$$

where $\Sigma_{ij}, (i, j = 1, 2)$ is the subsystem associated with the transfer matrix $\Sigma_{ij}(s) = D_{ij} + C_j(sI - A)^{-1}B_i$.

The H_∞ sub-optimal control problem is to find a stabilizing controller

$$\Sigma_c : \begin{cases} \dot{\eta} = A_k \eta + B_k y \\ u = C_k \eta + D_k y \end{cases}$$

where $\eta \in \mathbb{R}^{n_k}$ is the state of the controller, such that the resulting closed loop system has an H_∞-norm strictly less than an *a priori* given bound γ, if one exists. If γ is minimized, the problem is called the H_∞ optimal control problem. It is known that if the sub-optimal H_∞ control problem is solvable, then we can always find a suitable controller whose dynamic order is at most equal to the order of the system (1). However, in the singular case where the direct feedthrough matrix D_{21} does not have full row rank or in its dual case where D_{12} does not have full column rank we can guarantee *a priori* that a controller of lower order than the order of the system can be found, see [1–3].

Background: We can assume without loss of generality that the matrix $\begin{pmatrix} C_2 & D_{21} \end{pmatrix}$ has full row rank. For the singular problem where D_{21} does not have full row rank, we then obtain that

$$\operatorname{rank} \begin{pmatrix} C_2 & D_{21} \end{pmatrix} - \operatorname{rank} \begin{pmatrix} D_{21} \end{pmatrix} > 0.$$

From [1–3], we know that we can obtain a reduced-order H_∞ controller whose order is less than or equal to

$$n_k = n - \left[\operatorname{rank} \begin{pmatrix} C_2 & D_{21} \end{pmatrix} - \operatorname{rank} \begin{pmatrix} D_{21} \end{pmatrix} \right]. \tag{2}$$

A key step in deriving the reduced-order controller is the reduced-order observer based controller design [1], where the system with partially noise-free measurement outputs can be stabilized with an output feedback controller whose order is equal to the order of the system Σ minus the number of noise-free measurements. In [2], the continuous-time reduced-order H_∞ controller is also derived from the same standpoint, and the structure of the reduced-order H_∞ controller is clarified on the basis of the classical Riccati-based approach. On the other hand, the reduced-order H_∞ controller design problem can also be characterized in terms of LMIs and algorithms to derive the reduced-order controller of order n_k are presented not only for the continuous-time case but for the discrete-time case as well in [3]. Interestingly, from these results it can be conjectured that in continuous time there exists a reduced-order controller which has order lower than n_k if $\Sigma_{21}(s)$ has unstable invariant zeros on the positive real axis. This conjecture will be established in this paper by exploiting a

[1]This work was performed while the author was visiting the Dept. of Mathematics and Comp. Science, Eindhoven University of Technology, The Netherlands. The author was supported by the Telecommunications Advancement Foundation in Japan.

bilinear transform to reveal a relationship between the continuous-time and discrete-time results. Motivated by this conjecture, we consider an algorithm to obtain a reduced-order H_∞ controller of lower degree. In addition, the ideas in [2] are utilized to understand the mechanism of the controller order reduction as well as to extend this result to the case where $\Sigma_{21}(s)$ has unstable invariant zeros in $\mathbb{C}^+$.

Notation: We denote the set of positive real numbers by $\mathbb{R}^+$, the open left half complex plane by $\mathbb{C}^-$ and the open right half complex plane by $\mathbb{C}^+$. The class of stable real rational transfer functions is denoted by $\mathscr{RH}_\infty$. A generalized inverse of a matrix D is denoted by $D^\dagger$ and an orthogonal complement of D by $D^\perp$.

2 Results based on bilinear transform

First, let us define a bilinear transformation

$$\Gamma_c : s \mapsto z = \frac{s+\alpha}{s-\alpha}, \quad 0 < \alpha \in \mathbb{R}. \tag{3}$$

If α is not an eigenvalue of A, then Γ_c transforms the system Σ to a new system

$$\tilde{\Sigma} : \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}_1 w + \tilde{B}_2 u \\ z = \tilde{C}_1 x + \tilde{D}_{11} w + \tilde{D}_{12} u \\ y = \tilde{C}_2 x + \tilde{D}_{21} w + \tilde{D}_{22} u \end{cases},$$

where

$$\begin{aligned} \tilde{A} &= -\left(\alpha I + A\right)\left(\alpha I - A\right)^{-1} \\ \tilde{B}_i &= -\left[\left(\alpha I + A\right)\left(\alpha I - A\right)^{-1} + I\right] B_i \\ \tilde{C}_i &= C_i \left(\alpha I - A\right)^{-1},\ \tilde{D}_{ij} = \Sigma_{ij}(\alpha), (i,j = 1,2). \end{aligned}$$

It should be noted that the matrix $\tilde{D}_{21}$ has a lower rank than the matrix D_{21} if D_{21} has full rank and the following inequality holds.

$$\operatorname{rank}\left(\Sigma_{21}(\alpha)\right) < \operatorname{normrank}\left(\Sigma_{21}(s)\right) \tag{4}$$

This means that we can choose Γ_c such that $\tilde{D}_{21}$ has a lower rank than the normal rank of the associated transfer matrix if the system $\Sigma_{21}(s)$ has an invariant zero on the positive real axis. On the basis of this observation we deduce the following result.

Theorem 1
Suppose that the system $\Sigma_{21}(s)$ has unstable invariant zeros on the positive real axis and let one of these zeros be $\alpha > 0$. Then, there exists a continuous-time H_∞ controller of order

$$n_{k_c} = n - \left[\operatorname{rank}\begin{pmatrix} C_2 & D_{21}\end{pmatrix} - \operatorname{rank}\left(\Sigma_{21}(\alpha)\right)\right] \tag{5}$$

if the continuous-time H_∞ problem for the system Σ is solvable. Moreover, if D_{21} has full column rank, then we obtain $n_{k_c} < n_k$.

Lemma 1
Suppose that (A, B_2, C_2) is stabilizable and detectable. For $\alpha > 0$, there exists a matrix $N \in \mathbb{R}^{m_2 \times p_2}$ which satisfies

$$\det\left(\alpha I - A + B_2 N C_2\right) \neq 0. \tag{6}$$

By this lemma, even if we have $\det(\alpha I - A) = 0$, by using a static output feedback:

$$u = Ny + v \tag{7}$$

we can make the A-matrix of the new system Σ_F:

$$\begin{aligned} \dot{x} &= \left(A + B_2 N C_2\right) x + \left(B_1 + B_2 N D_{21}\right) w + B_2 v \\ z &= \left(C_1 + D_{12} N C_2\right) x + \left(D_{11} + D_{12} N D_{21}\right) w + D_{12} v \\ y &= C_2 x + D_{21} w \end{aligned}$$

nonsingular at α, where we assume $D_{22} = 0$ without loss of generality. Then we can apply the reduced-order controller design in Theorem 1 to Σ_F and then obtain a reduced-order controller. The preliminary static output feedback does not change the order of the resultant controller, and zeros of the system Σ_{21} are invariant under this preliminary feedback. Therefore by applying the above algorithm for the new system Σ_F, we can obtain a reduced-order controller of order n_{k_c}.

Remark 1
If $\alpha < 0$, Γ_c no longer preserves the stability of the discrete-time system. Thus this idea only works when $\Sigma_{21}(s)$ has invariant zeros on the positive real axis. On the other hand, in the case where $\Sigma_{21}(s)$ has stable invariant zeros, the controller order reduction problem is discussed in [4, 5].

Remark 2
We obtain the following algorithm to design a reduced-order H_∞ controller of order n_{k_c}. **(Step I)** *Transform the system Σ by the bilinear transform Γ to obtain the system $\tilde{\Sigma}$, where the parameter α is chosen such that the inequality (4) is satisfied. If the matrix A has an eigenvalue at α, apply the preliminary feedback (7) for Σ first, and then apply the transformation.* **(Step II)** *Solve the discrete-time H_∞ problem for $\tilde{\Sigma}$ by using the LMI algorithm presented in [3], and obtain a discrete-time H_∞ controller of order n_{k_c}.* **(Step III)** *Apply the inverse bilinear transform to this controller to obtain an n_{k_c}-th order continuous-time H_∞ controller (A_k, B_k, C_k, D_k). The resultant controller is obtained as $(A_k, B_k, C_k, D_k + N)$ where if the preliminary feedback is not used, N is put to zero.*

Remark 3
If the matrix D_{21} has full column rank, the difference between n_k and n_{k_c} is

$$n_k - n_{k_c} = S_{21}(s) - S_{21}(\alpha).$$

This indicates that the difference in the order between the reduced order H_∞ controller obtained here and the n_k-th order H_∞ controller is equal to the geometric multiplicity [6] of an unstable invariant zero of $\Sigma_{21}(s)$ at α. Thus the number in order reduction of the controller depends on the selection of the zero. If the geometric multiplicity of the zero is higher, then we can obtain a lower order H_∞ controller.

3 Analysis based on AREs

We have thus characterized a new class of reduced-order H_∞ controllers by using one unstable real zero of Σ_{21}. However in general there are many different zeros and these are in general complex. Moreover, by a consideration of continuity of a zero location and the controller structure, we can argue that if a zero is slightly changed from another one by some perturbation, we may still be able to reduce the order. Thus, on the basis of the background material presented in the previous section, we analyze the mechanism of the controller order reduction in the continuous-time singular H_∞ problem. The analysis is based on a solution obtained by using the fundamental two-Riccati equation approach [2, 7–10]. We introduce our recent work [2], which uses the two-ARE approach presented by Mita, et.al. [10], and extend its idea to analyze the controller-order reduction.

3.1 Preliminaries

Let us consider the continuous-time system described in (1). In addition to the conditions that (A, B_2) is stabilizable and (C_2, A) is detectable, we consider the H_∞ problem under the following assumptions

A1 D_{12} is of full column rank.

A2 D_{21} is of full column rank.

A3 $\Sigma_{21}(s)$ has invariant zeros in $\mathbb{C}^+$.

A4 $\Sigma_{12}(s)$ and $\Sigma_{21}(s)$ do not have invariant zeros on the imaginary axis.

The assumptions **A1**, **A2** and **A4** are made to simplify our analysis. The assumption **A3** captures the feature of the singular H_∞ problem we consider. The singular problem which refers to the case that these assumptions are not satisfied has been studied in [3, 11–15]. However, there is no study of the controller order reduction exploiting the existence of invariant zeros in $\mathbb{C}^+$. Without loss of generality, we can put assumptions on the matrices C_2, D_{11}, D_{21} and D_{22} as follows

B1 $D_{11} = O$ and $D_{22} = O$.

B2 $\begin{pmatrix} C_2 & D_{21} \end{pmatrix} = \begin{pmatrix} C_{21} & O \\ C_{22} & I_{m_1} \end{pmatrix}$, where $C_{21} \in \mathbb{R}^{(p_2-m_1)\times n}$ is of full row rank.

The assumption **B1** can be relaxed by using some standard techniques as described in [12, 16]. **B2** basically amount to choosing a suitable basis for the input and output spaces.

The invariant zeros of the system $\Sigma_{21}(s)$ can be made explicit by a suitable state-space transformation for the system Σ, i.e. $x = T\bar{x}$ with T invertible.

Lemma 2
Let us represent a set of invariant zeros of $\Sigma_{21}(s)$ with $\lambda(A_-) \cup \lambda(A_+)$ where $\lambda(A_-) \subset \mathbb{C}^-, \lambda(A_+) \subset \mathbb{C}^+$. Then we can choose a transformation matrix T such that the following equation holds in a new state-coordinate.

$$\begin{pmatrix} A - B_1 D_{21}^\dagger C_2 \\ \hdashline \left(D_{21}^\perp\right)^T C_2 \\ \hdashline D_{21}^\dagger C_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & O & O \\ A_{13} & A_{14} & O & O \\ A_{31} & A_{32} & A_- & O \\ A_{33} & A_{34} & O & A_+ \\ \hdashline I_{p_2-m_1} & O & O & O \\ \hdashline C_{22ll} & C_{22lr} & C_{22rl} & C_{22rr} \end{pmatrix},$$

where $\lambda(A_{14}) \subset \mathbb{C}^-$, the pair (A_+, C_{22rr}) is observable.

For notational ease we partition C_1 accordingly with $D_{21}^\dagger C_2$ as $C_1 = \begin{pmatrix} C_{1ll} & C_{1lr} & C_{1rl} & C_{1rr} \end{pmatrix}$.

3.2 Parameterization of the H_∞ controller

In this section, we derive the full-order H_∞ controller for the system Σ on the basis of the two-ARE approach. Here full order means the same order as the system Σ has. First, we introduce two AREs:

$$\begin{aligned} X\left(A - B_2 D_{12}^\dagger C_1\right) + \left(A - B_2 D_{12}^\dagger C_1\right)^T X \\ + X\left[\gamma^{-2} B_1 B_1^T - B_2 D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T\right] X \\ + C_1^T D_{12}^\perp \left(D_{12}^\perp\right)^T C_1 = O \end{aligned} \quad (8)$$

and

$$\begin{aligned} Y A_{ZH}^T + A_{ZH} Y \\ + Y\left[\gamma^{-2} C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right] Y = O. \end{aligned} \quad (9)$$

Here, A_{ZH} is defined as

$$A_{ZH} = A - B_1 D_{21}^\dagger C_2 + L_H \left(D_{21}^\perp\right)^T C_2 \quad (10)$$

where a matrix L_H is selected such that

$$\{\lambda\left(A_{ZH}\right)\} - \{\lambda\left(A_+\right)\} \subset \mathbb{C}^- \quad (11)$$

is satisfied. This condition implies that the observable subspace of the pair $\left(A - B_1 D_{21}^\dagger C_2, \left(D_{21}^\perp\right)^T C_2\right)$ is stabilized by L_H. If these AREs have solutions which stabilize the following matrices

$$A - B_2 D_{12}^{\dagger} C_1 + \left[\gamma^{-2} B_1 B_1^T - B_2 D_{12}^{\dagger} \left(B_2 D_{12}^{\dagger}\right)^T\right] X$$
$$A_{ZH} + Y \left[\gamma^{-2} C_1^T C_1 - \left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} C_2\right],$$

we call these solutions the stabilizing solutions. It is easy to show that each stabilizing solution of these AREs is unique and that the solution Y is independent of a specific choice for L_H, provided that (11) is satisfied.

By following the result presented in [10], we can obtain the class of all suboptimal H_∞ controllers, which are parametrized with two free parameters.

Lemma 3
Suppose that (A, B_2) is observable and (A, C_2) is detectable, and that assumptions **A1** *to* **A4** *are satisfied. Then an H_∞ controller exists if and only if the two AREs in (8) and (9) have stabilizing solutions $X \geq O, Y \geq O$ and these matrices satisfy*

$$\gamma^2 I - XY > O. \tag{12}$$

If there exists an H_∞ controller, the class of all suboptimal H_∞ controllers is given as follows

$$\mathscr{K}_\infty = \left\{K_\infty(s) \mid N(s), W(s) \in \mathscr{RH}_\infty, \|N(s)\|_\infty < \gamma\right\},$$

where $K_\infty(s)$ is defined as

$$K_\infty(s) = \left(\begin{array}{c|c} A_Y & \hat{B}_2 \\ \hline C_K(s) & \Pi^{-1} \end{array}\right)^{-1} \times \left(\begin{array}{c|c} A_Y & H_\infty \\ \hline C_K(s) & N(s) D_{21}^{\dagger} + W(s)\left(D_{21}^{\perp}\right)^T \end{array}\right),$$

where

$$A_Y = A + \gamma^{-2} Y C_1^T C_1 + H_\infty C_2$$
$$C_K(s) = -\Pi^{-1} F_\infty Z + N(s) D_{21}^{\dagger} \hat{C}_2 Z + W(s)\left(D_{21}^{\perp}\right)^T C_2$$
$$\hat{A} = A + \gamma^{-2} Y C_1^T C_1 + H_\infty C_2 + \hat{B}_2 F_\infty Z$$
$$\hat{B}_2 = B_2 + \gamma^{-2} Y C_1^T D_{12},\ \hat{C}_2 = \gamma^{-2} D_{21} B_1^T X + C_2$$
$$F_\infty = -D_{12}^{\dagger} C_1 - D_{12}^{\dagger}\left(B_2 D_{12}^{\dagger}\right)^T X$$
$$H_\infty = -B_1 D_{21}^{\dagger} - Y\left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} + L_H\left(D_{21}^{\perp}\right)^T$$
$$Z = \left(I - \gamma^{-2} Y X\right)^{-1},\ \Pi = \left(D_{12}^T D_{12}\right)^{-\frac{1}{2}}.$$

Remark 4
The H_∞ controller is represented with two free parameters. On the other hand, in the regular case where D_{21} is of full row rank, the H_∞ controller is represented with only one free parameter $N(s)$ [7]. If the matrix D_{21} is invertible, then the extra free parameter $W(s)$ disappears and we obtain the standard parametrization.

Remark 5
The order of the central solution, which is the controller obtained by putting all free parameters to zero, equals the order of the system Σ. The free parameters might be utilized to improve the controller performance. But also, the free parameters can be used to reduce the order of the controller. Controller order reduction using this philosophy is discussed in the next section.

3.3 n_k-th order H_∞ controller

By using the formula in Lemma 2, the unique stabilizing solution of the ARE in (9) can be represented by the unique stabilizing solution of the reduced-order ARE:

$$Y_r A_+^T + A_+ Y_r + Y_r\left(\gamma^{-2} C_{1rr}^T C_{1rr} - C_{22rr}^T C_{22rr}\right) Y_r = O, \tag{13}$$

where the solution Y_r stabilizes the matrix:

$$A_{Y_r} := A_+ + Y_r\left(\gamma^{-2} C_{1rr}^T C_{1rr} - C_{22rr}^T C_{22rr}\right).$$

Lemma 4
The positive semi-definite stabilizing solution of the ARE in (9) can be characterized by the stabilizing solution of the reduced-order ARE in (13) as follows

$$Y = \begin{pmatrix} O & O \\ O & Y_r \end{pmatrix} \in \mathbb{R}^{n \times n}, \text{ where } Y_r > O.$$

Remark 6
A direct consequence of the above lemma is that the stabilizing solution is independent of a specific choice for L_H, provided (11) is satisfied. It can be also verified that the solution of the ARE in (9) equals a zero matrix if the invariant zeros of $\Sigma_{21}(s)$ are all in $\mathbb{C}^-$. Then the matrix Z equals the identity matrix. On the other hand if $\Sigma_{21}(s)$ has invariant zeros in $\mathbb{C}^+$, the ARE deflates to the reduced-order ARE.

If we select the parameter $W(s)$ such that a full column rank matrix V satisfies

$$\begin{cases} A_Y V = V(A_{11} + L_{H_1}) \\ C_K(s) V = O \end{cases},$$

then we can find a reduced order controller.

Theorem 2
The class of the reduced-order H_∞ controllers is represented as

$$K_\infty(s) = \left(\begin{array}{c|c} \tilde{A}_Y & \tilde{B}_2 \\ \hline \tilde{C}_K(s) & \Pi^{-1} \end{array}\right)^{-1} \left(\begin{array}{c|c} \tilde{A}_Y & \tilde{H}_\infty \\ \hline \tilde{C}_K(s) & \Omega \end{array}\right), \tag{14}$$

where parameters are

$$\tilde{A}_Y = \left(V^\perp\right)^T A_Y V^\perp \in \mathbb{R}^{n_k \times n_k}$$
$$\tilde{C}_K(s) = \left(-\Pi^{-1} F_\infty Z + N(s) D_{21}^{\dagger} \hat{C}_2 Z\right) V^\perp$$
$$\tilde{B}_2 = \left(V^\perp\right)^T \hat{B}_2, \tilde{H}_\infty = \left(V^\perp\right)^T H_\infty$$
$$\Omega = \Pi^{-1} F_\infty Z V\left(D_{21}^{\perp}\right)^T + N(s) D_{21}^{\dagger}\left(I - \hat{C}_2 Z V\left(D_{21}^{\perp}\right)^T\right).$$

Remark 7
If the parameter $N(s)$ is put to zero, the order of the controller becomes n_k. The same order of the singular H_∞ controller is attained in [1] and [3]. On the other hand, the result obtained in this paper gives a nice interpretation of the structure of the reduced-order H_∞ controller. It still preserves the structure of the observer-based controller, which has the same order as the reduced-order observer as we also have an observer structure in [1]. If we utilize the parameter $N(s)$, we might obtain further order reduction. The next section discusses this topic.

3.4 Further order reduction of the H_∞ controller

As we have seen in Section 2, the reduced-order H_∞ controller obtained in the previous section may further reduce its order. This section investigates whether the reduced-order controller which is derived with the ARE approach can be further reduced in order. We will find that the further order reduction is related to unstable zeros of Σ_{21} and also to the structure of the zeros.

Let $V_m \in \mathbb{R}^{r\times m}$ be an arbitrary full column rank matrix which satisfies

$$A_{Y_r} V_m = V_m J \tag{15}$$

where $J \in \mathbb{R}^{m\times m}$ and m is an arbitrary number such that $m \le m_1$, and the number m_1 is the dimension of the disturbance w. If we can select an $N \in \{\|N\| < \gamma\}$ such that N satisfies

$$\tilde{C}_K(s)\begin{pmatrix} O \\ V_m \end{pmatrix} = O,$$

the parameter N further reduces the order of the H_∞ controller in (14).

Theorem 3
Suppose that the H_∞ control problem is solvable. Then if we can choose the matrix V_m in (15) such that the matrix J is represented as

$$J = -\alpha_i I_m \tag{16}$$

where $\alpha_i > 0$ is an unstable invariant zero of $\Sigma_{21}(s)$, the H_∞ controller can be reduced to the order $n_k - m$ while preserving the closed loop performance γ.

Remark 8
Since α_i is an unstable invariant zero of $\Sigma_{21}(s)$, we can see that the maximal number m which satisfies (15) and (16) amounts to the geometric multiplicity [6] of an unstable invariant zero of $\Sigma_{21}(s)$. Therefore the order of the controller given in this theorem coincides with the result given in Theorem 1 as

$$n_{k_c} = n_k - m,$$

provided that the matrix D_{21} has full column rank.

The following theorem gives a condition for the H_∞ controller reduction in the case where two distinct zero-modes in positive real number are included in J.

Theorem 4
Suppose that the H_∞ control problem is solvable. Furthermore, suppose that we can choose the matrix V_m in (15) such that the diagonal matrix J is represented as

$$J = -\begin{pmatrix} \alpha_i I_{m_i} & O \\ O & \alpha_j I_{m_j} \end{pmatrix}, \tag{17}$$

where $0 < \alpha_i < \alpha_j$ are unstable invariant zeros of $\Sigma_{21}(s)$ and $m = m_i + m_j$. Then, the H_∞ controller can be reduced to the order $n_k - m$ while preserving the closed loop performance γ if the following condition:

$$\alpha_j - \alpha_i \in \left\{\epsilon > 0 \mid 2\alpha_i F + \epsilon\tilde{F} > O\right\} \tag{18}$$

is satisfied, where $F > O$ is defined as

$$F = V_m^T Y_r^{-1}\hat{Z}V_m.$$

By using submatrices of F decomposed as $F = \begin{pmatrix} F_1 & F_2 \\ F_2^T & F_4 \end{pmatrix}$, $\tilde{F}$ is defined as $\tilde{F} = \begin{pmatrix} O & F_2 \\ F_2^T & 2F_4 \end{pmatrix}$.

Remark 9
Solving an LMI feasibility problem can easily check the condition (18). Since $F > O$, the distance between α_i and α_j becomes shorter the condition (18) is satisfied. This means that if the unstable zeros are located close each other, we can obtain a lower order H_∞ controller. Compared with the algorithm presented in Section 2, this result has an advantage that if the system Σ_{21} has distinct zeros on the positive real axis, we can further investigate lower order H_∞ controllers.

Next, we consider the case where the system $\Sigma_{21}(s)$ has complex zeros on the right half plane and the complex mode is included in J.

Theorem 5
Suppose that the H_∞ control problem is solvable. Furthermore, suppose that we can choose the matrix V_m in (15) such that the matrix J is represented as

$$J = -\text{blockdiag}\left(\begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \cdots, \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}\right)$$

where $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$ are respectively the real part and the imaginary part of an invariant zero of $\Sigma_{21}(s)$. Then, the H_∞ controller can be reduced to the order $n_k - m$ while preserving the closed loop performance γ if the following condition:

$$\beta_i \in \left\{\epsilon > 0 \mid \alpha_i F + \epsilon\tilde{F} > O\right\} \tag{19}$$

is satisfied. Here, F is a positive definite Hermitan matrix and is defined as

$$F = \hat{V}_m^* V_m^T Y_r^{-1} \hat{Z} V_m \hat{V}_m$$

where $\hat{V}_m \in \mathbb{C}^{m\times m}$ is a nonsingular matrix which satisfies

$$J\hat{V}_m = -\hat{V}_m \begin{pmatrix} \lambda_i I_{m/2} & O \\ O & \bar{\lambda}_i I_{m/2} \end{pmatrix} \qquad (20)$$
$$\lambda_i = \alpha_i + j\beta_i.$$

By using submatrices of F decomposed as $F = \begin{pmatrix} F_1 & F_2 \\ F_2^* & F_4 \end{pmatrix}$, $\tilde{F}$ is defined as $\tilde{F} = \begin{pmatrix} O & -jF_2 \\ jF_2^* & O \end{pmatrix}$.

Remark 10
By solving an LMI feasibility problem we can easily check the condition (19). Since $F > O$, we note that if the imaginary part of the zero is small enough, then the condition (19) is satisfied. This means that if the complex unstable zeros are located near the real axis, we can obtain a lower order H_∞ controller. Compared with the algorithm presented in Section 2, this result has an advantage that if the system Σ_{21} has complex zeros in $\mathbb{C}^+$, we can further investigate a lower order H_∞ controller.

4 Conclusion

In this paper we have established the existence of a new class of reduced-order H_∞ controllers in the continuous-time case. The reduced-order H_∞ controllers are characterized by unstable invariant zeros of the system Σ_{21}. An algorithm to obtain the reduced-order H_∞ controllers is presented on the basis of the LMI approach. On the other hand, the relation between the unstable zero and the controller order reduction is analyzed by using a controller parametrization obtained from the fundamental two-ARE approach. The mechanism of the controller order reduction is explained with finite pole-zero cancellations in the parametrized controller. Also, in the cases where the unstable zeros are distinct and are located in $\mathbb{R}^+$, or when they are located in $\mathbb{C}^+$ we obtain some conditions under which the order of the H_∞ controller is further reduced.

References

[1] A. A. Stoorvogel, A. Saberi, and B. M. Chen, "A reduced order observer based controller design for H_∞ optimization," *IEEE Trans. Autom. Contr.*, vol. 39, no. 2, pp. 355–360, 1994.

[2] T. Watanabe, *Designing Low-order Robust Controllers with Solving Some Fundamental H_∞ Control Problems.* PhD thesis, Tokyo Institute of Technology, 1999.

[3] X. Xin, L. Guo, and C. Feng, "Reduced-order controllers for continuous and discrete-time singular H^∞ control problems based on LMI," *Automatica*, vol. 32, no. 11, pp. 1581–1585, 1996.

[4] K. C. Goh and M. G. Safonov, "Connection between plant zeros and H_∞ controller order reduction," *Proc. Amer. Contr. Conf.*, pp. 2175–2179, 1993.

[5] J. Sefton and K. Glover, "Pole/zero cancellations in the general H_∞ problem with reference to a two block design," *Syst. Contr. Lett.*, vol. 14, pp. 295–306, 1990.

[6] A. G. J. Macfarlane and N. Karcanias, "Poles and zeros of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory," *Int. J. Contr.*, vol. 24, no. 1, pp. 33–74, 1976.

[7] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Trans. Autom. Contr.*, vol. 34, no. 8, pp. 831–847, 1989.

[8] K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov, "A characterization of all solutions to the four block general distance problem," *SIAM J. Contr. and Optim.*, vol. 29, no. 2, pp. 283–324, 1991.

[9] H. Kimura, Y. Lu, and R. Kawatani, "On the structure of H_∞ control systems and related extensions," *IEEE Trans. Autom. Contr.*, vol. 36, no. 6, pp. 653–667, 1991.

[10] T. Mita, J. B. Matson, and B. D. O. Anderson, "A complete and simple parametrization of controllers for a nonstandard H_∞ problem," *Automatica*, vol. 34, no. 3, pp. 369–374, 1998.

[11] A. A. Stoorvogel, "The H_∞ control problem with zeros on the boundary of the stability domain," *Int. J. Contr.*, vol. 63, no. 6, pp. 1029–1053, 1996.

[12] A. A. Stoorvogel, *The H_∞ Control Problem: A State Space Approach.* Prentice-Hall, Inc., 1992.

[13] C. Scherer, *The Riccati Inequality and State-Space H_∞-Optimal Control.* PhD thesis, The University of Würzburg, 1990.

[14] C. Scherer, "H_∞ control by state-feedback for plants with zeros on the imaginary axis," *SIAM J. Contr. and Optim.*, vol. 30, no. 1, pp. 123–142, 1992.

[15] S. Hara, T. Sugie, and R. Kondo, "H^∞ control problem with $j\omega$-axis zeros," *Automatica*, vol. 28, pp. 55–70, 1992.

[16] M. G. Safonov, D. J. N. Limebeer, and R. Y. Chiang, "Simplifying the H^∞ theory via loop-shifting, matrix-pencil and descriptor concepts," *Int. J. Contr.*, vol. 50, no. 6, pp. 2467–2488, 1989.