

A Simplified Algorithm for Balanced Realization of Laguerre Network Models

Quan-Gen Zhou and Edward J. Davison
Department of Electrical and Computer Engineering
University of Toronto
10 King's College Road
Toronto, Ontario, Canada M5S 3G4
Email: zhou@control.utoronto.ca
ted@control.utoronto.ca

Abstract

In this paper, a simplified algorithm for constructing an internally balanced realization of a Laguerre network model is presented. Both continuous-time and discrete-time cases are treated in a unified framework. The algorithm does not require the computation of controllability and observability grammians, which makes it highly efficient, compared to existing procedures. An example is given to illustrate the method.

1 Introduction

Laguerre networks have been used in system synthesis and modeling since the early work of Lee [1] and Wiener [2]. Recently, there has been renewed interest in using Laguerre networks to approximate dynamical systems (see, e.g., [3], [4] and [5]). In comparison with conventional methods, Laguerre network modeling offers some appealing advantages. For instance, it naturally leads to a linear-in-the-parameters model, which considerably simplifies the algorithm for parameter estimation. Moreover, Laguerre modeling falls into the category of output error methods, and thanks to the orthonormal properties of the Laguerre basis functions, unbiased and consistent estimates of the model parameters can be obtained under fairly practical conditions [3]. Laguerre network models, however, are not as convenient for control design as more conventional models, such as state variable models. Furthermore, it is known that Laguerre modeling can result in a high order model for moderately damped systems. Thus, reduction of the Laguerre model may be necessary, for instance, by using a balanced truncation method [6]. In this case, an efficient algorithm for balanced realization of Laguerre models is desirable.

While standard procedures for balanced realization may be applied to Laguerre models, the computation is involved. A standard procedure, for instance, would in-

clude first constructing a minimal realization of the Laguerre model and then solving a pair of Lyapunov equations for the controllability and observability grammians [6]. However, owing to the special structure of Laguerre models, simple schemes may be found to solve the problem more efficiently. In [7], the problem was considered for a multivariable continuous-time system, and an algorithm was given without requiring the solution of Lyapunov equations. While it is more efficient than standard procedures, the algorithm still requires the computation of controllability and observability grammians.

In this paper, we shall present a simple and straightforward solution to the problem of balanced realization of Laguerre models for multivariable systems. Both continuous-time and discrete-time cases will be considered in a unified framework by using bilinear transformations. The resultant algorithm obtained does not require the computation of controllability and observability grammians, which makes it computationally highly efficient, compared to existing procedures.

2 An Efficient Algorithm

Consider a multivariable, linear, time-invariant, finite-dimensional, stable and strictly proper system with r inputs and q outputs, whose transfer function matrix is given in the form of a Laguerre network model, i.e.,

$$G_c(s) = \sum_{k=0}^{N-1} C_k \frac{\sqrt{2\lambda}}{s + \lambda} \left(\frac{s - \lambda}{s + \lambda} \right)^k, \quad \lambda > 0 \quad (1)$$

for the continuous-time case, and

$$G_d(z) = \sum_{k=0}^{N-1} C_k \frac{\sqrt{1 - \lambda^2}}{z - \lambda} \left(\frac{1 - \lambda z}{z - \lambda} \right)^k, \quad |\lambda| < 1 \quad (2)$$

for the discrete-time case, where N is the model order, C_k , $k = 0, 1, \dots, N - 1$, are the Laguerre coefficient matrices, and λ is the Laguerre parameter which specifies

the poles of the Laguerre model, either continuous-time or discrete-time. We shall give an efficient algorithm for constructing a balanced realization of the system by exploiting the special structures of the Laguerre models.

First, we shall use bilinear transformations to convert the Laguerre models, both continuous-time and discrete-time, to the form of an FIR model. For the continuous-time case, let us introduce the operator, $p = T_c(s)$, where $T_c(s)$ is the bilinear transformation defined by

$$T_c(s) = \frac{s + \lambda}{s - \lambda}$$

and also define

$$H_c(p) = G_c(T_c^{-1}(p))$$

where $T_c^{-1}(p)$ is the inverse transformation of $T_c(s)$, given by

$$T_c^{-1}(p) = \lambda \frac{p + 1}{p - 1}$$

Then, from (1),

$$\begin{aligned} H_c(p) &= \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\lambda}} C_k (1 - p^{-1}) p^{-k} \\ &= \frac{1}{\sqrt{2\lambda}} C_0 + \sum_{k=1}^{N-1} \frac{1}{\sqrt{2\lambda}} (C_k - C_{k-1}) p^{-k} \\ &\quad - \frac{1}{\sqrt{2\lambda}} C_{N-1} p^{-N} \end{aligned} \quad (3)$$

For the discrete-time case, we use $w = T_d(z)$, with $T_d(z)$ defined by

$$T_d(z) = \frac{z - \lambda}{1 - \lambda z}$$

and similarly define

$$H_d(w) = G_d(T_d^{-1}(w))$$

where $T_d^{-1}(w)$ is the inverse transformation of $T_d(z)$, given by

$$T_d^{-1}(w) = \frac{w + \lambda}{1 + \lambda w}$$

Then, from (2),

$$\begin{aligned} H_d(w) &= \sum_{k=0}^{N-1} \frac{1}{\sqrt{1 - \lambda^2}} C_k (\lambda + w^{-1}) w^{-k} \\ &= \frac{\lambda}{\sqrt{1 - \lambda^2}} C_0 \\ &\quad + \sum_{k=1}^{N-1} \frac{1}{\sqrt{1 - \lambda^2}} (\lambda C_k + C_{k-1}) w^{-k} \\ &\quad + \frac{1}{\sqrt{1 - \lambda^2}} C_{N-1} w^{-N} \end{aligned} \quad (4)$$

The Laguerre models, (1) and (2), are now transformed into the same form of an FIR model, which also gives

a Laurent series expansion at the origin for the transformed transfer function matrices, $H_c(p)$ and $H_d(w)$, respectively. Note that the bilinear transformation $T_c(s)$ maps the open left half s -plane into the open unit disc in the p -plane, and the transformation $T_d(z)$ maps the open unit disc in the z -plane into the open unit disc in the w -plane. We shall therefore define stability of the transformed transfer function matrices as corresponding to all their poles lying inside the open unit disc, and also, accordingly, define the controllability and observability grammians of their realizations in the same way as for discrete-time models.

In [8], the generalized Markov parameters of a transfer function matrix are defined as the coefficients of the Laurent series expansion, at the origin, of its bilinearly transformed counterpart. It can then be seen from (3) and (4) that these parameters of $G_c(s)$ and $G_d(z)$, with respect to the bilinear transformations, $T_c(s)$ and $T_d(z)$, respectively, are completely given by the finite number of coefficients of the transformed FIR models, or the N Laguerre coefficient matrices of $G_c(s)$ and $G_d(z)$, respectively. Let $\{H_k\}$ denote the sequence of these parameters of the system, either continuous-time or discrete-time. The parameters can be found from (3) and (4) as follows:

$$H_k = \begin{cases} \frac{1}{\sqrt{2\lambda}} C_0 & k = 0 \\ \frac{1}{\sqrt{2\lambda}} (C_k - C_{k-1}) & 1 \leq k \leq N - 1 \\ -\frac{1}{\sqrt{2\lambda}} C_{N-1} & k = N \\ 0 & k > N \end{cases} \quad (5)$$

for the continuous-time case, and

$$H_k = \begin{cases} \frac{\lambda}{\sqrt{1 - \lambda^2}} C_0 & k = 0 \\ \frac{1}{\sqrt{1 - \lambda^2}} (\lambda C_k + C_{k-1}) & 1 \leq k \leq N - 1 \\ \frac{1}{\sqrt{1 - \lambda^2}} C_{N-1} & k = N \\ 0 & k > N \end{cases} \quad (6)$$

for the discrete-time case. Noting that the least common denominator of $H_c(p)$ or $H_d(w)$ is of degree N , the following generalized Hankel matrix can be formed:

$$H = \begin{bmatrix} H_1 & H_2 & \cdots & H_N \\ H_2 & H_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_N & 0 & \cdots & 0 \end{bmatrix} \quad (7)$$

which has an upper triangular structure. The method in [8] may then be used to find a minimal realization of the system. Note, however, that the resultant realization obtained in general will not be internally balanced. To obtain a balanced realization of the system, we need to make modifications to the general algorithm which was given in [8]. Specifically, we shall use a singular value decomposition procedure based on [9]. In addition, the matrices of the resultant realization of the system will be adjusted so as to allow preservation of

the controllability and observability grammians. We summarize the procedures in the following algorithm.

Algorithm 1 (Balanced Realization)

1. Obtain the generalized Markov parameters of $G_c(s)$ or $G_d(z)$ according to (5) or (6).
2. Form the generalized Hankel matrix, H , as defined by (7).
3. Find orthogonal matrices, K and L , such that

$$H = K \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} L = [H_{i+j-1}]$$

where $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$.

4. Construct a minimal realization of $H_c(p)$ or $H_d(w)$ according to

$$\begin{aligned} A_h &= \Sigma^{-\frac{1}{2}} K_1^T \tilde{H} L_1^T \Sigma^{-\frac{1}{2}} \\ B_h &= \Sigma^{\frac{1}{2}} L_1 E_r^T \\ C_h &= E_q K_1 \Sigma^{\frac{1}{2}} \\ D_h &= H_0 \end{aligned}$$

where \tilde{H} is formed by shifting H to the left by r columns, i.e., $\tilde{H} = [H_{i+j}]$; L_1 is defined by the first n rows of L , and K_1 by the first n columns of K ; E_r is defined to be the $1 \times N$ block matrix $[I_r \ 0 \ 0 \ \dots \ 0]$, and E_q is defined similarly.

5. Determine a balanced realization of the system by

$$\begin{aligned} A_c &= \lambda \Gamma_c (A_h + I_n) \\ B_c &= \sqrt{2\lambda} \Gamma_c B_h \\ C_c &= -\sqrt{2\lambda} C_h \Gamma_c \\ D_c &= 0 \end{aligned}$$

where $\Gamma_c = (A_h - I_n)^{-1}$, for the continuous-time case, or

$$\begin{aligned} A_d &= \Gamma_d (A_h + \lambda I_n) \\ B_d &= \sqrt{1 - \lambda^2} \Gamma_d B_h \\ C_d &= \sqrt{1 - \lambda^2} C_h \Gamma_d \\ D_d &= 0 \end{aligned}$$

where $\Gamma_d = (I_n + \lambda A_h)^{-1}$, for the discrete-time case.

Note that, due to the properness of the system, the existence of the matrices, Γ_c and Γ_d , is always guaranteed. Also note that the formula for computing D_c or D_d , as defined by the bilinear transformation, $T_c(s)$ or $T_d(z)$, is in conformity with the assumption of strict properness of the system, which can be seen as follows.

In the continuous-time case, from the general formula (see, e.g., [8]),

$$D_c = D_h - C_h \Gamma_c B_h \quad (8)$$

On the other hand, by definition,

$$H_c(p) = C_h (pI_n - A_h)^{-1} B_h + D_h$$

Thus,

$$H_c(1) = C_h (I_n - A_h)^{-1} B_h + D_h \quad (9)$$

However, from the Laurent series expansion of $H_c(p)$ and also (5),

$$H_c(1) = \sum_{k=0}^{\infty} H_k = 0 \quad (10)$$

Substituting (9) and (10) into (8) then gives

$$D_c = 0$$

Similarly, in the discrete-time case, from the general formula,

$$D_d = D_h - \lambda C_h \Gamma_d B_h \quad (11)$$

whereas, by definition,

$$H_d(w) = C_h (wI_n - A_h)^{-1} B_h + D_h$$

and so,

$$H_d\left(-\frac{1}{\lambda}\right) = -\lambda C_h (I_n + \lambda A_h)^{-1} B_h + D_h \quad (12)$$

But, now from the Laurent series expansion of $H_d(w)$ and (6),

$$H_d\left(-\frac{1}{\lambda}\right) = \sum_{k=0}^{\infty} (-1)^k \lambda^k H_k = 0 \quad (13)$$

It then follows from (11), (12) and (13) that

$$D_d = 0$$

By following [8], it can be shown that the realization as given by the algorithm is a minimal realization of the system. In fact, as we shall see in the next section, the realization is internally balanced as well.

It may be seen that the main computation of the algorithm is only a singular value decomposition for constructing the quadruple, (A_h, B_h, C_h, D_h) , which can easily be carried out by using standard algorithms (see, e.g., [10]). Thus, it is computationally highly efficient, in comparison with other methods.

3 Property of Balance

We now proceed to examine the realization of the system as given by Algorithm 1, and it turns out that the realization is indeed internally balanced.

We shall begin with a brief discussion on bilinear transformations. It is well known that special types of bilinear transformations do not change the Hankel singular values. In [11], a class of so-called “unit circle” bilinear transformations were introduced, and it was shown that “unit circle” transformations preserve the Hankel singular values. By following the results in [11], it can further be shown that, for a pair of specially calibrated realizations which are related by a “unit circle” transformation, the controllability and observability grammians are also preserved. Now, on returning back to our problem, it is a simple matter to verify that the bilinear transformation, $T_c(s)$, is a “unit circle” transformation, and for the pair of realizations of $H_c(p)$ and $G_c(s)$, as given by Algorithm 1, the transformation preserves the controllability and observability grammians. With respect to the transformation, $T_d(z)$, it can be shown that, while it is not a “unit circle” transformation, the transformation does preserve the controllability and observability grammians for the pair of realizations of $H_d(w)$ and $G_d(z)$, as given by Algorithm 1. We put together these facts in the following lemma.

Lemma 1 *For the pair of realizations of $H_c(p)$ and $G_c(s)$ ($H_d(w)$ and $G_d(z)$), as defined by Algorithm 1, the bilinear transformation, $T_c(s)$ ($T_d(z)$), preserves the controllability and observability grammians.*

Proof As noted above, the first part of the claims re $H_c(p)$ and $G_c(s)$ follows from the properties of “unit circle” transformations. Therefore, we only need to deal with the second part re $H_d(w)$ and $G_d(z)$. Let P_d and Q_d be the controllability and observability grammians, respectively, of the realization of $G_d(z)$, as given by Algorithm 1. Then, by definition, the grammians, P_d and Q_d , must satisfy the following Lyapunov equations:

$$P_d - A_d P_d A_d^T = B_d B_d^T \quad (14)$$

and

$$Q_d - A_d^T Q_d A_d = C_d^T C_d \quad (15)$$

We first show that the controllability grammian, P_d , is preserved under the bilinear transformation, $T_d(z)$. Multiplying (14) by $\frac{1}{\sqrt{1-\lambda^2}}(I_n + \lambda A_h)$ from the left, and by $\frac{1}{\sqrt{1-\lambda^2}}(I_n + \lambda A_h^T)$ from the right, and also substituting the formulae for A_d and B_d , as given by Algorithm 1, into the equation, yields

$$\begin{aligned} & \frac{1}{1-\lambda^2}(I_n + \lambda A_h)P_d(I_n + \lambda A_h^T) \\ & - \frac{1}{1-\lambda^2}(A_h + \lambda I_n)P_d(A_h^T + \lambda I_n) = B_h B_h^T \end{aligned}$$

Rearranging the equation then gives

$$P_d - A_h P_d A_h^T = B_h B_h^T$$

Note that the controllability and observability grammians of a realization of $H_d(w)$ are defined in the same way as for discrete-time systems. Thus, the above equation is actually the Lyapunov equation for the controllability grammian of the realization of $H_d(w)$, as given by Algorithm 1. This shows that P_d as defined by the equation is preserved under the transformation, $T_d(z)$.

With regards to the observability grammian, Q_d , note that the matrix, A_d , as given by Algorithm 1, can be rewritten as

$$A_d = (A_h + \lambda I_n)(I_n + \lambda A_h)^{-1} \quad (16)$$

Then, by using the same procedures as above, it can be shown that the bilinear transformation $T_d(z)$ also preserves the observability grammian.

Recall that a realization is called internally balanced if its controllability and observability grammians are equal and diagonal [6]. Therefore, by Lemma 1, it is clear that if the realization of $H_c(p)$ ($H_d(w)$) is balanced, so will be the realization of $G_c(s)$ ($G_d(z)$). The remaining question is then that, will the realization of $H_c(p)$ ($H_d(w)$) as given by Algorithm 1 be internally balanced?

In [12], it was defined that, a discrete-time system quadruple, (A, B, C, D) , is finitely balanced for n steps, if it satisfies

$$\sum_{i=0}^{n-1} (A^T)^i C^T C A^i = \sum_{i=0}^{N-1} A^i B B^T (A^T)^i = \Lambda \quad (17)$$

where Λ is a diagonal matrix. It is then interesting to note that, by its construction, the realization of $H_c(p)$ ($H_d(w)$) as given by Algorithm 1 is finitely balanced for n steps. Indeed, let \mathcal{C}_h be the controllability matrix of the realization of $H_c(p)$ ($H_d(w)$), and \mathcal{O}_h be the observability matrix. It can be shown [13] that

$$\mathcal{C}_h \mathcal{C}_h^T = \mathcal{O}_h^T \mathcal{O}_h = \Sigma \quad (18)$$

which immediately implies (17).

In general, a finitely balanced realization may not necessarily be internally balanced, in the sense as defined by [6]. However, in our case, the generalized Markov parameters of $G_c(s)$ ($G_d(z)$) disappear after a finite number of steps. By following the same line as in [7], it can then be shown that the realization of $H_c(p)$ ($H_d(w)$) as given by Algorithm 1 is indeed internally balanced. Now, from Lemma 1, the bilinear transformation $T_c(s)$ ($T_d(z)$) preserves both the controllability and observability grammians. Therefore, the realization of $G_c(s)$ ($G_d(z)$) as given by Algorithm 1 will also be internally balanced and in fact have the same balanced grammians as the realization of $H_c(p)$ ($H_d(w)$). In summary, we have the following result.

Theorem 1 *The realizations of $H_c(p)$ and $G_c(s)$ ($H_d(w)$ and $G_d(z)$), as given by Algorithm 1, are both internally balanced, and in addition, they share the same balanced grammians.*

The algorithm provides an efficient solution to the problem of constructing a balanced realization for Laguerre models. The most appealing advantage of the algorithm is that it does not require the computation of controllability and observability grammians, which makes it computationally highly efficient.

4 An Example

Consider a 2×2 , third order Laguerre model, either continuous-time or discrete-time, where the Laguerre parameter $\lambda = 0.5$ and the coefficient matrices are given by

$$C_0 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Algorithm 1 was used to construct a balanced realization of the model, and the resultant system matrices obtained were as follows:

$$A_c = \begin{bmatrix} -0.9844 & -0.6146 & -0.2213 \\ -0.2421 & -0.3192 & -0.4007 \\ 0.5139 & 0.5362 & -0.1964 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -2.5655 & -2.1484 \\ -0.0752 & -1.2123 \\ 0.6047 & 0.4816 \end{bmatrix},$$

$$C_c = \begin{bmatrix} -2.8242 & -0.9787 & -0.5277 \\ -1.7947 & -0.7194 & 0.5649 \end{bmatrix}, D_c = 0$$

for the continuous-time case, and

$$A_d = \begin{bmatrix} 0.7625 & 0.1880 & -0.1149 \\ -0.0439 & 0.4619 & 0.5651 \\ -0.1870 & -0.1302 & 0.2756 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 1.7036 & 1.5114 \\ -0.5084 & 0.8104 \\ 0.8601 & 0.2753 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 1.9329 & -0.3796 & 0.7299 \\ 1.2161 & -0.8782 & -0.1590 \end{bmatrix}, D_d = 0$$

for the discrete-time case.

It may be verified that the resultant realizations obtained are indeed internally balanced.

5 Conclusions

In this paper, a simplified algorithm for constructing a balanced realization of a Laguerre model has been presented. The algorithm does not require the computation of controllability and observability grammians, nor the solution of Lyapunov equations. Thus, it is computationally highly efficient, compared to existing procedures. As a matter of fact, the only major computation in the algorithm is a singular value decomposition procedure.

References

- [1] Y.W. Lee. Synthesis of electrical networks by means of the Fourier transforms of Laguerre functions. *J. Maths. and Phys.*, 11:83–113, 1932.
- [2] N. Wiener. The theory of prediction. In E.F. Beckenbach, editor, *Modern Mathematics for the Engineer*. McGraw-Hill, Scarborough, CA, 1956.
- [3] C.C. Zervos, P.R. Bélanger, and G.A. Dumont. On PID controller tuning using orthonormal series identification. *Automatica*, 24(2):165–175, 1988.
- [4] B. Wahlberg. System identification using Laguerre models. *IEEE Trans. Automat. Contr.*, AC-36(5):551–562, 1991.
- [5] B. Wahlberg and P.M. Mäkilä. On approximation of stable linear dynamical systems using Laguerre and Kautz functions. *Automatica*, 32(5):693–708, 1996.
- [6] B.C. Moore. Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Trans. Automat. Contr.*, AC-26(1):17–32, 1981.
- [7] Q.-G. Zhou and E.J. Davison. Balanced realization of Laguerre network models. In *Proc. Amer. Contr. Conf.*, pages 2874–2878, Chicago, Illinois, 2000.
- [8] Q.-G. Zhou and E.J. Davison. Construction of minimal realizations based on the generalized markov parameters. In *Proc. 38th IEEE Conf. Dec. Contr.*, pages 942–947, Phoenix, Arizona, 1999.
- [9] H.P. Zeiger and A.J. McEwen. Approximate linear realization of given dimension via Ho's algorithm. *IEEE Trans. Automat. Contr.*, AC-19(2):153, 1974.
- [10] The Math Works, Natick, Mass. *MATLAB, High Performance Numeric Computation and Visualization Software: Reference Guide*, 1992.
- [11] B. Clapperton, F. Crusca, and M. Aldeen. Bilinear transformation and generalized singular perturbation model reduction. *IEEE Trans. Automat. Contr.*, 41(4):589–593, 1996.
- [12] W. Gawronski. *Balanced Control of Flexible Structures*. Springer, New York, 1996.
- [13] C.-T. Chen. *Linear System Theory and Design*. Holt, Rinehart and Winston, New York, 1994.