Optimal control with discontinuous running cost: eikonal equation and shape-from-shading

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Abstract

We consider a class of Hamilton-Jacobi equations with discontinuous coefficients that contains as examples the eikonal equation with discontinuous refraction index, or the Hamilton-Jacobi equation of shape-from-shading with discontinuous intensity function. In general the Dirichlet problem for such equations does not have unique solution, however we can characterize the minimal and maximal solution and, under appropriate compatibility conditions, prove existence of a unique solution in the sense of viscosity solutions.

1 Introduction

In this paper we want to discuss the following Dirichlet problem for the Bellman equation of optimal control

$$\begin{array}{ll} \lambda\, u(x)+\sup_{a\in A}\{-f(x,a)\cdot Du(x)-h(x,a)\} = g(x), & x\in\Omega \\ u(x)=0, & x\in\partial\Omega \end{array} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^N$, $\lambda \geq 0$ and $g:\Omega\to[0,+\infty[$ will always be locally bounded and Borel measurable, hence possibly discontinuous. In the study of (1.1) it is fundamental to consider the following control system

$$\left\{\begin{array}{l} \dot{y}(t)=f(y(t),a(t)) \\ y(0)=x \end{array}\right. \tag{1.2}$$

where $f:\mathbb{R}^N\times A\to\mathbb{R}^N$ is a continuous function and A is a compact subset of a metric space. In order to simplify the assumptions and avoid some technical difficulties, the set of admissible controls $\mathcal{A}$ in (1.2) is choosen to be the set of relaxed controls, i.e.

$$\mathcal{A}=\{a(\cdot): a:\mathbb{R}_+\to P(D) \text{ measurable}\},$$

and $P(D)$ is the set of Radon probability measures on $D\subset\mathbb{R}^N$ a compact set (see the book by Warga [11]). Optimal control problems for the system (1.2) will be defined with respect to a running cost function $h:\mathbb{R}^N\times A\to\mathbb{R}$ and the functional

$$J(t,x,a)=\int_0^t e^{-\lambda s}[h(y_x(s),a(s))+g(y_x(s))]ds. \tag{1.3}$$

In the following $h:\mathbb{R}^N\times A\to[0,+\infty[$ will be supposed to be at least continuous. In fact, some more stringent assumptions are needed on the data h, f, and throughout the paper we will suppose that, for all $R>0$, they satisfy

$$\left\{\begin{array}{ll} |f(x,a)-f(y,a)|\leq L|x-y|, & x,y\in\mathbb{R}^N,\ a\in A, \\ |h(x,a)-h(y,a)|\leq L_R|x-y|, & |x|,|y|\leq R,\ a\in A, \end{array}\right. \tag{1.4}$$

where L, L_R, C are positive constants. As a consequence, for given $a\in\mathcal{A}$ and $x\in\mathbb{R}^N$, there is a unique global solution of (1.2), that we denote by $y_x(t;a)\equiv y(\cdot)$.

A motivation for studying equation (1.1) with discontinuous coefficient g can be found in the applications. For instance in geometric optics, to describe the propagation of light, the eikonal equation appears

$$\sum_{i,j=1}^{N} a_{ij}(x)u_{x_ix_j}(x)=g(x),$$

where $a=\sigma\sigma^t$ and g has the meaning of the refraction index of the medium. As well known, refraction law applies across surfaces of discontinuity of g. Another example can be found in image processing and the shape-from-shading model. In this case we come up with the equation

$$\sqrt{1+|Du(x)|^2}=\frac{1}{I(x)}(\equiv g(x))$$

and the object to reconstruct is the graph of the unknown function u. In this case $I(x)\in]0,1]$ represents the intensity of light reflected by the object and it is discontinuous when the object has hedges. With a bit of convex analysis, both equations above can be rewritten in the form (1.1). Further motivation appears directly in control theory when discontinuous functions are used to represent targets (in this case g will be a characteristic function) or state constraints (when g is instead an indicator functions).

Being nonsmooth in general, solutions of (1.1) will be defined in the sense of the theory of viscosity solutions. An important point to analyze then comes up. Indeed with this notion of solution, when an equation has a discontinuous coefficient, we cannot interpret g in a pointwise sense. This is due to the fact that value functions

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of optimal control are natural candidate solutions for Bellman equations. For equation (1.1) it is easily recognized that, even when g has nice discontinuities, value functions will not always satisfy the equation in the viscosity sense if g is interpreted pointwise. We will check this fact by means of examples in the next section. In order to define viscosity solutions we use appropriate semicontinuous envelopes of g, applying an idea that already appears in Ishii [2]. Then the standard dynamic programming principle method easily adapts to show that value functions indeed solve the corresponding Bellman equation. The main drawback of this notion of solution becomes the fact that equations with different g's become equivalent from the point of view of the definition of solution. Uniqueness of solutions is therefore the main issue to investigate. This problem is however a common feature with other notions of solution such as the almost everywhere sense. It has to be noticed in fact that indeed uniqueness fails in many examples. As a matter of fact, uniqueness of solutions is related to the compatibility of the vector field f, the control problem corresponding to (1.1) and the set of discontinuities of g. Informally, we will show that the existence of an optimal trajectory of the control problem, whose set of times spent on the discontinuity set of g has measure zero is a necessary and sufficient condition for uniqueness of solutions of (1.1). More precisely, necessary and sufficient conditions for uniqueness can be stated on a minimizing sequence of controls and the corresponding trajectories. In general however, we will analyze the lack of uniqueness and give explicit representation formulas for the minimal and maximal solutions of (1.1).

Hamilton-Jacobi equations with discontinuous coefficients have been previously studied. Lions [3] studies the boundary value problem for the equation

$$\lambda\, u(x) + H(Du(x)) = g(x), \tag{1.5}$$

in a bounded domain with H convex and g lower semicontinuous, and shows the existence of a Lipschitz continuous almost everywhere solution. Newcomb-Su [5] studied the boundary value problem for the eikonal equation, i.e. (1.5) with $\lambda = 0$ and strictly positive g, using a different, although rather indirect, notion of solution that they call Monge solution. They prove a comparison principle and uniqueness in the class of continuous Monge solutions of the Dirichlet problem. Indeed the Monge solution characterizes the minimal among all viscosity solutions and it is not always the most natural among solutions, for instance in image processing.

2 Viscosity solutions and value functions

We start this section discussing the notion of viscosity solution for the Hamilton-Jacobi equation with discontinuous terms

$$\lambda u(x) + H(x, Du(x)) = g(x), \quad x \in \Omega, \tag{2.1}$$

where we denote the Hamiltonian

$$H(x,p) := \sup_{a\in A} \left\{-f(x,a)\cdot p - h(x,a)\right\}. \tag{2.2}$$

For a general reference on viscosity solutions in optimal control the reader can consult the book by Bardi and Capuzzo-Dolcetta [1]. In order to define viscosity solutions we need upper and lower semicontinuous envelopes of a locally bounded function $v : D \to \mathbb{R}$. They are, respectively,

$$\begin{array}{l} v^*(x) = \lim_{r\to 0^+} \sup_{y\in D,\ |y-x|\le r} v(y), \\ v_*(x) = \lim_{r\to 0^+} \inf_{y\in D,\ |y-x|\le r} v(y). \end{array}$$

Definition 2.1 *A lower semicontinuous function* $U : \Omega \to \mathbb{R}$ *(resp. upper semicontinuous* $V : \Omega \to \mathbb{R}$*) is a viscosity super- (resp. sub-) solution of (2.1) if for all* $\varphi \in \mathcal{C}^1(\Omega)$ *and a local minimum point* x *of* $(U-\varphi)$*, (resp. a local maximum point* x *of* $(V-\varphi)$*), we have*

$$\begin{array}{l} \lambda U(x) + H(x, D\varphi(x)) \ge g_*(x), \\ (resp.\ \lambda V(x) + H(x, D\varphi(x)) \le g^*(x)). \end{array}$$

A continuous function U *is a viscosity solution of (2.1) if it is both a supersolution and a subsolution. A solution* $U : \overline{\Omega} \to \mathbb{R}$ *of (2.1) solves the boundary value problem (1.1) if moreover it attains the boundary condition.*

Asking the boundary condition of a Dirichlet boundary value problem to be satisfied pointwise usually requires stringent assumptions on the data, as we will see at the end of this section. One might relax this request and study the case where the boundary condition is satisfied in a *generalized* sense. This extention creates however additional technical difficulties especially to determine uniqueness of solution and goes beyond the scope of this paper. In the case of continuous Hamiltonians, this theory was made precise by the author [6] and we refer the reader to the forthcoming paper [9], where the details in the case of discontinuous coefficients will be analyzed and we will also allow discontinuous viscosity solutions.

Note that the discontinuous term g is not dealt with pointwise. For equations of the form (1.1), that have an interpretation in control theory and calculus of variations, a key fact that we need to cope with when we define viscosity solutions is that value functions of the corresponding optimal control problems should solve the equation. The following examples give us a reason for using semicontinuous envelopes, showing that if g is dealt with pointwise, the theory is not particularly satisfactory.

Example 2.2 *Here we discuss the solvability of the simple equation*

$$|u'| = g(x), \quad x \in [-2,2],\ u(-2) = u(2) = 0 \qquad (2.3)$$

that can be represented in the form (1.1) with $A = [-1,1]$, $f(x,a) = a$ *and* $h \equiv 0$. *We will consider two cases where our analysis eventually will show uniqueness of viscosity solutions, as defined above. We will try and define viscosity solutions dealing with* g *in a pointwise way. Let us first consider the case when* $g(x) = 1$ *for* $x > 0$ *and* $g(x) = 0$ *for* $x < 0$. *We do not restrict ourselves to any particular value for* $g(0)$ *at the moment except that* g *will be nonnegative. With a control-theoretic interpretation, a sensible guess is*

$$u_o(x) = \begin{cases} 1 - |x-1|, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

This is indeed the value function of the optimal control problem corresponding to equation (2.3) whatever way we choose $g(0) \geq 0$. *In order to check that* u_o *is in fact a viscosity solution, the most delicate point is the origin. If* $u_o - \varphi$ *attains a minimum at* 0, *then* $\varphi'(0) \in [0,1]$. *Therefore if the coefficient* g *is treated pointwise, we come up with the condition* $g(0) \leq 0$. *Thus only a lower semicontinuous* g *would be allowed. There are no smooth functions such that* $u_o - \varphi$ *attains at zero a local maximum point.*

On the other hand, let us consider a discontinuous term such that $g(x) = 2 - x$, *for* $x > 0$ *and* $g(x) = 1$, *for* $x < 0$. *Again we can check directly if the function*

$$u(x) = \begin{cases} x^2/2 - 2x + 2, & x \geq 0, \\ x + 2, & x < 0, \end{cases}$$

i.e. the value function of the control problem corresponding to (2.3) when $g(0) \geq 1$, *is a viscosity solution of the equation. If* $u - \varphi$ *attains at* 0 *a maximum point then necessarily* $\varphi'(0) \in [-2,1]$. *It then turns out that* u *is a viscosity solution if and only if* $g(0) \geq 2$ *and therefore only an upper semicontinuous term* g *would be acceptable in this case.*

In order to see how easily uniqueness can fail without proper assumptions on g, *now that we accepted that envlopes of functions should be used, let us consider the same equation (2.3) as above with the choice* $g(x) = 2\chi_{\mathbb{Q}}$, *twice the charactristic function of the rationals. Then one easily checks that both* $u_1 \equiv 0$ *and* $u_2(x) = 2 - 2|x|$ *are viscosity solutions.*

We continue this section by discussing Lipschitz regularity of the value function of the control problem corresponding to the boundary value problem (1.1) and existence of solutions of (1.1). To define precisely the optimal control problem, let us consider the exit-time of the trajectories of the system (1.2) from Ω, i.e. for all $a \in \mathcal{A}$

$$\tau_x(a) = \inf\{t \geq 0 : y(t,x,a) \in (\mathbb{R}\backslash\Omega)\}.$$

We will show that, under reasonable assumptions, the value function

$$V(x) = \inf_{a \in \mathcal{A}} \{ \int_0^{\tau_x} e^{-\lambda s}\, [h(y_x(s), a(s)) + g(y_x(s))]\, ds\} \qquad (2.4)$$

is indeed locally Lipschitz continuous in $\bar{\Omega}$. We will consider the following assumption.

(STLC) For $x_o \in \overline{\Omega}$ there are $L, r > 0$ such that for $x \in \Omega \cap B(x_o, r)$ and $\forall z \in \bar{\Omega} \cap B(x_o, r)$, $\exists a_{x,z} \in \mathcal{A}$ and $T > 0$ such that $y_x(t; a_{x,z}) \in \Omega$, $\forall t \in]0,T[$, $y_x(T; a_{x,z}) = z$, and $T \leq L|x - z|$.

The previous condition is the so called small-time-local-controllability of the system at the points of the domain $\overline{\Omega}$. It is well known for instance that it holds at $x_o \in \Omega$ if the vector space spanned by $f(x_o, A)$ is the whole of $\mathbb{R}^N$. Note that V is locally bounded if $\lambda > 0$ and g, h are bounded. If (SLTC) holds then V is always locally bounded if Ω is bounded.

Proposition 2.3 *Assume (STLC) and let the functions* h, g *be bounded. If* $\lambda = 0$ *suppose in addition that* Ω *is bounded. Then the value function* V *in (2.4) is locally Lipschitz continuous in* Ω *and takes up the boundary condition continuously, i.e.*

$$\lim_{\Omega \ni x \to z} V(x) = 0, \quad z \in \partial\Omega.$$

Then it a viscosity solution of (2.1) and a solution of the boundary value problem (1.1).

Proof. We will show the regularity of the value function V. The fact that then V is a viscosity solution of the equation is a standard consequence of the method of dynamic programming and the ideas of viscosity solutions theory and we skip it, see e.g. ([1]).

Let $x_o \in \Omega$, we will show that V is Lipschitz continuous in $B(x_o, r) \cap \Omega$, where r is as in (STLC). Consider $x \in B(x_o, r) \cap \Omega$ and $z \in B(x_o, r) \cap \bar{\Omega}$. Let $\varepsilon > 0$ be fixed. Then there exists $\bar{a}$ such that

$$V(z) + \varepsilon \geq \int_0^{\tau_z(\bar{a})} e^{-\lambda s}\, [h(y_z(s;\bar{a}), \bar{a}(s)) + g(y_z(s;\bar{a}))]\, ds$$

Note that if $z \in \partial\Omega$, then $\tau_z(a) = 0$ for any $a \in \mathcal{A}$ and the estimates above and below will be even easier. We now define the following control

$$a(t) = \begin{cases} a_{x,z}(t) & t \leq T \\ \bar{a}(t - T) & t > T \end{cases}$$

Then we obtain, by a change of variables,

$$\begin{aligned} & V(x)-V(z)-\varepsilon \\ \le & \textstyle\int_0^{\tau_x(a)} e^{-\lambda s}\left[h(y_x(s;a),a(s))+g(y_x(s;a))\right]ds \\ - & \textstyle\int_0^{\tau_z(\bar a)} e^{-\lambda s}\left[h(y_z(s;\bar a),\bar a(s))+g(y_z(s;\bar a))\right]ds \\ \le & (e^{-\lambda T}-1)\textstyle\int_0^{\tau_z(\bar a)} e^{-\lambda s}\left[h(y_z(s;\bar a),\bar a(s))+g(y_z(s;\bar a))\right]ds \\ + & \textstyle\int_0^{T} e^{-\lambda s}\left[h(y_x(s;a_{x,z}),a_{x,z}(s))+g(y_x(s;a_{x,z}))\right]ds \\ & \le (4+\lambda)ML_R|x-z|, \end{aligned} \tag{2.5}$$

where M is a bound for h, g in $\bar\Omega$. If $z \in \Omega$, exchanging the roles of z and x, and because ε is arbitrary, then we obtain a local Lipschitz estimate for V in $B(x_o, r) \cap \Omega$. If instead $z \in \partial\Omega$ we similarly obtain

$$0 \ge \limsup_{\Omega \ni x \to z} V(x).$$

However by the assumptions we know that $0 \le V$ in Ω and we conclude. □

3 Optimality principles for viscosity solutions and minimal and maximal solutions

In this section we discuss the fact that, viscosity super and subsolutions of (2.1) can be characterized by some implicit representation formulas that use the data of the Hamiltonian along the solutions of (1.2) and how this can be applied to understand uniqueness or lack of uniqueness in the boundary value problem. This idea can be viewed as a weak form of the method of characteristics. We call such representation formulas *optimality principles.* Value functions of control problems satisfy optimality principles basically by the dynamic programming principle, but this property can be extended to all viscosity super and subsolutions of (2.1).

Definition 3.1 *We say that u satisfies the upper optimality principle in Ω with respect to the optimal control problem for system (1.2), set of admissible controls $\mathcal{A}$ and running cost $h+g$ if*

$$u(x) = \inf_{a\in\mathcal{A}} \sup_{t\in[0,\tau_x[} \left\{ \textstyle\int_0^t e^{-\lambda s}\left[h(y_x(s),a(s))+g_*(y_x(s))\right] + e^{-\lambda t}u(y_x(t)) \right\},$$

for all $x \in \Omega$.

We say that u satisfies the lower optimality principle in Ω if

$$u(x) = \inf_{a\in\mathcal{A}} \inf_{t\in[0,\tau_x[} \left\{ \textstyle\int_0^t e^{-\lambda s}\left[h(y_x(s),a(s))+g^*(y_x(s))\right] + e^{-\lambda t}u(y_x(t)) \right\},$$

for all $x \in \Omega$.

Optimality principles for Hamilton-Jacobi equations have been discussed in various ways and generalities, see for instance Lions [3], Lions-Souganidis [4], Swiech [10] and the references therein. Usually in the literature they appear as inequalities rather than equalities as here. The study of the formulation of Definition 3.1 and its implication to the lack of uniqueness for Hamilton-Jacobi equations of several control problems, starts in the papers by the author [7], [8], see also the references therein. The following extension, that applies to equations with discontinuous terms is however proved in detail the paper by the author [9].

Theorem 3.2 *1. Let $g : \Omega \to [0,+\infty[$ be lower semicontinuous and let U be a lower semicontinuous viscosity supersolution of (2.1), bounded from below. Then U satisfies the upper optimality principle in Ω.*

2. Let $g : \Omega \to [0,+\infty[$ be upper semicontinuous. Let U be an upper semicontinuous viscosity subsolution of (2.1). Then U satisfies the lower optimality principle in Ω.

We now address existence of maximal subsolutions and minimal supersolutions of (1.1), in particular to find their explict representation as value functions of optimal control problems. To this end we introduce the two following value functions of optimal control problems corresponding to the boundary value problem (1.1), namely respectively

$$V_m(x) = \inf_{a\in\mathcal{A}} \int_0^{\tau_x(a)} e^{-\lambda t}\left(h(y_x(s),a(s)) + g_*(y(t))\right)dt, \tag{3.1}$$

$$V_M(x) = \inf_{a\in\mathcal{A}} \int_0^{\tau_x(a)} e^{-\lambda t}\left(h(y_x(s),a(s)) + g^*(y(t))\right)dt. \tag{3.2}$$

Proposition 2.3 can be applied to both functions above. In the assumptions of that proposition, by definition of viscosity solution and of semicontinuous envelope of a function it is immediate to check that they are both Lipschitz continuous solutions of the same boundary value problem (1.1). Obviously $V_m(x) \le V_M(x)$, for $x \in \Omega$, and in general they are different, certainly in all cases when the jump set of g has nonempty interior, as one easily checks for instance for the control problem corresponding to the third case of Example 2.2. This explains structural lack of uniqueness of our problem.

By using optimality principles for our equation, we can obtain the following weak comparison result.

Proposition 3.3 *1. Let $U : \overline{\Omega} \to \mathbb{R}$ be a viscosity supersolution, bounded from below of the partial differential equation in (1.1) such that $U(x) \ge 0$ on $\partial\Omega$. We require U to be nonnegative if $\lambda = 0$. Then*

$$U(x) \ge V_m(x), \quad \text{for all } x \in \Omega.$$

Thus if moreover h, g are bounded, (SLTC) holds, and Ω is bounded if $\lambda = 0$, V_m is the minimal viscosity supersolution of (1.1) bounded from below (or nonnegative, if $\lambda = 0$).

2. Let $U : \overline{\Omega} \to \mathbb{R}$ be an upper semicontinuous viscosity subsolution of the partial differential equation in (1.1) such that $U(x) \leq 0$ on $\partial\Omega$. We also assume that U is bounded from above if $\lambda > 0$ or that $g(x) \geq c > 0$ for all $x \in \overline{\Omega}$ if $\lambda = 0$. Then

$$U(x) \leq V_M(x), \quad x \in \Omega.$$

Thus, if h, g are bounded, (SLTC) holds and Ω is bounded when $\lambda = 0$, then V_M is the maximal viscosity subsolution of (1.1) attaining the boundary condition.

Proof. The proof is a consequence of the optimality principles for the pde in (1.1). Let U be as in 1. Then by Theorem 3.2, U verifies the upper optimality principle in Ω. Therefore taking the limit as $t \to \tau_x(a)$ we obtain

$$\begin{aligned}
&U(x) = \inf_{a\in\mathcal{A}} \sup_{t\in[0,\tau_x(a)[} \\
&\{\int_0^t e^{-\lambda s}\left[h(y_x(s), a(s)) + g_*(y_x(s))\right] ds \\
&+ e^{-\lambda t} U(y_x(t))\} \geq V_m(x) = \inf_{a\in\mathcal{A}} \\
&\{\int_0^{\tau_x(a)} e^{-\lambda s}\left[h(y_x(s), a(s)) + g_*(y_x(s))\right] ds \\
&+ e^{-\lambda\tau_x(a)} U(y_x(\tau_x(a)))\}.
\end{aligned}$$

In the above, one just needs to be a little careful when $\tau_x(a) = +\infty$ and consider the cases $\lambda > 0$ and $\lambda = 0$ separately.

In the assumptions of 2., U satisfies the lower optimality principle in Ω. Then by taking again $t \to \tau_x(a) = +\infty$ we get

$$\begin{aligned}
&U(x) = \inf_{a\in\mathcal{A}} \inf_{t\in[0,+\infty]} \\
&\{\int_0^t e^{-\lambda s}\ \left[h(y_x(s), a(s)) + g^*(y_x(s))\right] ds + e^{-\lambda t} U(y_x(t))\} \\
&\leq V_M(x) = \inf_{a\in\mathcal{A}} \\
&\{\int_0^{\tau_x(a)} e^{-\lambda s}\left[h(y_x(s), a(s)) + g^*(y_x(s))\right] ds \\
&\quad + \chi_{\{\tau_x(a)<+\infty\}}(\tau_x(a))\ e^{-\lambda\tau_x(a)} U(y_x(\tau_x(a)))\}.
\end{aligned}$$

If $\lambda = 0$ we can reduce to the case $\tau_x(a) < +\infty$ because, since $g \geq c > 0$, trajectories that do not reach the boundary $\partial\Omega$ in finite time are never optimal. □

The consequence is the following result.

Corollary 3.4 *Assume (SLTC) and suppose that h, g are bounded. Let $U : \overline{\Omega} \to \mathbb{R}$ be bounded if $\lambda > 0$ or nonnegative if $\lambda = 0$. In the latter case also assume Ω bounded and $g(x) \geq c > 0$ for all $x \in \overline{\Omega}$. If U is a viscosity solution of the boundary value problem (1.1), then $V_m \leq U \leq V_M$ in Ω.*

The previous result gives explicit representation formulas for the minimal supersolution and the maximal subsolution, as value functions of optimal control problems related to the boundary value problem that we want to solve. Moreover, the question of uniqueness for (1.1) is characterized by a mere control theoretic question: is it true that V_m and V_M coincide? This is the problem that we address in the final section of this paper.

4 Uniqueness of viscosity solutions

We now address the uniqueness problem for (1.1). From what we obtained so far, it is clear that uniqueness does not hold, in general. We now introduce the key condition on the data for uniqueness to hold. It is related to the introduced value function V_m.

(JC)$_x$ at $x \in \Omega$ there exists a minimizing sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\begin{aligned}
&\lim_{n\to+\infty} \int_0^{\tau_x(a_n)} e^{-\lambda t}\ (h(y_x(t), a_n(t)) + g_*(y_x(t)))\, dt \\
&\quad = V_m(x) \quad \text{and} \\
&\lim_{n\to+\infty} \int_0^{\tau_x(a_n)} e^{-\lambda t}\ (g^*(y_x(t)) - g_*(y_x(t)))\, dt = 0.
\end{aligned}$$

The following theorem holds.

Theorem 4.1 *Suppose that (SLTC) holds and that the data g, h are bounded, and that Ω is bounded if $\lambda = 0$. At $x \in \Omega$, we have $V_m(x) = V_M(x)$ if and only if (JC)$_x$ holds. Therefore in the conditions of Corollary 3.4 a unique solution of (1.1) exists if and only if (JC)$_x$ holds for all $x \in \Omega$.*

Proof. Let us suppose first that $(a_n)_n \subset \mathcal{A}$ is a sequence with the property (JC)$_x$. So

$$\begin{aligned}
&V_m(x) = \lim_{n\to+\infty} \\
&\int_0^{\tau_x(a_n)} e^{-\lambda s}\left[h(y_x(s), a_n(s)) + g_*(y_x(s))\right] ds = \\
&\lim_{n\to+\infty} \{\int_0^{\tau_x(a_n)} e^{-\lambda s}\ \left[h(y_x(s), a_n(s)) + g^*(y_x(s))\right] ds \\
&\quad \geq V_M(x) \geq V_m(x).
\end{aligned}$$

Therefore $V_m(x) = V_M(x)$.

Conversely, let us suppose that $V_m(x) = V_M(x)$. We now choose a minimizing sequence of admissible controls $(a_n)_n \subset \mathcal{A}$ such that

$$\begin{aligned}
&V_M(x) = \\
&\lim_{n\to+\infty} \{\int_0^{\tau_x(a_n)} e^{-\lambda s}\ \left[h(y_x(s), a_n(s)) + g^*(y_x(s))\right] ds.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&V_M(x) \geq \limsup_{n\to+\infty} \\
&\{\int_0^{\tau_x(a_n)} e^{-\lambda s}\left[h(y_x(s), a_n(s)) + g_*(y_x(s))\right] ds \\
&\geq \liminf_{n\to+\infty} \\
&\{\int_0^{\tau_x(a_n)} e^{-\lambda s}\left[h(y_x(s), a_n(s)) + g_*(y_x(s))\right] ds \\
&\geq V_m(x) = V_M(x),
\end{aligned}$$

and hence (JC)$_x$ holds. □

Remark 4.2 *Condition $(JC)_x$ can be made slightly more readable. By our using relaxed controls, the value function V_m has optimal trajectories at any point. If $a \in \mathcal{A}$ is an optimal trajectory for $V_m(x)$ and $g^*(y_x(\cdot;a)) = g_*(y_x(\cdot,a))$ almost everywhere, then it is clear that $(JC)_x$ holds with the constant sequence $a_n \equiv a$.*

If on the other hand $V_m(x) = V_M(x)$ and $a \in \mathcal{A}$ is optimal for $V_M(x)$, then the proof of Theorem 4.1 shows that a is also optimal for $V_m(x)$ and $g^*(y_x(\cdot;a)) = g_*(y_x(\cdot,a))$ almost everywhere. Existence of optimal trajectories for V_M is doubtful in general, however if optimal trajectories at x for $V_M(x)$ exist, we can say that $V_m(x) = V_M(x)$ if and only if there is an optimal trajectory of $V_m(x)$ such that $g^*(y_x(\cdot;a)) = g_*(y_x(\cdot,a))$ almost everywhere.

By the dynamic programming principle, when the necessary and sufficient condition is given in terms of optimal controls and trajectories, it is equivalent to assume $(JC)_x$ at points where g has a jump in order to obtain the uniqueness of the solution of (1.1) by Theorem 4.1.

The reader can now go back and revisit some of the statements in Example 2.2 in view of the results in this section. The case of equations in a single variable when g has a finite number of jumps is easily analyzed by studying the control problem. We will next discuss an explicit example in higher dimensions, although here in a somewhat informal way, for more details see [9]. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set with nonempty boundary and suppose that

$$\overline{\Omega} = \cup_{i=1}^m \overline{\Omega}_i$$

where each Ω_i is an open, connected domain with Lipschitz boundary, $\overline{\Omega}_i \cap \overline{\Omega}_j = \partial\Omega_i \cap \partial\Omega_j$ if $i \neq j$, and each $x \in \Omega$ belongs to at most two subdomains $\overline{\Omega}_i$. We consider the boundary value problem for the eikonal equation

$$\begin{array}{ll} |Du(x)| = g(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{array} \tag{4.1}$$

where the discontinuous coefficient satisfies $g(x) \geq c$, $c > 0$. Moreover we suppose that in Ω_i near the boundary of the subdomain, g assumes a constant value g_i and that, for $x \in \partial\Omega_i$,

$$g(x) = \liminf_{(\Omega \setminus K) \ni y \to x} g(y),$$

where $K = \cup_{i=1}^m \partial\Omega_i$.

Theorem 4.3 *The boundary value problem (4.1) in the setting just described has a unique viscosity solution.*

Sketch of a proof. The idea of the proof of uniqueness is to exploit Theorem 4.1 and in particular Remark 4.2. For the complete details of the argument, see [9]. The key fact is that the system corresponding to the equation above has vector field $f(x,a) = a$, $a \in B_1(0)$. Therefore if an optimal trajectory for $V_m(x)$ intersects the set of jumps of g for a set of times of positive measure, we can modify it in a way to avoid hitting the set of jumps of g except for a finite number of times. Using geometric measure theory this can be done accurately, and we can construct an approximating minimizing sequence with the properties required by Theorem 4.1 and uniqueness follows. □

References

[1] M. Bardi, I. Capuzzo-Dolcetta, **Optimal control and viscosity solutions of Hamilton-Jacobi equations**, Birkhäuser 1997.

[2] H. Ishii, *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*, Ann. Sc. Norm. Sup. Pisa (IV) **16** (1989), 105–135.

[3] P.L. Lions, **Generalized solutions of Hamilton-Jacobi equations**, Pitman 1982.

[4] P.L. Lions, P.E. Souganidis, *Differential games and directional derivatives of viscosity solutions of Bellman's and Isaacs' equations* I and II, Siam J. Control Optim. **23** (1985), 566–583; **24** (1986), 1086–1089.

[5] R.T. Newcomb II, J. Su, *Eikonal equations with discontinuities*, Diff. Integral Equations **8** (1995), 1947–1960.

[6] P. Soravia, *Discontinuous viscosity solutions to Dirichlet problems for Hamilton-Jacobi equations with convex hamiltonians*, Commun. Partial Diff. Eq., **18** (1993), 1493–1514.

[7] P. Soravia, *Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. I. Equations of unbounded and degenerate control problems without uniqueness*, Advances Diff. Equations **4** (1999), 275–296.

[8] P. Soravia, *Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. II. Equations of control problems with state constraints*, Diff. Integral Equations **12** (1999), 275–293.

[9] P. Soravia, *Boundary vavlue problems for Hamilton-Jacobi equations with discontinuous Lagrangian*, preprint Universitá di Padova (2000).

[10] A. Swiech, *Sub- and superoptimality principles of dynamic programming revisited*, Nonlinear Anal. **26** (1996), 1429–1436.

[11] J. Warga, **Optimal control of differential and functional equations**, Academic Press 1972.