High gain observer for a class of implicit systems

Hassan HAMMOURI
Laboratoire d'Automatique et de Génie des Procédés,
UCB-Lyon 1, 43 bd du 11 Novembre 1918, 69622 Villeurbanne, France
E-mail: hammouri@lagep.univ-lyon1.fr

Nicolas MARCHAND
Laboratoire des Signaux et Systèmes
Supelec, Plateau de Moulon, 3 rue Joliot-Curie, 91190 Gif sur Yvette, France
E-mail: Nicolas.Marchand@lss.supelec.fr

Abstract

Under some observability assumptions (uniform observability), a high gain observer for a class of implicit dynamical systems is given in this paper. Numerically, the computation of trajectories of such implicit systems usually necessitates the use of an optimization algorithm together with an ODE numerical method. This complicates the synthesis of an observer. The observer design proposed here leads to a classical dynamical system defined on some $\mathbb{R}^N$, with $N \geq n$, n being the dimension of the state space of the implicit system.

Keywords : *Observer, Nonlinear Systems, Implicit Systems.*

1 Problem statement

In this paper, the class of implicit systems of the following form will be considered:

$$\begin{cases} \dot{x} \;=\; f_0(x,\rho) + \sum_{i=1}^{p} u_i f_i(x,\rho) \\ \varphi(x,\rho) = 0 \\ y \;=\; h(x,\rho) \end{cases} \tag{1}$$

where $(x,\rho) \in \mathbb{R}^n \times \mathbb{R}^d$, the f_i's and φ are assumed to be sufficiently smooth and:

$$\left.\frac{\partial \varphi}{\partial \rho}\right|_{x,\rho} \text{ is full rank } \forall (x,\rho) \in \mathbb{M} \tag{2}$$

where $\mathbb{M}$ is the set of zeros of φ:

$$\mathbb{M} := \left\{(x,\rho) \in \mathbb{R}^n \times \mathbb{R}^d, \text{ s.t. } \varphi(x,\rho) = 0\right\} \tag{3}$$

Clearly, from condition (2), $\mathbb{M}$ becomes a smooth manifold.

The computation of trajectories of such systems usually needs the use of optimization techniques either to explicit φ, or to make sure that $\varphi(x,\rho)$ effectively vanishes. This can for instance take the form of an ODE routine to integrate the first part of the system together with a optimization routine to solve the second part of the problem. These systems are often encountered in various fields such as the control of nonholonomic mechanical systems or of chemical reactors. Assumption (2) is known to be a characterization of the implicity called Hessenberg index one [1] (see also the implicit function theorem).

Now, set $z := \binom{x}{\rho}$, system (1) becomes equivalent to:

$$\begin{cases} \dot{z} \;=\; F_0(z) + \sum_{i=1}^{p} u_i F_i(z) \\ y \;=\; h(z) \\ z \in \mathbb{M} \end{cases} \tag{4}$$

with, for $0 \leq i \leq p$:

$$F_i(z) := \begin{pmatrix} f_i(x,\rho) \\ -\left(\left.\frac{\partial \varphi}{\partial \rho}\right|_{x,\rho}\right)^{-1} \left(\left.\frac{\partial \varphi}{\partial x}\right|_{x,\rho}\right) f_i(x,\rho) \end{pmatrix} \tag{5}$$

System (4) is called uniformly observable (observable independently on the input) if for any input u defined on any interval $[0,T]$ and for every initial states $x \neq \bar{x}$, there exists a time $t \in [0,T]$, such that $h(x(t;x,u)) \neq h(x(t;\bar{x},u))$ where $x(t;x,u)$ and $x(t;\bar{x},u)$ are respectively the trajectories of system (4) with input u and initial conditions x and $\bar{x}$. For such systems, it has been shown [2, 3, 4] that the map $\Phi : z \rightarrow (h(z), L_{F_0}(h)(z), \ldots, L_{F_0}^{n-1}(h)(z))$ is a local diffeomorphism (almost everywhere) which transforms system (4) in the following canonical form:

$$\begin{cases} \dot{\zeta} \;=\; A\zeta + \tilde{f}_0(\zeta) + \sum_{i=1}^{p} u_i \tilde{f}_i(\zeta) \\ y \;=\; C\zeta \end{cases} \tag{6}$$

with:

$$A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$$
$$\tilde{f}_0 = (0, \dots, 0, \tilde{f}_{0n})^T$$
$$\tilde{f}_{ij}(\zeta) = \tilde{f}_{ij}(\zeta_1, \dots, \zeta_j)$$

It is proved in [3], that the following system:

$$\dot{\hat{\zeta}} = A\hat{\zeta} + \tilde{f}_0(\hat{\zeta}) + \sum_{i=1}^{p} u_i \tilde{f}_i(\hat{\zeta}) - S_\theta^{-1} C^T (C\hat{\zeta} - y)$$

is an exponential observer for system (6) as soon as the fields F_i are global lipschitz with a lipschitz constant depending only upon the upper bound $\|u\|_\infty$.

From a purely mathematical point of view, the observer for system (4) can be obtained as follows: let $T\phi$ be the tangent map from the tangent space $T\mathbb{M}$ into $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, the observer for (4) takes the form:

$$\begin{cases} \dot{\hat{z}} = F_0(\hat{z}) + \sum_{i=1}^{n} u_i F_i(\hat{z}) - (T_{\hat{z}}\phi)^{-1} S_\theta^{-1} C^T (h(\hat{z}) - y) \\ \hat{z}(0) \in \mathbb{M} \end{cases} \quad (7)$$

In practice, this observer may work only if:

i) $\hat{z}(0) \in \mathbb{M}$ (that is $\varphi(\hat{x}(0), \rho(0)) = 0$)

ii) there is no parameter uncertainty

iii) the measurements are not noisy

The aim here consists in robustify the above observer so that the initialisation of $\hat{z}$ can be taken in some tubular neighbourhood of the manifold $\mathbb{M}$.

This paper is organized as follows. In section 2, some assumptions and preliminary results are given required by the observer's construction. Section 3 is dedicated to the main result of this paper.

2 Assumptions and preliminary results

Some notations and preliminary results used in the following to state the main result are given here.

Assumptions 1 *One shall assume that the following assumptions hold for system (1):*

i) There exists two compact sets $K \subset K' \subset \mathbb{M}$ and a bounded subset $\mathcal{U} \subset \mathcal{L}_\infty(\mathbb{R}^+, \mathbb{R}^p)$ such that for every $u \in \mathcal{U}$ and every associated trajectory $z_u(.)$ of system (1) issued from K, one has:

$$z_u(t) \in K' \quad \forall t \geq 0$$

ii) The map:

$$\phi : z \quad \rightarrow \quad \left(h(z), L_{F_0}(h)(z), \dots, L_{F_0}^{n-1}(h)(z)\right)^T$$

is a diffeomorphism from a bounded open set $\Omega \subset \mathbb{M}$ containing K' into its range.

iii) For $j = 0, \dots, n-1$, $i = 1, \dots, p$, one has on $\mathbb{M}$:

$$dL_{F_i} L_{F_0}^j(h) \wedge dL_{F_0}^j(h) \wedge dL_{F_0}^{j-1}(h) \wedge \dots \wedge dh = 0$$

where $\wedge$ denotes the exterior product of differential forms.

Remark 2.1 *Conditions 1.ii) and 1.iii) express the uniform observability of system (4). More precisely, ii) and iii) imply that system (4) restricted to Ω can be steered into the canonical form (6). Hence, the restriction of system (4) to Ω becomes observable independently of the input (see [2, 3]).*

The candidate observer proposed here is based on the following construction.

First, remark that Assumption 1.ii) together with condition (2) imply the existence of a tubular neighbourhood $\mathcal{T}_\mathbb{M}$ of the manifold $\mathbb{M}$ such that the map $\breve{\phi} : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ defined for all $z = (x, \rho)$ by:

$$\breve{\phi}(z) = \left(h(z), L_{F_0}(h)(z), \dots, L_{F_0}^{n-1}(h)(z), \varphi^T(z)\right)^T$$

is a diffeomorphism from $\mathcal{T}_\mathbb{M}$ into its range.

For all $j \in \{1, \dots, n\}$, consider the following construction:

- Ω_j and K_j are subsets of $\mathbb{R}^n$ defined by:

$$\Omega_j := \left\{ \left(h(z), L_{F_0}(h)(z), \dots, L_{F_0}^{j-1}(h)(z)\right), \ z \in \Omega \right\}$$
$$K_j := \left\{ \left(h(z), L_{F_0}(h)(z), \dots, L_{F_0}^{j-1}(h)(z)\right), \ z \in K \right\}$$

- let $\chi_j : \mathbb{R}^j \rightarrow [0, 1]$ be $\mathcal{C}^\infty$ functions such that:
  i) $\text{Supp}(\chi_j) \subset \Omega_j$ (where $\text{Supp}(\chi_j)$ denotes the closure of $\{x;\ \chi_j(x) \neq 0\}$)
  ii) $\chi_j(\xi_1, \dots, \xi_j) = 1$ for $(\xi_1, \dots, \xi_j) \in K_j$
- For $i = 0, \dots, p$, let the fields F_{ei} be defined by:

$$\left.\frac{\partial \breve{\phi}}{\partial z}\right|_z F_{e0}(z) = \begin{pmatrix} L_{F_0}(h)(z) \\ \vdots \\ L_{F_0}^{n-1}(h)(z) \\ \chi_n(h(z), \dots, L_{F_0}^{n-1}(h)(z)) L_{F_0}^n(h)(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and for $1 \leq i \leq p$:

$$\frac{\partial \breve{\phi}}{\partial z}\bigg|_z F_{e_i}(z) = \begin{pmatrix} \chi_1(h(z))L_{F_i}(h)(z) \\ \chi_2(h(z), L_{F_0}(h)(z))L_{F_i}(L_{F_0}(h))(z) \\ \vdots \\ \chi_n(h(z), ..., L_{F_0}^{n-1}(h)(z))L_{F_i}(L_{F_0}^{n-1}(h))(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Clearly, F_{e_i} is uniquely defined and coincide with F_i on K'.

The extended system is then defined by:

$$\begin{cases} \dot{z} &= F_{e_0}(z) + \sum_{i=1}^{p} u_i F_{e_i}(z) \\ y &= h(z) \\ z \in \mathbb{M} \end{cases} \tag{8}$$

Remarks 2.2

i) Let $u \in \mathcal{U}$ and $z_u(.)$ be a trajectory of system (4) issued from K, then $z_u(.)$ is also a trajectory of (8) and conversely.

ii) The F_{ei}'s are global lipschitz fields of $\mathbb{R}^{n+d}$.

Now, consider the following change of coordinates $\varsigma = \binom{\xi}{\eta} := \breve{\phi}(z) = (h(z), L_{F_0}(h)(z), \ldots, L_{F_0}^{n-1}(h)(z), \varphi^T(z))^T$ defined on $\mathcal{T}_{\mathbb{M}}$; then system (8) takes the following form:

$$\begin{cases} \dot{\xi} &= A\xi + \tilde{f}_{e_0}(\xi, \eta) + \sum_{i=1}^{p} u_i \tilde{f}_{e_i}(\xi, \eta) \\ \dot{\eta} &= 0 \\ y &= C\xi \end{cases} \tag{9}$$

where

- A and C are defined by:

$$A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}$$

- f_{e_0} is defined by:

$$\tilde{f}_{e_0}(\varsigma) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{F_0}^{n}(h)(\phi^{-1}(\varsigma)) \end{pmatrix}$$

- from Assumption 1.ii) and the above construction, the f_{ei}'s are such that $\forall i \in \{1, \ldots, p\}$, $\forall j \in \{1, \ldots, n\}$, $\forall \varsigma \in \breve{\phi}(\mathbb{M})$, one has:

$$\tilde{f}_{e_{i_j}}(\varsigma) = \tilde{f}_{e_{i_j}}(\varsigma_1, \ldots, \varsigma_j) \tag{10}$$

3 Main result

With the above notations and definitions, our main result can be stated:

Theorem 3.1 *Assume that system (1) fulfills Assumptions 1, then:*

i) for every compact $\hat{K}$ of $\mathbb{R}^{n+d}$, there exists a real positive number θ and a positive definite matrix Ω so that for any initialisation $\hat{\varsigma}_0 = (\hat{\xi}_0, \hat{\eta}_0)$ in $\hat{K} \bigcap \mathcal{T}_{\mathbb{M}}$, the following system:

$$\begin{cases} \dot{\hat{\xi}} = A\hat{\xi} + \tilde{f}_{e_0}(\hat{\xi}, \hat{\eta}) + \sum_{i=1}^{p} u_i \tilde{f}_{e_i}(\hat{\xi}, \hat{\eta}) \\ \qquad -S_\theta^{-1} C^T (C\hat{\xi} - y) \\ \dot{\hat{\eta}} = -\Omega \hat{\eta} \end{cases} \tag{11}$$

is an exponential observer for system (4), where S_θ is defined by:

$$\theta S_\theta + A^T S_\theta + S_\theta A = C^T C \tag{12}$$

ii) In the original set of coordinates, observer (11) takes the form:

$$\begin{cases} \dot{\hat{x}} = f_{e_0}(\hat{x}, \hat{\rho}) + \sum_{i=1}^{p} u_i f_{e_i}(\hat{x}, \hat{\rho}) \\ \quad - \left(\frac{\partial \phi}{\partial x}\Big|_{\hat{x}, \hat{\rho}} \right)^{-1} S_\theta^{-1} C^T (h(\hat{x}, \hat{\rho}) - y) \\ \dot{\hat{\rho}} = - \left(\frac{\partial \varphi}{\partial \rho}\Big|_{\hat{x}, \hat{\rho}} \right)^{-1} \Bigg[\Omega \varphi(\hat{x}, \hat{\rho}) \\ \quad + \left(\frac{\partial \varphi}{\partial x}\Big|_{\hat{x}, \hat{\rho}} \right) \Big[f_{e_0}(\hat{x}, \hat{\rho}) + \sum_{i=1}^{p} u_i f_{e_i}(\hat{x}, \hat{\rho}) \\ \quad - \left(\frac{\partial \phi}{\partial x}\Big|_{\hat{x}, \hat{\rho}} \right)^{-1} S_\theta^{-1} C^T (h(\hat{x}, \hat{\rho}) - y) \Big] \Bigg] \end{cases} \tag{13}$$

Remarks 3.1

i) Observer (13) only requires the jacobian of diffeomorphism ϕ. Consequently, the above scheme does not require to explicit the solution ρ of $\varphi(x, \rho) = 0$.

ii) The dynamic of $\hat{\eta}$ of equation (11) can be seen as a continuous time transcription of an optimization Newtown's algorithm. The above scheme mixes a classical high gain observer with an optimization routine. Hence, such an approach can be successfully used with any other exponential observer.

Proof :

Here, the trajectories of the system (4) are assumed to be into K' (Assumption 1.i)). This in particular implies that $\xi(t)$ belongs to a bounded subset of $\mathbb{R}^n$.

Let $\varepsilon := \Delta_\theta(\hat{\xi}-\xi)$, where Δ_θ is the $n\times n$ diagonal matrix $\text{diag}(\frac{1}{\theta},\ldots,\frac{1}{\theta^n})$. Let $V := V_1 + V_2$ be a candidate Lyapunov function with:

$$V_1 := \varepsilon^T S_1 \varepsilon \quad \text{and} \quad V_2 := \hat{\eta}^T\hat{\eta}$$

By posing $\Lambda_e(\xi,\eta,u) := \tilde{f}_{e_0}(\xi,\eta)+\sum_{i=1}^{p} u_i \tilde{f}_{e_i}(\xi,\eta)$, similar calculations as in [3] yields:

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \dot{V}_2\\ &= 2\varepsilon^T S_1\Big[\theta(A - S_1^{-1}C^TC)\varepsilon \qquad (14)\\ &+\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u) - \Lambda_e(\xi,0,u)\right)\Big] - 2\hat{\eta}^T\Omega\hat{\eta}\end{aligned}$$

Using (12):

$$\begin{aligned}\dot{V} &= -\theta V_1 - \varepsilon C^TC\varepsilon - 2\hat{\eta}^T\Omega\hat{\eta} \qquad (15)\\ &+2\varepsilon^T S_1\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u) - \Lambda_e(\xi,0,u)\right)\end{aligned}$$

The main difficulty that occurs here is that, contrary to the usual high gain observer's proof [3], Λ_e does not have any triangular structure elsewhere that on the manifold $\mathbb{M}$. Indeed, the estimate $\binom{\hat{\xi}}{\hat{\eta}}$ can not be assumed to remain on the manifold though $\binom{\xi}{0}$ does. Writing

$$\begin{aligned}&\Lambda_e(\hat{\xi},\hat{\eta},u) - \Lambda_e(\xi,0,u) =\\ &\quad\Lambda_e(\hat{\xi},\hat{\eta},u)-\Lambda_e(\hat{\xi},0,u)-\Lambda_e(\xi,0,u)+\Lambda_e(\hat{\xi},0,u)\end{aligned}$$

equality (15) becomes:

$$\begin{aligned}\dot{V} &= -\theta V_1 - \varepsilon C^TC\varepsilon\\ &+2\varepsilon^T S_1\Delta_\theta\left(\Lambda_e(\hat{\xi},0,u) - \Lambda_e(\xi,0,u)\right)\\ &-2\hat{\eta}^T\Omega\hat{\eta} + 2\varepsilon^T S_1\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u) - \Lambda_e(\hat{\xi},0,u)\right)\end{aligned}$$

Using Schwarz inequality and noting by $\lambda_{\min}(\Omega)$ the smallest eigenvalue of Ω, it gives:

$$\begin{aligned}&\dot{V} \leq -\theta V_1 - 2\lambda_{\min}(\Omega)V_2 + 2\sqrt{V_1}\Big[\qquad (16)\\ &\sqrt{\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},0,u)-\Lambda_e(\xi,0,u)\right)\right]^T S_1\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},0,u)-\Lambda_e(\xi,0,u)\right)\right]} +\\ &\sqrt{\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u)-\Lambda_e(\hat{\xi},0,u)\right)\right]^T S_1\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u)-\Lambda_e(\hat{\xi},0,u)\right)\right]}\Big]\end{aligned}$$

Similarly as in [3], using:

- the triangular form (10) (the i^{th} component of $\Lambda_e(\xi,0,u)$depends only upon $(\xi_1,\ldots,\xi_i)$),
- the uniform bound on the controls (Assumption 1.i)),
- the global lipschitz property with of the fields F_{e_i},

one deduces:

$$\begin{aligned}&\sqrt{\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},0,u)-\Lambda_e(\xi,0,u)\right)\right]^T S_1\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},0,u)-\Lambda_e(\xi,0,u)\right)\right]}\\ &\qquad\leq k\,\|\varepsilon\|\\ &\qquad\leq \frac{k}{\lambda_{\min}(S_1)}\sqrt{V_1} \qquad (17)\end{aligned}$$

where k is a positive constant which does not depend on θ.

Using the mean value theorem and noticing that $V_2 = \|\hat{\eta}\|^2$, there exists a continuous function $k'(\theta,\hat{\xi},\hat{\eta},u)$ such that:

$$\begin{aligned}&\sqrt{\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u)-\Lambda_e(\hat{\xi},0,u)\right)\right]^T S_1\left[\Delta_\theta\left(\Lambda_e(\hat{\xi},\hat{\eta},u)-\Lambda_e(\hat{\xi},0,u)\right)\right]}\\ &\qquad\leq k'(\theta,\hat{\xi},\hat{\eta},u)\sqrt{V_2} \qquad (18)\end{aligned}$$

Using inequalities (17) and (18) in (16) gives:

$$\begin{aligned}\dot{V} \;\leq\; &-\left(\theta - \frac{2k}{\lambda_{\min}(S_1)}\right)V_1 - 2\lambda_{\min}(\Omega)V_2\\ &+k'(\theta,\hat{\xi},\hat{\eta},u)\sqrt{V_1}\sqrt{V_2} \qquad (19)\end{aligned}$$

For any compact set $\hat{K}$, let:

$$\alpha := \sup\left\{V(\xi,\eta,\hat{\xi},\hat{\eta});\ (\xi,\eta)\in K,\ (\hat{\xi},\hat{\eta})\in\hat{K}\right\}$$

$$\hat{K}' := \left\{(\hat{\xi},\hat{\eta})\in\mathbb{R}^n\times\mathbb{R}^d; V(\xi,\eta,\hat{\xi},\hat{\eta})\leq\alpha, (\xi,\eta)\in K\right\}$$

$$\bar{k}'(\theta) := \sup_{\substack{(\hat{\xi},\hat{\eta})\in\hat{K}'\\ u\in\mathcal{U}\\ t\geq 0}} k'(\theta,\hat{\xi},\hat{\eta},u(t))$$

Then, for any initialisation $\hat{\varsigma}_0 = \binom{\hat{\xi}_0}{\hat{\eta}_0}\in\hat{K}$, it is sufficient to chose θ and Ω such that:

$$\begin{aligned}\theta \;&>\; \frac{2k}{\lambda_{\min}(S_1)}\\ \lambda_{\min}(\Omega) \;&\geq\; \frac{\bar{k}'(\theta)^2}{8\left(\theta - \frac{2k}{\lambda_{\min}(S_1)}\right)}\end{aligned}$$

Indeed, with the above choices, one has:

$$\begin{aligned}\dot{V} \;&\leq\; -\left(\sqrt{\left(\theta - \frac{2k}{\lambda_{\min}(S_1)}\right)V_1} + \sqrt{2\lambda_{\min}(\Omega)V_2}\right)^2\\ &\leq\; -\min\left(\theta - \frac{2k}{\lambda_{\min}(S_1)}, 2\lambda_{\min}(\Omega)\right)V\end{aligned}$$

This last inequality proves the exponential decrease of V.

Finally, the form (13) can simply be obtained using $\breve{\phi}^{-1}$ and noticing that $\hat{\eta} = \varphi(\hat{x},\hat{\rho})$, which gives:

$$\dot{\hat{\rho}} = -\left(\left.\frac{\partial\varphi}{\partial\rho}\right|_{\hat{x},\hat{\rho}}\right)^{-1}\left.\frac{\partial\varphi}{\partial x}\right|_{\hat{x},\hat{\rho}}\dot{\hat{x}} - \left(\left.\frac{\partial\varphi}{\partial\rho}\right|_{\hat{x},\hat{\rho}}\right)^{-1}\Omega\varphi(\hat{x},\hat{\rho})$$

4 Conclusion

In this paper, a high gain observer for a class of implicit systems moving onto a manifold $\mathbb{M}$ is proposed. The observer is proved to be exponentially convergent in a tubular neighbourhood of the manifold. The proposed scheme does not require to explicit the implicit relation defining $\mathbb{M}$ or even to use optimization techniques as it is usually needed. It would be interesting for further investigations to see if the proposed scheme can be generalized to higher implicity degrees as it can be done for some numerical methods [1].

References

[1] M. U. Ascher and R. L. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. Society for Industrial & Applied Mathematics, July 1998.

[2] J. P. Gauthier and G. Bornard, “Observability for any input of a class of nonlinear systems,” *IEEE Trans. on Automatic Control*, vol. 26, no. 4, pp. 922–926, 1981.

[3] J. P. Gauthier, H. Hammouri, and S. Othmann, “A simple observer for nonlinear systems. application to bioreactors,” *IEEE Trans. on Automatic Control*, vol. 37, no. 6, pp. 875–880, 1992.

[4] H. Hammouri, K. Busawon, and M. Farza, “Nonlinear observer for local uniform observable systems,” internal report, LAGEP - 43 bd du 11 Novembre 1918, 69622 Villeurbanne, France, 1997.