Optimal Decentralized Controllers for Spatially Invariant Systems

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Abstract

We consider the problem of optimal $\mathcal{H}_2$ design of semi-decentralized controllers for a special class of spatially distributed systems. This class includes spatially invariant and distributed discrete-time systems with an inherent temporal delay in the interaction of neighbouring sites. Such a structure arises naturally from spatio-temporal discretizations of many physical systems described by partial differential equations. We consider the problem of optimal design of distributed controllers that have the same information passing delay structure as the plant. We show how the YJBK parametrization of such stabilizing controllers yields a convex parametrization for this class. We then show how the optimal $\mathcal{H}_2$ problem can be solved exactly.

1 Introduction

We consider spatially distributed systems where all signals are functions of discrete spatial and temporal indeces, e.g. $\{u(i,k)\}$, where both i and k are integers and we interpret each as the spatial and temporal index respectively. The theory of such spatio-temporal systems has been worked out in some detail. We consider only spatially distributed systems with the additional property that the dynamics are spatially invariant. For recent work on this class and some of the background for the present work, we refer the reader to [1, 2] and the references therein.

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2 Problem Definition

Consider the standard configuration for disturbance attenuation in Figure 1 where the plant G and the controller K are spatially and temporally invariant systems. The particular structure of interest is when in the spatial and temporal transform domain G_{22} is of the form

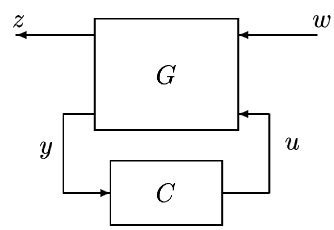

Figure 1: The Standard Problem

$$G_{22}(z,\lambda) = \sum_{i=-\infty}^{\infty} g_i(\lambda) z^i \tag{1}$$

with

$$g_i(\lambda) = \lambda^{|i|} \tilde{g}_i(\lambda)$$

where λ corresponds to the temporal one-sided transform variable, z corresponds to the spatial two-sided transform variable, and $\tilde{g}_i(\lambda)$ is a transfer funtion corresponding to a temporally causal system. The interpretation of this structure is that the input u_i to the ith system g_i affects the output y_j of the jth system g_j which is $|j-i|$ spatial locations away with a (time) delay proportional to their spatial distance $|j-i|$, i.e., with a delay of $|j-i|$ time steps.

Given such a G_{22} we are looking for stabilizing controllers with the same structure as G_{22}. Namely, we want K as

$$K(z,\lambda) = \sum_{i=-\infty}^{\infty} k_i(\lambda) z^i \tag{2}$$

with

$$k_i(\lambda) \;=\; \lambda^{|i|}\tilde{k}_i(\lambda)$$

Thus we are imposing an implicit decentralized structure on K since now the measurements of the jth location will be made available at the ith station after $|j-i|$ time steps.

The problem of interest is to design such a K which is stabilizing and minimizes the $\mathcal{H}_2$ norm of the closed loop.

3 Problem Solution

We consider here only the case where G_{22} is stable. The following proposition shows that by employing the Youla parametrization the decentralization constraints on K transform to convex constraints on the Youla parameter Q.

Proposition 3.1 *All stabilizing K with the desired structure are given by*

$$K \;=\; -Q(I-G_{22}Q)^{-1},$$

with Q stable given by

$$Q(z,\lambda) \;=\; \sum_{i=-\infty}^{\infty} q_i(\lambda)z^i$$

and with q_i of the form

$$q_i(\lambda) \;=\; \lambda^{|i|}\tilde{q}_i(\lambda)$$

where $\tilde{q}_i$ is stable.

Proof: That all stabilizing, possibly without the structure, K are given as $K = -Q(I-G_{22}Q)^{-1}$ follows the same arguments as in the finite dimenstional case [3]. We will thus prove the structural property of Q only. We view G_{22} as the following mapping

$$\begin{pmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \\ \cdots & g_1 & g_0 & g_{-1} & \cdots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{pmatrix}.$$

Grouping together the outputs at each time step $0,1,2,\ldots$ as

$$y(0) = \begin{pmatrix} \vdots \\ y_{-1}(0) \\ y_0(0) \\ y_1(0) \\ \vdots \end{pmatrix}, \qquad y(1) = \begin{pmatrix} \vdots \\ y_{-1}(1) \\ y_0(1) \\ y_1(1) \\ \vdots \end{pmatrix}, \ldots$$

and similarly for u we can view $y = G_{22}u$ as a standard matrix

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} G_0 & & & \\ G_1 & G_0 & & \\ G_2 & G_1 & G_0 & \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \end{pmatrix}$$

where G_i are band operators. In particular, we have

$$G_0 \;=\; \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ldots & 0 & \hat{g}_0(0) & 0 & \ldots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

i.e., diagonal,

$$G_1 = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ldots & 0 & \hat{g}_1(0) & \hat{g}_0(1) & \hat{g}_1(0) & 0 & \ldots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

i.e., 3-diagonal, and generalizing, G_i is (2i+1)-diagonal, where $\hat{g}_i = \{\hat{g}_i(t)\}_{t=0}^{\infty}$ denotes the pulse response of the system associated with $g_i(\lambda)$. Representing similarly K

$$K \;=\; \begin{pmatrix} K_0 & & \\ K_1 & K_0 & \\ & \ddots & \ddots \end{pmatrix}$$

it is easy to verify that K is required to have K_0 diagonal, K_1 3-diagonal,...,K_i (2i+1)-diagonal, etc. By considering the parametrization $K = -Q(I - G_{22}Q)^{-1}$ it is now clear that K has the required structure if and only if Q has precisely the same structure, namely,

$$Q \;=\; \begin{pmatrix} Q_0 & & \\ Q_1 & Q_0 & \\ & \ddots & \ddots \end{pmatrix}$$

with Q_i $2i+1$-diagonal which completes the proof. ∎

With the above parametrizatiofproblem of interest becomes

$$\inf_Q \|H - UQV\|$$

with

$$Q(z,\lambda) \;=\; Q_0(z) + Q_1(z)\lambda + \ldots$$

where

$$Q_0(z) = q_{0,0}$$
$$Q_1(z) = q_{1,-1}z^{-1} + q_{1,0} + q_{1,1}z$$
$$Q_2(z) = q_{2,-2}z^{-2} + q_{2,-1}z^{-1} + q_{2,0} + q_{2,1}z + q_{22}z^2$$
$$\ldots$$

with $q_{i,j}$ scalars and with H, U, V are stable maps that depend only on G. For simplicity, let us further assume that $V = I$; this does not change the main idea of our approach.

Do a inner-outer for $U(z,\lambda)$ fixing z

$$U(z,\lambda) \;=\; U_{\text{in}}(z,\lambda)U_{\text{out}}(z,\lambda)$$

Let

$$R := U_{\text{out}}Q = R_0(z) + R_1(z)\lambda + \cdots$$

then

$$\begin{aligned}\inf_Q \|H - UQ\| &= \inf_Q \left\|\tilde{U_{\text{in}}}H - U_{\text{out}}Q\right\| \\ &= \inf_R \left\|\tilde{U_{\text{in}}}H - R\right\|.\end{aligned}$$

Let

$$U_{\text{out}}(z,\lambda) \;=\; U_0(z) + U_1(z)\lambda + \cdots$$

It thus becomes a problem of characterizing the decentralization constraint on R. Clearly the problem is convex but not finite dimensional. Instead of solving this problem we look at a relaxation of it. Namely, we require that only the first N coefficients in $Q(\lambda, z) = Q_0(z) + Q_1(z)\lambda + \ldots$ are constrained to correspond to band-operators Q_i with $(2i+1)$-diagonal for $i = 0, 1, \ldots, N-1$, where N is arbitrary. This equivalently amounts to relaxing the controller structure in exactly the same manner. The interpretation is that if the spatial distance $|i - j|$ between location i and j is greater or equal than N the time delay in obtaining measurement y_j for the use in control decision u_i at location i is N (i.e., it does not grow indefinetly.) Of course, if $|i - j| < N$ the delay is $|i - j|$ time-steps.

To further clarify the ideas, let us first look at the one-step delay problem, i.e., $N = 1$. The only constraint there is that

$$Q_0(z) = q_{0,0} = \text{ scalar independent of } z$$

The following shows how R is affected.

Proposition 3.2 *With the constraint* $Q_0(z) = q_{0,0}$ *as above,*

$$R = U_{\text{out}}Q \quad\Leftrightarrow\quad R_0(z) = q_{0,0}U_0(z),\; q_{0,0} \text{ is a scalar}$$

Proof: $(\Rightarrow)$ obvious $(\Leftarrow)$ Let R be s.t. $R_0(z) = q_{0,0}U_0(z)$, $q_{0,0}$ scalar. Consider

$$Q \;=\; q_{0,0} + U_{\text{out}}^{-1}(R - U_{\text{out}}q_{0,0})$$

Note that $R - U_{\text{out}}q_{0,0}$ is of the form $\lambda\tilde{R}(z,\lambda)$ and hence

$$Q(z,0) \;=\; q_{0,0}.$$

Moreover,

$$U_{\text{out}}Q \;=\; R.$$

■

Hence, only $R(z,0) = R_0(z)$ is affected. Denoting

$$X = \tilde{U_{\text{in}}}H = \cdots X_{-1}(z)\lambda^{-1} + X_0(z) + X_1(z)\lambda + \cdots$$

by applying a standard projection thorem we get that the optimal value of R is given by

$$\begin{aligned}R^* &= \Pi_{\mathcal{H}_2(S_A)}[X] \\ &= \Pi_{(S_A)}[X_0(z)] + X_1(z)\lambda + X_2(z)\lambda^2 + \cdots\end{aligned}$$

where

$$S_A \;=\; \{r(z) : r(z) = q_{0,0}U_0(z) \text{ for some scalar } q_{0,0}\},$$

$$\begin{aligned}\mathcal{H}_2(S_A) = \{&\cdots X_{-1}(z)\lambda^{-1} + X_0(z) + X_1(z)\lambda + \cdots : \\ &X_0(z) = q_{0,0}U_0(z), \text{ for some scalar } q_{0,0}\}\end{aligned}$$

To find $\Pi_{S_A}[X_0(z)]$ amounts to finding $q^*_{0,0}$ such that

$$< X_0(z) - q^*_{0,0}U_0(z), q_{0,0}U_0(z) > \;=\; 0 \text{ for any } q_{0,0}.$$

Therefore

$$q^*_{0,0} \;=\; < X_0(z), U_0(z) > / < U_0(z), U_0(z) >$$

For N-step delay the same albeit more complicated procedure holds. The key again is that only the the N-coefficients of R are affected as the following shows

Proposition 3.3 *With the constraint Q_i corresponding to 2i+1 diagonal operator for $i = 0, 1, \ldots, N-1$ it holds that*

$$R = U_{\text{out}} Q$$

if and only if

$$\begin{pmatrix} R_0(z) \\ \vdots \\ R_{N-1}(z) \end{pmatrix} = \begin{pmatrix} U_0(z) & & & \\ U_1(z) & U_0(z) & & \\ \vdots & & \ddots & \\ U_{N-1}(z) & \ldots & \ldots & U_0(z) \end{pmatrix} L(z) q \quad (3)$$

where $L(z)$ is given as

see appendix

and the vector q is of the form

$$q^T := \begin{pmatrix} q_0^T & q_1^T & \ldots & q_{N-1}^T \end{pmatrix} \quad (4)$$

with

$$q_i^T = \begin{pmatrix} q_{i,-i} & \ldots & q_{i,0} & \ldots & q_{i,i} \end{pmatrix} \quad (5)$$

Proof: ($\Rightarrow$) obvious ($\Leftarrow$) Let R be s.t. it satisfies the condition of the above proposition for some scalar vector q. For $i = 0, \ldots, N-1$ define $Q_i(z) := \sum_{j=-i}^{j=i} q_{i,j} z^j$ and consider

$$Q_N(z, \lambda) := \sum_{i=0}^{i=N-1} Q_i(z) \lambda^i$$

Define

$$Q = Q_N + U_{\text{out}}^{-1} (R - U_{\text{out}} Q_N)$$

Note that $R - U_{\text{out}} Q_N$ is of the form $\lambda^N \tilde{R}(z, \lambda)$ and hence Q is of the required form and moreover,

$$U_{\text{out}} Q = R.$$

■

According to the parameterization (3) of R, computing the optimal solution in this case amounts to calculating

$$R^* = \Pi_{\mathcal{H}_2(S_A)}[X] =$$

$$\chi_0(z) + \chi_1(z)\lambda + \ldots + \chi_{N-1}(z)\lambda^{N-1} + \ldots$$

where

$$\begin{pmatrix} \chi_0(z) \\ \chi_1(z) \\ \vdots \\ \chi_{N-1}(z) \end{pmatrix} = \Pi_{S_A} \begin{pmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{N-1}(z) \end{pmatrix}$$

with

$$S_A =$$

$$\left\{ r(z) : r(z) = \sum_{\substack{i = 0, N-1 \\ j = -i, i}} q_{i,j} V_{i,j}(z) \text{ for some } q \right\}$$

where the vector of scalars q is of the form (4),(5) and

$$V_{i,j}(z) = \begin{pmatrix} \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} i \\ U_0(z) \\ U_1(z) + U_0(z) \\ \vdots \\ U_{N-1-i} + \ldots + U_0 \end{pmatrix} z^j$$

Then, by a standard projection result, the vector q^* can be obtained as the solution of a set of linear equations

$$A q^* = p \quad (6)$$

where the components of the matrix A and the vector p are shown in the **appendix**.

4 Example

Consider the one dimensional "damped" heat equation on an infinite spatial domain

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} - \varepsilon y(x,t) + u(x,t)$$

and its finite difference approximation with time and space discretization steps equal to T and L, respectively.

$$\frac{y(i,k+1) - y(i,k)}{T} = \frac{y(i+1,k) - 2y(i,k) + y(i-1,k)}{L^2} -$$

$$-\varepsilon y(i,k) + u(i,k).$$

Taking the appropriate transforms one obtains the transfer function

$$G(z, \lambda) = \frac{y(z,\lambda)}{u(z,\lambda)} = \frac{T\lambda}{1 - \frac{\gamma}{2}\left(z^{-1} + 2\alpha + z\right)\lambda}$$

where

$$\gamma = 2T/L^2 \quad ; \quad \alpha = L^2/(2T) - \varepsilon L^2/2 - 1$$

It can be easily checked that the dynamics of such a system are asymptotically stable under the following conditions:

$$\begin{cases} \gamma < 1/(1+\alpha) \\ \alpha > 1 - 1/\gamma \end{cases}$$

which in turn correspond to

$$\begin{cases} \varepsilon > 0 \\ T < \dfrac{1}{2/L^2 + \varepsilon/2} \end{cases}$$

Moreover, $G(z,\lambda)$ is of the form (1),

$$G(z,\lambda) = \sum_{n=1}^{\infty} T \left(\frac{\gamma}{2}\right)^{n-1} \left(z^{-1} + 2\alpha + z\right)^{n-1} \lambda^n$$

We want to compute a decentralized controller for optimal $\mathcal{H}_2$ attenuation of an additive disturbance on the system output with weighting function

$$W(z,\lambda) = \frac{W_0 \lambda}{1 - \frac{c}{2}\left(z^{-1} + 2a + z\right)\lambda}$$

which is of the same structure as the plant itself. We assume $W(z,\lambda)$ to be asymptotically stable as well, i.e.

$$\begin{cases} c < 1/(1+a) \\ a > 1 - 1/c \end{cases} .$$

With the stabilizing controller parameterization

$$K(z,\lambda) = \frac{Q(z,\lambda)}{1 - G(z,\lambda)Q(z,\lambda)}$$

with $K(z,\lambda)$ and $Q(z,\lambda)$ of the prescribed form, the problem can be stated as

$$\min_Q \|(1 - GQ)W\| = \min_Q \|H - UQ\|$$

where

$$H(z,\lambda) = \frac{\lambda}{1 - r(z)\lambda} \quad ; \quad U(z,\lambda) = \frac{T\lambda^2}{(1-\rho(z)\lambda)(1-r(z)\lambda)}$$

and

$$\rho(z) = \frac{\gamma}{2}\left(z^{-1} + 2\alpha + z\right) \quad ; \quad r(z) = \frac{c}{2}\left(z^{-1} + 2a + z\right).$$

An inner-outer factorization of $U(z,\lambda)$ yields

$$U_{\text{in}}(z,\lambda) = \lambda^2 \quad ; U_{\text{out}}(z,\lambda) = \frac{T}{(1-\rho(z)\lambda)(1-r(z)\lambda)}$$

hence

$$X(z,\lambda) = U_{\text{in}}^{\sim}(z,\lambda)H(z,\lambda) = \frac{1}{\lambda(1 - r(z)\lambda)} =$$

$$= X_{-1}(z)\lambda^{-1} + X_0(z) + X_1(z)\lambda + X_2(z)\lambda^2 + \ldots =$$

$$= \lambda^{-1} + r(z) + r^2(z)\lambda + r^3(z)\lambda^2 + \ldots$$

and

$$U_{\text{out}}(z,\lambda) = U_0(z) + U_1(z)\lambda + U_2(z)\lambda^2 + \ldots =$$

$$= T + T(r(z) + \rho(z))\lambda + T(r^2(z) + r(z)\rho(z) + \rho^2(z))\lambda^2 + \ldots$$

i.e.

$$U_i(z) = T \sum_{j=0}^{i} r^{i-j}(z)\rho^j(z).$$

Let us compute the decentralized controller assuming a two-step delay, i.e. $N = 2$. We get

$$V_{0,0}(z) = T[1 \; r(z)+\rho(z)]^T \;\; ; \;\; V_{1,i}(z) = T[0 \; 1]^T z^i \;\; ; \;\; i = -1, 0, 1$$

hence, by computing the inner products,

$$A = T^2 \begin{pmatrix} 1 + 2(\frac{\gamma}{2} + \frac{c}{2})^2 + (\alpha\gamma + ac)^2 & \frac{\gamma}{2} + \frac{c}{2} & \alpha\gamma + ac & \frac{\gamma}{2} + \frac{c}{2} \\ \frac{\gamma}{2} + \frac{c}{2} & 1 & 0 & 0 \\ \alpha\gamma + ac & 0 & 1 & 0 \\ \frac{\gamma}{2} + \frac{c}{2} & 0 & 0 & 1 \end{pmatrix}$$

$$p = T \begin{pmatrix} ac + c^2[a(\gamma + c) + (a^2 + 1/2)(\alpha\gamma + ac)] \\ ac^2 \\ c^2(a^2 + 1/2) \\ ac^2 \end{pmatrix}$$

Solving (6) we have

$$q^* = T^{-1} \begin{pmatrix} ac \\ ac^2/2 - ac\gamma/2 \\ c^2/2 - \alpha\gamma ac \\ ac^2/2 - ac\gamma/2 \end{pmatrix}$$

We note that the optimal $\mathcal{H}_2$ solution to this problem, computed without invoking the considered structure for the controller, is given by

$$Q^* = U_{\text{out}}^{-1} \Pi_{(\mathcal{RH}_2)}[H U_{\text{in}}^{-1}] =$$

$$= T^{-1}(1 - r(z)\lambda)(1 - \rho(z)\lambda)\left(\frac{r(z)}{1 - r(z)\lambda}\right)$$

$$= T^{-1} r(z)(1 - \rho(z)) =$$

$$T^{-1}\frac{c}{2}(z^{-1} + 2a + z) - T^{-1}\frac{c\gamma}{4}(z^{-1} + 2a + z)(z^{-1} + 2\alpha + z)\lambda$$

which doesn't happen to have the prescribed decentralized structure.

References

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5 Appendix

$$L(z) := \begin{pmatrix} 1 & & & & \\ 1 & (z^{-1}\ 1\ z) & & & \\ 1 & (z^{-1}\ 1\ z) & (z^{-2}\ z^{-1}\ 1\ z\ z^2) & & \\ \vdots & \vdots & \vdots & \ldots & \\ 1 & (z^{-1}\ 1\ z) & (z^{-2}\ z^{-1}\ 1\ z\ z^2) & \ldots & (z^{-N+1}\ \ldots\ z^{-1}\ 1\ z^1\ \ldots\ z^{N-1}) \end{pmatrix}$$

$$A = \begin{pmatrix} < V_{0,0}, V_{0,0} > & < V_{1,-1}, V_{0,0} > & < V_{1,0}, V_{0,0} > & < V_{1,1}, V_{0,0} > & < V_{2,-2}, V_{0,0} > & \ldots \\ < V_{0,0}, V_{1,-1} > & < V_{1,-1}, V_{1,-1} > & < V_{1,0}, V_{1,-1} > & < V_{1,1}, V_{1,-1} > & < V_{2,-2}, V_{1,-1} > & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$p = \begin{pmatrix} < (X_0 \ldots X_{N-1})^T, V_{0,0} > \\ < (X_0 \ldots X_{N-1})^T, V_{1,-1} > \\ < (X_0 \ldots X_{N-1})^T, V_{1,0} > \\ < (X_0 \ldots X_{N-1})^T, V_{1,1} > \\ \vdots \end{pmatrix}$$