# Extremum Seeking Loops with Assumed Functions

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**Abstract**

Extremum seeking (also *peak-seeking*) controllers are designed to operate at an unknown set-point that extremizes the value of a performance function. This performance function is approximated by an assumed function with a finite number of parameters. These parameters, which are estimated on-line, are assumed to change slowly compared to the plant and compensator dynamics. Philosophically, the approach of assuming a function is in contrast with traditional approaches that use time scale separation between gradient computation and function minimization and the system dynamics. To analyze our current scheme, quadratic functions or exponentials of quadratic functions are assumed as approximations to the performance function. This allows the peak-seeking control loop to be reduced to a linear system. For this loop, compensators can be designed and robust performance and stability analysis of the loop due to parameter uncertainty in the assumed performance functions can be obtained.

## 1 Introduction

The usual chore of a control system is to operate a system in a desired manner at some known desired set-point. The job of an extremum-seeking (also known as peak-seeking) controller includes the additional task of finding the setpoint at which to operate. This desired state is found by optimizing, on-line, some function of the state of the system, which provides a performance metric for the state. The state that maximizes the performance function is the desired operating point of the system.

Investigation of this class of problem dates back at least to 1922 [1]. A subsequent flurry of interest arose in the 1950s and 1960s [2, 3]. A recent rejuvenation of the field has been witnessed in the form of applications to pressure-maximizing compressors, drag-reducing flight formations [4, 5], and efficient fuel-burning in IC engines [6, 7, 8]. The approaches reported by these authors separate the problem by timescale, assuming that the plant dynamics and associated stabilizing controller are fast with respect to the outer-loop peak-seeking scheme. Philosophically, rather than assume a time-scale separation between the optimization function and the plant dynamics, a function with a finite number of parameters is here assumed to approximately fit the desired performance function. These parameters are assumed to be slow when compared to the system plant and compensator dynamics. In particular, if the desired performance function is quadratic, then a linear loop results for which compensators can be designed.

In section 2, this topic is examined and the controllers developed. In section 3 the stability and performance robustness of these controllers are characterized. In section 4, the quadratic is generalized to an exponential of a quadratic form. The performance and stability analysis remains valid but with some modification of the conditions of section 3. The negative exponential, which is not convex but is unimodal, appears to be a good approximation for characterizing the induced moment used as the performance function in the drag reduction problem of [4]. In section 5, a new implementation for peak-seeking control is presented, in which the effect of the nonlinear function on the dynamics is included, and an analysis of its convergence is given in section 6.

## 2 Quadratic Functions and Controller Synthesis

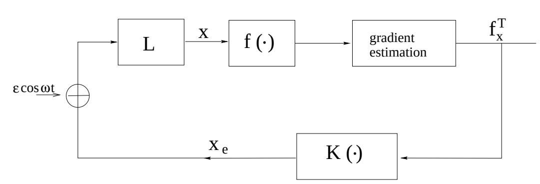


**Figure 1:** An Extremum-Seeking Loop.

The peak-seeking loop we study is shown in Figure 1. The gradient estimation is described in detail in [4]; we give only a brief outline here. In the block diagram, the system $L(\cdot)$ represents the plant, assumed to include a well-designed tracking controller such that input commands are tracked reliably and quickly by the output $x$. The function denoted $f(\cdot)$ is the perfor-

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mance measure which we try to optimize. The value of $f(\cdot)$, assumed to be available, is processed by the gradient estimation block, the output of which is fed to the outer-loop compensator $K(\cdot)$. The dither signal $\varepsilon \cos \omega t$, where $\varepsilon$ is a suitably small amplitude and $\omega$ a chosen frequency, is added to the output of the compensator to facilitate the gradient estimation. The function of compensator $K(\cdot)$ is to drive the plant output to a value such that the perfomance function $f(\cdot)$ is maximized. The essential multiple-time scale assumption of this loop is that the output $x$ and the dither is slow with respect to the plant.

In [5], it was assumed that the real-valued function $f(x) : \mathbb{R}^n \to \mathbb{R}$ satisfies the restriction

$$0 > \phi I \geq f_{xx}(\cdot) \geq \beta I \tag{1}$$

in the region of interest. Here, we will instead make the approximation that

$$f(x) = \frac{1}{2} x^T M x + b^T x + c \tag{2}$$

where $M \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are the parameters of the quadratic function. It follows that the slope at any given point $x$ is given by

$$f_x = Mx + b \tag{3}$$

(we slightly abuse standard notation by taking the gradient to be a column vector). Note that this approximation fulfills the restriction (1) so long as the matrix $M$ remains within the bounds specified for $f_{xx}(\cdot)$.

In the two-time scale analysis presented in [5], the tracking loop dynamics are fast compared to the outer gradient loop. Here, $M$ and $b$ are viewed as slowly varying parameters of the outer loop, so that in the time-scale of the tracking loop their rates of change are almost zero. A desired objective would then be to design a controller that is robust to $M$ varying within bounds and independent of $b$, with assured stability. This is the goal of this and the following sections.

### 2.1 An Equivalent Linear Peak-seeking Loop

In this section, only $b$ and $M$ are assumed to be slow with respect to the plant dynamics since the output $x$ is no longer restricted to be on a slow time scale. Then, using (2) and (3), the combined effect of the extremal functional and the gradient estimator in Figure 1 is replaced by a single linear block $Mx + b$. Notice that by the manipulation performed, we have replaced the nonlinearity in the loop with a linear element.

Our objective in designing a controller for the loop is to drive the gradient (i.e. the error) to zero. The closed loop system must be stable for the permissible variations in $M$ (since $b$ appears as a reference signal, it does not affect the stability of the loop). To achieve these objectives we first design a tracker using conventional LQ theory (section 2.2) and then establish stability and performance robustness of the loop using a notion of affine quadratic stability (AQS). To achieve this result we replace the previous restriction on the curvature with the assumption that the individual elements and rates of $f_{xx}(= M)$ lie within closed intervals. This assumption is explicitly used in section 3 and is implicitly assumed in the stability of the peak-seeking controller of section 5.

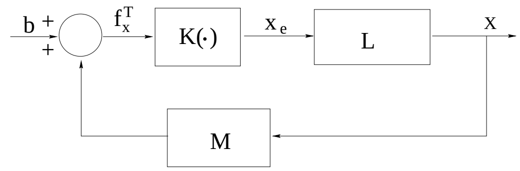


**Figure 2:** An Equivalent Linear Feedback Scheme

### 2.2 Compensator Design

In the peak-seeking loop of Figure 2, if the closed loop system is stable, then for an integral compensator $\left(K(s) = \bar{k}(s)/s\right)$ the gradient $f_x$ converges to zero for arbitrary input $b$. This loop clearly has much flexibility. In fact, the tracking loop $L$ (i.e. the output tracks the input $x_e$ in Figure 2) can be designed along with the gradient compensator $K$. This is done in the following.

We assume that the plant $L$ of Figure 2 can be written as

$$\begin{aligned} \dot{y} &= Ay + Bu & (4) \\ x &= Cy & (5) \end{aligned}$$

where $y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is the control vector, and $x \in \mathbb{R}^p$ is the output of the system and is the argument of the function $f$. Note that $f_x$ is an error signal. To design a compensator where we ensure that $f_x$ converges to zero, we use a dynamic model where $b$ does not explicitly appear in the dynamics. This requires that $\dot{u}$ rather than $u$ be considered the control for the synthesis process. We therefore design the controller using the augmented state space

$$x_{\rm a} \triangleq \begin{pmatrix} \dot{y} \\ f_x \end{pmatrix} \tag{6}$$

The dynamic system for controller design is

$$\dot{x}_{\rm a} = A_{\rm a} x_{\rm a} + B_{\rm a} \dot{u} \tag{7}$$

where

$$A_{\rm a} \triangleq \begin{bmatrix} A & 0 \\ M_n C & 0 \end{bmatrix}, \qquad B_{\rm a} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{8}$$

and $M_n$ is the nominal value at which the LQ tracker gains are designed. By recasting the state-space in the format as above, we have made the tracking controller gains independent of $b$. The optimal state-feedback law is given by (see [9])

$$\dot{u}^* = -K_{sf}\, x_{\rm a} = -K_y\, \dot{y} - K_f\, f_x \tag{9}$$

where $K_{sf}$ is the state-feedback gain matrix and partitioned as $\left[\begin{array}{cc} K_y & K_f \end{array}\right]$. The casting of the LQ problem naturally brings in an integrating effect in the control action of the form

$$u^*(t) = \int_0^t -K_{sf}\, x_{\rm a} dt \;=\; -K_y\, y - K_f \int_0^t f_x\, dt \tag{10}$$

The peak-seeking control loop has been generalized to that of Figure 3. This has been accomplished because the time-scale separation of the compensator $K$ and the plant $L$ no longer has to be assumed in the loops of Figure 2 and Figure 3.

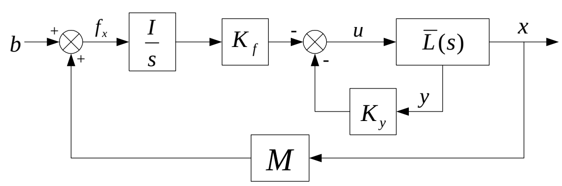


**Figure 3:** General Compensator Structure for Peak-Seeking Controller with Quadratic Performance Function

At this point we address one technicality to satisfy all the requirements to obtain a solution to the LQ problem.

**Theorem 1** *$(A_{\rm a}, B_{\rm a})$ is stabilizable if and only if $(A, B)$ is stabilizable and $\Lambda \triangleq \left[\begin{array}{cc} A & B \\ C & 0 \end{array}\right]$ has full row rank.*

**Proof:** Without loss of generality, $M_n$ is assumed to be the identity matrix.
*Sufficiency*: Suppose $(A, B)$ is stabilizable. Then

$$\text{rank}\left[\begin{array}{cc} A - \lambda I_n & B \end{array}\right] = n \quad \forall\, Re(\lambda) \geq 0$$

Hence, for all $\lambda$ such that $Re(\lambda) \geq 0$ and $\lambda \neq 0$

$$\begin{aligned} &\text{rank}\left[\begin{array}{cc} A_{\rm a} - \lambda I_{n+p} & B_{\rm a} \end{array}\right] \\ &= \text{rank}\left[\begin{array}{ccc} A - \lambda I_n & 0 & B \\ C & -\lambda I_p & 0 \end{array}\right] = n + p \end{aligned} \tag{11}$$

If $\Lambda$ has full row rank, (11) becomes

$$\text{rank}\left[\begin{array}{cc} A_{\rm a} - \lambda I_{n+p} & B_{\rm a} \end{array}\right] = n + p \quad \forall Re(\lambda) \geq 0$$

which implies $(A_{\rm a}, B_{\rm a})$ is stabilizable.
*Necessity*: Suppose that $(A_{\rm a}, B_{\rm a})$ is stabilizable. Then

$$\text{rank}\left[\begin{array}{ccc} A - \lambda I_n & 0 & B \\ C & -\lambda I_p & 0 \end{array}\right] = n + p \quad \forall Re(\lambda) \geq 0 \tag{12}$$

Therefore

$$\text{rank}\left[\begin{array}{cc} A - \lambda I_n & B \end{array}\right] = n \quad \forall\, Re(\lambda) \geq 0$$

which implies $(A, B)$ is stabilizable. By setting $\lambda = 0$, (12) becomes rank $\Lambda = n + p$ ∎

**Remark 1**: The condition that $\Lambda$ has full row rank implies the following:

1) $C$ has full row rank.
2) rank $B \geq$ rank $C$. If rank$B = m$, then $m \geq p$.
3) System $(C, A, B)$ should not have an invariant zero[10] at the origin, which would cancel out the integrator pole.
4) It is obvious from the full row rank condition that $\left[\begin{array}{cc} A - sI & B \\ C & 0 \end{array}\right]$ has full row normal rank. This implies that the output $x$ is completely controllable. ▮

The LQR procedure yields a state feedback control law of the form (9) where $K_{sf} = -R^{-1} B_{\rm a}^T S$ and $S$ is the unique positive semi-definite solution to the Algebraic Riccati equation

$$S A_{\rm a} + A_{\rm a}^T S - S B_{\rm a} R^{-1} B_{\rm a}^T S + Q = 0 \tag{13}$$

where it is assumed that $R > 0$, $Q \geq 0$, $(A_{\rm a}, B_{\rm a})$ is stabilizable and $(Q, A_{\rm a})$ is detectable. The closed loop system is governed by the dynamics

$$A_{cl} \triangleq A_{\rm a} - B_{\rm a} R^{-1} B_{\rm a}^T S \tag{14}$$

By partitioning $Q$ and $S$ as $\left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{array}\right]$ and $\left[\begin{array}{cc} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{array}\right]$ respectively, (13) becomes

$$\begin{aligned} 0 &= Q_{11} + S_{12} M_n C + S_{11} A + C^T M_n S_{12}^T \\ &\quad + A^T S_{11} - S_{11} B R^{-1} B^T S_{11} \qquad (15) \\ 0 &= Q_{12}^T + S_{22} M_n C + S_{12}^T A \\ &\quad - S_{12}^T B R^{-1} B^T S_{11} \qquad (16) \\ 0 &= Q_{22} - S_{12}^T B R^{-1} B^T S_{12} \qquad (17) \end{aligned}$$

The state feedback control law becomes

$$\dot{u} = -R^{-1} B^T S_{11} \dot{x} - R^{-1} B^T S_{12} f_x \tag{18}$$

**Remark 2**: Using Remark 1 and applying lemma 2.14 in [10] to (17), the feedback gain for $f_x$ in (18) has the form

$$R^{-1} B^T S_{12} = R^{-\frac{1}{2}} U\, Q_{22}^{\frac{1}{2}} \tag{19}$$

where $U \in \mathbb{R}^{m\times p}$ satisfies $U^T U = I$. By choosing $Q_{22}$ such that $Q_{22} = M^{-2}$, (18) becomes

$$\dot{u} = -R^{-1}B^T S_{11}\dot{x} - R^{-\frac{1}{2}} U M_n^{-1} f_x$$

where the second term in right hand side describes the Newton-Raphson scheme in function optimization. This demonstrates that the outer loop extremizes the quadratic function (2) by the Newton-Raphson method while the inner loop stabilizes the dynamic plant $L$ in Figure 3. ∎

## 3 Affine Quadratic Stability and Performance Robustness

At this stage we have completed the design of the controller. However, the stability of the system needs to be preserved for variations about the nominal value of $M$. As described before, such variations are natural as the system seeks to minimize the gradient, which is a dynamically changing entity. These variations enter affinely in the closed loop matrix $A_{cl}$ (14) through the $A_{\rm a}$ matrix. Explicitly, the closed loop matrix $A_{cl}$ can be written as

$$\begin{aligned} A_{cl} &= A_0 + m_{11}A_{11} + \ldots m_{1p}A_{1p} \\ &\quad + m_{22}A_{22} + \ldots + m_{pp}A_{pp} \end{aligned} \qquad (20)$$

where

$$M = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1p} \\ m_{12} & m_{22} & \ldots & m_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ m_{1p} & \ldots & \ldots & m_{pp} \end{pmatrix} \qquad (21)$$

and the $A_i$s are appropriate matrices of dimension $R^{(n+p)\times(n+p)}$. Assume bounds on the variation of $m_{ij}$s and the rates of variation as

$$m_{ij} \in [\underline{m}_{ij}, \overline{m}_{ij}] \quad \dot{m}_{ij} \in [\underline{\dot{m}}_{ij}, \overline{\dot{m}}_{ij}] \qquad (22)$$

Note that the bounds of $m_{ij}$ are precisely the intervals of the permissible variations in the elements of the curvature. The nominal value of $M$ is

$$m_{ij_n} = 0.5 * [\underline{m}_{ij} + \overline{m}_{ij}] \qquad (23)$$

These lower and upper limits of the $m_{ij}$s and their rates define a rectangular polytope in the n-dimensional space.

To analyse the stability of the system, we consider a Lyapunov function of the form

$$V(x_{\rm a}) \triangleq x_{\rm a}^T P(m) x_{\rm a} \qquad (24)$$

where $P(\cdot)$ is a function of the entries $m_{ij}$ of the matrix $M$. Using results from [11, 12], we can state the following sufficient condition for stability.

**Theorem 2** *Let* $m_{ij} \in [\underline{m}_{ij}, \bar{m}_{ij}]$, $\dot{m}_{ij} \in [\underline{\dot{m}}_{ij}, \bar{\dot{m}}_{ij}]$ $i,j = 1,\ldots n$. *Then a sufficient condition for the stability of* $A_{cl}$ *is*

1. *There exist* $(\frac{p(p+1)}{2} + 1)$ *symmetric matrices* $P_0, \{P_{ij} : i = 1,\ldots,p, j(\geq i) = 1,\ldots p\}$ *such that*
$$A(m)^T P(m) + P(m)A(m) + P(\dot{m}) - P_0 < 0 \qquad (25)$$
*for the corners of the polytope, where*
$$P(m) \triangleq P_0 + \sum_{i,j(\geq i)=1}^{p} m_{ij}P_{ij}$$
*and*
$$\sum_{i,j(\geq i)=1}^{p} [A_{ij}^T P_{ij} + P_{ij}A_{ij}] \geq 0 \qquad (26)$$
2. $A_{cl}(m_{ij_n})$ *is stable* ($m_{ij_n}$ *denotes the nominal value of* $m_{ij}$).

In addition to guaranteeing stability to the type of variations considered, we now consider robust performance requirements. Consider the system once again as $\dot{x}_{\rm a} = A_{cl}x_{\rm a}$ with an output of interest

$$z = C_z x_{\rm a} \qquad (27)$$

The performance measure is assumed to be

$$J_{rp} \triangleq E(\int_0^\infty \|z\|_2^2 \, dt)$$

where the expectation is associated with the initial state $x_{a_0}$, assumed to be a random vector.

To ensure $J_{rp} < \nu^2$, $\nu \in \mathbb{R}$, we consider the following argument. Construct a Lyapunov function as before, but which additionally satisfies

$$\frac{dV(x_{\rm a})}{dt} + \|z(t)\|_2^2 < 0$$

By integrating from zero to infinity and since the system is stable, the inequality reduces to

$$\int_0^\infty \|z\|_2^2 \, dt < x_{\rm a_0}^T P(m) x_{a_0} \qquad (28)$$

Characterizing a worst-case performance bound in the LQG ($H_2$) sense can be interpreted as a worst-case energy bound with initial condition $x_{\rm a_0}$ where $E(x_{\rm a_0}x_{\rm a_0}^T) = I$. Taking the expectation on both sides of 28 the inequality becomes

$$\begin{aligned} J_{up} &= E[\int_0^\infty \|z\|_2^2 \, dt] < E[x_{a_0}^T P(m) x_{a_0}] \\ &= \mathrm{Tr}(E[P(m)x_{\rm a_0}x_{\rm a_0}^T]) = \mathrm{Tr}(P(m)) < \nu^2 \end{aligned} \qquad (29)$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of the matrix. Ensuring $\mathrm{Tr}(P(m)) < \nu^2$ ensures $J_{up} < \nu^2$. Therefore, the sufficient conditions for ensuring robust performance are obtained by rewritting theorem 2, but adding the additional condition that Tr $(P(m)) < \nu^2$.

## 4 Exponential Functions

The ideas presented in the previous sections can be adapted to other functions. Consider an exponential function of the variable $x \in R^p$ given as

$$f(x) = e^{\theta(\frac{1}{2}x^T Mx + bx + c)} \tag{30}$$

where $\theta$ is a given parameter which can be either positive or negative. It follows that the slope at any given point $x$ is given by

$$f_x = \theta(Mx + b)f(x) \tag{31}$$

To recover the linear feedback scheme obtained in the quadratic case, the idea is to cast the gradient $f_x(\cdot)$ as $\theta(M'x + b')$ where

$$M' \triangleq \theta M f(x), \quad b' \triangleq \theta b f(x)$$

The matrix $M'$ is now treated as uncertain and control synthesis is carried out in the same way as before. Note that if $\theta < 0$, then the function is not concave, but is unimodal . We search for a maximum of the function. This shape is very close to the moment induced by the trailing vortex in the drag reduction problem in [4].

## 5 Implementation of Peak-seeking Controller Based on Function Approximation.

The essential concept is that to avoid a multiple-time scale decomposition for peak-seeking controllers, the functional form must be assumed and the parameters for these functions are assumed to change slowly relative to the the system dynamics. Sections 2 and 4 consider two functions which enable linear analysis and controller synthesis of the peak seeking loop. In this section the mechanization of the peak-seeking control loop based on assumed functions is presented. Figure 4 represents a block diagram of our peak-seeking control loop. Embedded in the structure is the structure of Figure 3. Note that with the compensator of (10), the gradient goes to zero. The more complex control structure of Figure 4 is due to the need to estimate the parameters of an assumed approximation of the function $f(\cdot)$, where $x$ is assumed known. The function $f$ is approximated as in section 3. The underlying assumption is that these parameters change slowly compared to the plant dynamics and compensator. This in practice depends upon how well a multi-dimensional parabola or exponential fits the actual function $f(\cdot)$. However, our initial analysis of this loop assumes this type of time scale separation (Section 3). Note that this new approach where the functional form is assumed is in marked contrast to the dynamic separation of previous work [8, 7, 5]. Also, note the dashed line in Figure 4. This represents an explicit feedback of the nonlinear function to be optimized back into the system dynamics. This can occur in some systems and makes a global analysis more difficult (see Section 6). However, for local analysis about the optimum point, since the gradient is assumed small, the analysis should revert back to that associated with the loop of Figure 3.

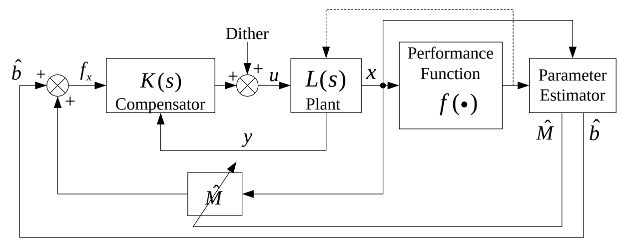


**Figure 4:** Mechanization of Peak-Seeking Controller

## 6 Extremal-seeking Loop Assuming Quadratic Function Fed Back to the Plant

If we assume that the parameters of the quadratic function are known, but the quadratic function is a physical quantity that forces the system dynamics as represented by the dashed line in Figure 4, then it is shown that the effect of the nonlinearity in stability analysis must be considered. The dynamics given in (4) must be augmented by the forcing function $f(x)$ as

$$\dot{y} = Ay + Bu + \Gamma f(x) \tag{32}$$

where $\Gamma$ is a vector that defines the way $f(x)$ enters into the dynamics. In the drag reduction problem, $f$ is the moment induced by the presence of the vortex wake from a leading aircraft. A generalization can be made by noting that both the lift and moment on the trailing aircraft are being maximized.

Using the state space of (6) and its propagation equations (7), modified by (32) and stabilized by the control law of (9), the new dynamic equations are

$$\dot{f}_x = M_n C\dot{y} \tag{33}$$
$$\ddot{y} = A\dot{y} - BK_f f_x - BK_y\dot{y} + \Gamma f_x^T C\dot{y} \tag{34}$$

The essential nonlinearity here is the scalar $f_x^T C\dot{y}$. A stable equilibrium point is $f_x = 0$, $\dot{y} = 0$. Note that in the vicinity of the equilibrium point, the effect of the nonlinearity is second order. This result is fortuitous, since convergence in the vicinity of the extremal is assured. Note that this nonlinearity is due to the presence of the integrator in the compensator. Nevertheless, the presence of this special nonlinearity must be examined in the presence of large excursions from the equilibrium point to understand the global dynamic behavior of the current peak-seeking controller.

Re-write the system (33,34) using $x_{\rm a}$ as defined in (6) as

$$\dot{x}_{\rm a} = A_{cl}x_{\rm a} + \bar{\Gamma}x_{\rm a}^T\bar{C}x_{\rm a}$$

where

$$\bar{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix}$$

The stability of this closed-loop system can be examined using the Lyapunov function $V = x_{\mathrm{a}}^T S x_{\mathrm{a}}$ where $S$ is the solution of the Riccati equation (13). We then have

$$\begin{aligned} \dot{V} &= \dot{x}_{\mathrm{a}}^T S x_{\mathrm{a}} \\ &= x_{\mathrm{a}}^T \left( S A_{cl} + A_{cl}^T S + \bar{C}^T x_{\mathrm{a}}^T \bar{\Gamma}^T S + S \bar{\Gamma} x_{\mathrm{a}} \bar{C} \right) x_{\mathrm{a}} \\ &= -x_{\mathrm{a}}^T \left\{ S \begin{bmatrix} B R^{-1} B^T & 0 \\ 0 & 0 \end{bmatrix} S - \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \right. \\ &\quad - \begin{bmatrix} C^T f_x^T \Gamma S_{11} & C^T f_x^T \Gamma S_{12} \\ 0 & 0 \end{bmatrix} \\ &\quad \left. - \begin{bmatrix} S_{11} \Gamma^T f_x C & 0 \\ S_{12}^T \Gamma^T f_x^T C & 0 \end{bmatrix} \right\} x_{\mathrm{a}} \qquad (35) \end{aligned}$$

Near the point at which $f_x = 0$, this equation is dominated by the first two terms, which guarantee convergence. This confirms the earlier statement that convergence is guaranteed near the equilibrium point.

## 7 Conclusions

By recognizing the affine nature of the gradient of a quadratic function, we have presented an analysis and synthesis procedure for a new class of extremal seeking controllers. For this new class, the time-scale separation between the system dynamics and the gradient search need not be assumed. However, the parameters which make up the performance function are assumed to be slowly varying with respect to the plant and compensator dynamics. For this class, general compensators can be designed. For the robust stability and performance of such controllers, it is shown that the existence of certain matrices $P_i$ is a sufficient condition. Finally, an adaptive loop for which the parameters of the performance function are estimated is discussed. This structure forms the basis of our current theoretical and applied work in peak-seeking control.


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