A Convexifying Algorithm for the Design of Structured Linear Controllers[1]

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Abstract

This paper addresses the design of linear controllers with special structure imposed on the gain matrix. This problem is called a SLC (Structured Linear Control) problem. The SLC problem includes fixed order output feedback control, decentralized control, joint plant and control design, and many other linear control problems. A theoretical framework that allows one to pursue the solution of SLC problems is provided. Although the obtained conditions are nonconvex, it is shown that solving a SLC problem involving standard control objectives such as stability, bounds on the H_2 or H_∞ norms, and real positiveness is not harder than solving a standard unstructured static output feedback problem.

A convexifying algorithm that might be used to solve the SLC problem is also developed. At each iteration a certain function is added to the constraints in order to make them convex. At convergence, the artificially introduced convexifying functions reduce to zero, guaranteeing the feasibility of the original problem. Local optimality can be guaranteed.

Some examples illustrate how the SLC framework and the convexifying algorithm can improve the solutions of control problem with suboptimal solutions available.

1 Introduction

Control problems are typically formulated as optimization problems. A few of them can be stated as *convex* optimization problems, in which case powerful algorithms can be used to find solutions. Recently, a class of convex optimization problems known as *semidefinite programs* has received a lot of attention, becoming an invaluable tool for the formulation of control problems. Part of this framework is known by the name *linear matrix inequalities* (LMI), which has been extensively used to provide suitable formulation for several system and control problems [1]. The solution to these problems can be calculated with the help of efficient interior-point polynomial algorithms, with many implementations available.

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In the LMI framework, a linear control synthesis problem is usually solved with the help of auxiliary parametrizations [1]. That is, the controller parameters are not available for optimization, being obtained via some nonlinear one-to-one transformation of the optimization variables. Calling $H(s)$ the transfer function from a given input to a given output, one can, roughly speaking, solve several linear control problem in the form

$$\min_{K} f(H(s)), \tag{1}$$

where $f(\cdot)$ is some appropriately defined cost function. Notice that the controller gain K in problem (1) is an unconstrained variable. The structure of the controller is imposed by the properties of the problem solution. As a consequence of this fact, fundamental problems such as the design of controllers with arbitrary order [2] or with particular structures, such as decentralization [3], remain unsolved. In other words, linear control problems of the form

$$\min_{K\in\Psi} f(H(s)), \tag{2}$$

where the controller K is subject to constraints are hard to handle. Even convexity of Ψ provides little or no help in solving problem (2).

In this paper, we refer to the nonconvex problem (2), where the set Ψ is a convex set, as an SLC (Structured Linear Control) problem. The SLC framework is able to deal with all sorts of convex controller constraints, including fixed order output feedback control design [2], decentralization [3] or norm bounds on the controller gains. It is also appropriate for the simultaneous design of plant and controller [4] where some matrices of the plant depend affinely on a set of parameters. Multiobjective linear control problems [5] where several specifications must be feasible or minimized can also be treated as an SLC problem (mathematically, a multiobjective linear control problem can be formulated with the help of multiple plants and multiple controllers, by imposing a convex constraint on all controllers that force them to coincide). The SOF (Static Output Feedback) problem [2], which has been studied by many authors, is also an SLC.

The first purpose of this paper is to describe a framework to deal with the SLC problem. We show that for standard control objectives such as stabilization, H_2 and H_∞ performance, and positive realness, SLC problems can be formulated as optimization problems involving LMI with an additional nonconvex equality constraint. While we do not claim a complete solution to the SLC problem, we show that solving an SLC problem is not harder than solving a SOF problem. Moreover, the nonconvex constraint that appears in the SLC problem is of the same nature as the one that appears in the SOF problem, which means that almost all algorithms available for the SOF problem [6] can be modified to work with SLC problem formulation given in this paper with little or no effort.

The second objective is to develop a *convexifying algorithm* that might be able to solve the SLC problem. We introduce the concept of a *convexifying potential functional*. These potentials are added to nonconvex constraints in order to render a nonconvex optimization problem convex. The algorithm works by iterating on a sequence of convex problems and the driving force that ensures convergence to this mechanism is the ability to zero the potential at each iteration. Convergence to a local optimum to the SLC problem can be guaranteed. Examples illustrate how available suboptimal solutions to control problems can be significantly improved.

2 A framework for the design of SLC

Let a linear system be described by the state space equations

$$\delta x = Ax + B_w w + B_u u, \tag{3}$$

$$z = C_z x + D_{zw} w + D_{zu} u, \tag{4}$$

$$y = C_y x + D_{yw} w, \tag{5}$$

where δ represents the time-derivative operator for continuous-time systems or the unitary time-shift operator for discrete-time systems. We assume that the state vector has dimension n and that all matrices have been appropriately defined. In this model, u represents the control input vector, w is a vector of exogenous perturbations, y is the measured output and z is the controlled output. Consider the following controller structure

$$\delta x_c = A_c x_c + B_c y, \tag{6}$$

$$u = C_c x_c + D_c y, \tag{7}$$

where the controller state has dimension n_c. Defining the controller matrices in the compact form

$$K := \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \tag{8}$$

and the augmented state vector as $\tilde{x} := \begin{bmatrix} x^T & x_c^T \end{bmatrix}^T$, one can show [1] that the closed loop connection of the controller (6–7) with (3–5) provides the linear system

$$\delta \tilde{x} = \mathcal{A}(K)\tilde{x} + \mathcal{B}(K)w, \tag{9}$$

$$z = \mathcal{C}(K)\tilde{x} + \mathcal{D}(K)w, \tag{10}$$

where

$$\mathcal{A}(K) := \mathbf{A} + \mathbf{B}K\mathbf{B}, \qquad \mathcal{B}(K) := \mathbf{D} + \mathbf{B}K\mathbf{E}, \tag{11}$$

$$\mathcal{C}(K) := \mathbf{C} + \mathbf{H}K\mathbf{M}, \qquad \mathcal{D}(K) := \mathbf{F} + \mathbf{H}K\mathbf{E}, \tag{12}$$

are all *affine mappings* on the variable K and matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{H}$ are constant matrices that depend on the open-loop dynamics (3–5).

When $n_c = 0$ and K is unconstrained, the above problem reduces to the standard SOF problem. Two particular unconstrained configurations are of importance: a) *full-order dynamic output feedback*: when the order of the controller is equal to the order of the plant $(n_c = n)$; and b) *static state feedback*: when $n_c = 0$ and the system output is a full rank linear transformation of the state vector. For these, stabilization, some norm minimization and robust control problems have been solved using LMI [1].

In the SLC context one is able to pursue the design of complex control problems by appropriately constraining the controller matrix K. For instance, decentralized controllers of arbitrary order for the augmented system (9–10) can be obtained if K is constrained to the following structure

$$K := \begin{bmatrix} \operatorname{diag}[D_c] & \operatorname{diag}[C_c] \\ \operatorname{diag}[B_c] & \operatorname{diag}[A_c] \end{bmatrix} \tag{13}$$

where $\operatorname{diag}[\cdot]$ indicates that $(\cdot)$ has a block-diagonal structure. A less obvious SLC problem appears in the simultaneous plant/control design problem [4]. Assuming that matrices $A(p)$, $B_w(p)$, $C_z(p)$ and $D_{zw}(p)$ depend affinely on some plant parameters p, the augmented controller $\tilde{K} := (K, p)$ still provides affine mappings $\mathcal{A}(\tilde{K})$, $\mathcal{B}(\tilde{K})$, $\mathcal{C}(\tilde{K})$ and $\mathcal{D}(\tilde{K})$ to be used in system (9–10). Even if the augmented controller $\tilde{K}$ is unstructured, the joint problem is structured since $\tilde{K}$ is not a single unstructured block matrix.

2.1 Discrete-time systems

A real and square matrix is said to be Schur if all its eigenvalues lie inside the unitary circle. The following lemma provides the basic algebraic tool.

Lemma 1 *Assume that $\mathcal{Q}(K)$ is a symmetric matrix. There exists a symmetric matrix $P > 0$ such that*

$$\mathcal{S}(K)^T P \mathcal{S}(K) - U^T P U + \mathcal{Q}(K) + \mathcal{R}(K)^T \mathcal{R}(K) < 0 \tag{14}$$

if, and only if, there exists symmetric matrices X and Y such that

$$\begin{bmatrix} U^T X U - \mathcal{Q}(K) & \mathcal{S}(K)^T & \mathcal{R}(K)^T \\ \mathcal{S}(K) & Y & \mathbf{0} \\ \mathcal{R}(K) & \mathbf{0} & \mathbf{I} \end{bmatrix} > 0, \quad X = Y^{-1}. \tag{15}$$

Proof: Using Schur complement, inequalities (14) can be put in the equivalent form

$$\begin{bmatrix} U^T P U - \mathcal{Q}(K) & \mathcal{S}(K)^T & \mathcal{R}(K)^T \\ \mathcal{S}(K) & P^{-1} & \mathbf{0} \\ \mathcal{R}(K) & \mathbf{0} & \mathbf{I} \end{bmatrix} > 0$$

from where (15) comes with $X := Y^{-1} := P$. ∎

Starting from inequality (14), which has a product between the mapping $\mathcal{S}(K)$ and the instrumental variable P, the above lemma shows how to build a matrix inequality with larger dimension (15) where this product is no longer present. Moreover, if $\mathcal{S}(K)$, $\mathcal{R}(K)$ and $\mathcal{Q}(K)$ are affine on K, so is the transformed inequality (15).

Theorem 2 *The following statements are true:*

i) Matrix $\mathcal{A}(K)$ is Schur if, and only if, there exist $K \in \Psi$, and symmetric matrices X and Y such that

$$\begin{bmatrix} X & \mathcal{A}(K)^T \\ \mathcal{A}(K) & Y \end{bmatrix} > 0, \quad X = Y^{-1}. \tag{16}$$

ii) Matrix $\mathcal{A}(K)$ is Schur and $\|\mathcal{C}(K)\,[z\mathbf{I} - \mathcal{A}(K)]^{-1}\,\mathcal{B}(K)\|_2^2 < \mu$ if, and only if, there exist $K \in \Psi$, and symmetric matrices X and Y such that

$$\begin{bmatrix} X & \mathcal{A}(K)^T & \mathcal{C}(K)^T \\ \mathcal{A}(K) & Y & \mathbf{0} \\ \mathcal{C}(K) & \mathbf{0} & \mathbf{I} \end{bmatrix} > 0, \quad X = Y^{-1}, \tag{17}$$

$$\begin{bmatrix} W & \mathcal{B}(K)^T \\ \mathcal{B}(K) & Y \end{bmatrix} > 0, \quad \text{trace}\,[W] < \mu. \tag{18}$$

iii) Matrix $\mathcal{A}(K)$ is Schur and $\|\mathcal{C}(K)\,[z\mathbf{I} - \mathcal{A}(K)]^{-1}\,\mathcal{B}(K) + \mathcal{D}(K)\|_\infty^2 < \mu$ if, and only if, there exist $K \in \Psi$, and symmetric matrices X and Y such that

$$\begin{bmatrix} X & \mathbf{0} & \mathcal{A}(K)^T & \mathcal{C}(K)^T \\ \mathbf{0} & \mu\mathbf{I} & \mathcal{B}(K)^T & \mathcal{D}(K)^T \\ \mathcal{A}(K) & \mathcal{B}(K) & Y & \mathbf{0} \\ \mathcal{C}(K) & \mathcal{D}(K) & \mathbf{0} & \mathbf{I} \end{bmatrix} > 0, \; X = Y^{-1}. \tag{19}$$

iv) Matrix $\mathcal{A}(K)$ is Schur and the transfer function $\mathcal{C}(K)\,[z\mathbf{I} - \mathcal{A}(K)]^{-1}\,\mathcal{B}(K) + \mathcal{D}(K)$ is ESPR (Extended Strictly Positive Real) if, and only if, there exist $K \in \Psi$, and symmetric matrices X and Y such that

$$\begin{bmatrix} X & \mathcal{C}(K)^T & \mathcal{A}(K)^T \\ \mathcal{C}(K) & \mathcal{D}(K) + \mathcal{D}(K)^T & \mathcal{B}(K)^T \\ \mathcal{A}(K) & \mathcal{B}(K) & Y \end{bmatrix} > 0, \; X = Y^{-1}. \tag{20}$$

Proof: The proof of this theorem follows from Lemma 1 by replacing U and the functionals $\mathcal{S}(K)$, $\mathcal{Q}(K)$ and $\mathcal{R}(K)$ according to Table 1. The second inequality in (17) and the first inequality in (18) come from

$$\mu > \text{trace}\left[\mathcal{B}(K) Y^{-1} \mathcal{B}(K)\right] > \|\mathcal{C}(K)\,[z\mathbf{I} - \mathcal{A}(K)]^{-1}\,\mathcal{B}(K)\|_2^2$$

by a Schur complement argument. ■

This theorem shows that it is possible to rewrite four largely used control objectives in a form that preserve the affine dependence on the controller matrices. As the controller matrices are available for optimization, these constraints can be used to provide solutions to the SLC problem.

The characterizations given in Theorem 2 are necessary and sufficient. However the price paid for that is the introduction of the nonconvex constraint $X = Y^{-1}$. It can be shown by a Schur complement argument that this nonconvex constraint can be equivalently cast as a rank constraint involving X and Y. In any case, the nonconvexity is restricted to the instrumental variables X and Y and does not directly involve the controller parameter K. This kind of constraint also appears in several formulations of the SOF problem [2]. The conclusion is that solving the SLC problem for the control objectives stated in Theorem 2 is not more difficult than solving an unconstrained SOF problem.

2.2 Continuous-time systems

While for discrete-time systems the quadratic characteristic of Lyapunov based inequalities seems to provide in Lemma 1 and Theorem 2 an almost "natural" guidance to the development of the new conditions. A slightly more involved manipulation is required for continuous-time systems. Indeed, more than one strategy could be used to preserve the affine dependence on the controller parameters. The one described below tries to mimic the behavior of Lemma 1.

Lemma 3 *Assume that $\mathcal{Q}(K)$ is a symmetric matrix. There exists a symmetric matrix $P > 0$ such that*

$$\mathcal{S}(K)^T P U + U^T P \mathcal{S}(K) + \mathcal{Q}(K) + \mathcal{R}(K)^T \mathcal{R}(K) < 0 \tag{21}$$

if, and only if, there exists symmetric matrices X and Y, and a scalar $\alpha > 0$ such that $X = \alpha^2 Y^{-1}$ and

$$\begin{bmatrix} U^T X U - \mathcal{Q}(K) & \mathcal{S}(K)^T + \alpha U^T & \mathcal{R}(K)^T \\ \mathcal{S}(K) + \alpha U & Y & \mathbf{0} \\ \mathcal{R}(K) & \mathbf{0} & \mathbf{I} \end{bmatrix} > 0 \tag{22}$$

The proof of this lemma is omitted for brevity. Using Lemma 3 the same Table 1 provides the continuous-time analog of the conditions discussed in Theorem 2. The reader is referred to the full version of this paper [7] for a detailed treatment of the continuous-time case.

3 The convexifying algorithm

In this section we describe a new class of algorithm. While most algorithms in the literature are aimed at the feasibility problem, this new algorithm enable us to pursue the improvement of solutions for suboptimal control optimization problems that are available.

Definition 1 (Potential matrix functional) *A first-order differentiable matrix functional $G(x,\xi)$ defined for all x and ξ in a convex set Φ is called a* potential functional *if*

i) $G(x,\xi) \geq 0$ for all $x, \xi \in \Phi$,

	U	$\mathcal{S}(K)$	$\mathcal{Q}(K)$	$\mathcal{R}(K)$
$i)$	$\mathbf{I}$	$A(K)$	$\mathbf{0}$	$\mathbf{0}$
$ii)$	$\mathbf{I}$	$A(K)$	$\mathbf{0}$	$C(K)$
$iii)$	$\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}$	$\begin{bmatrix}A(K) & B(K)\end{bmatrix}$	$\begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mu\mathbf{I}\end{bmatrix}$	$\begin{bmatrix}C(K) & D(K)\end{bmatrix}$
$iv)$	$\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}$	$\begin{bmatrix}A(K) & B(K)\end{bmatrix}$	$\begin{bmatrix}\mathbf{0} & -C(K)^T \\ -C(K) & -D(K)-D(K)^T\end{bmatrix}$	$\mathbf{0}$

Table 1: Substitution table

ii) $G(x,x) = \mathbf{0}$ *for all* $x \in \Phi$*,*

iii) $\nabla G(x,x) = \mathbf{0}$ *for all* $x \in \Phi$*.*

Notice that condition *iii*) just states that the differentiable functional $G(\cdot)$ attains its minimum at every point $G(x,x)$. We are especially interested in potential functionals with the following property.

Definition 2 (Convexifying potential matrix functional) *A first-order differentiable potential matrix functional* $G(x,\xi)$ *is said to be a* convexifying potential matrix functional *if, given a first order differentiable nonconvex matrix functional* $F(x)$ *defined for all* $x \in \Phi$*,* $F(x) + G(x,\xi)$ *is a convex matrix functional for all* $x,\xi \in \Phi$*.*

We will be looking for potentials that are able to "convexify" a given functional, for instance, by adding to the Hessian of the original functional some positive amount. For a given nonconvex functional $F(\cdot)$, there might exist many candidates for convexifying potentials. Independently of a particular choice and using only the properties in the definitions, we are in position to state a very simple algorithm that converges to a solution that satisfies the necessary optimality conditions to the nonconvex optimization problem

$$\min_{x\in\Omega} f(x), \quad \Omega := \{x \in \Phi \,:\, F(x) \leq 0\}. \tag{23}$$

Where it is assumed that $f(x)$ is a scalar and first order differentiable convex function bounded from below on the convex set Ω, and that $F(x)$ is a nonconvex matrix functional.

Algorithm 1 *Let* $\varepsilon > 0$*,* $x_0 \in \Omega$ *and a convexifying potential matrix functional* $G(x,\xi)$ *be given. Set* $k \leftarrow 0$ *and iterate:*

1. *Solve the convex optimization problem*

$$\begin{aligned} x_{k+1} &:= \arg\min_x \{f(x) \,:\, x \in \Omega_k\}, \\ \Omega_k &:= \{x \,:\, F(x) + G(x,x_k) \leq 0\} \end{aligned} \tag{24}$$

2. *If* $\|x_{k+1} - x_k\| < \varepsilon$*, stop. Otherwise, set* $k \leftarrow k+1$ *and go back to Step 1.*

The convex problem (24) is significantly simpler than (23), and we assume that its solution can be obtained by some available convex programming technique. Furthermore, there is no need to perform any sort of line search, which simplifies implementation of Algorithm 1. The next theorem proves that the stationary solution generated by Algorithm 1 satisfies the necessary optimality conditions for problem (23). For a proof see [7].

Theorem 4 *Given a scalar and first order differentiable convex function* $f(x)$ *and a convexifying potential matrix functional* $G(x,\xi)$ *defined for all* x,ξ *in the convex set* Φ*, Algorithm 1 generates a sequence of feasible points that converges to a solution satisfying the necessary optimality conditions for problem (23).*

Notice that at an stationary solution $x_{k+1} = x_k$ and the value of the convexifying potential functional $G(x_{k+1},x_k)$ in (24) reduces to zero, guaranteeing the feasibility of the original problem.

When $F(x)$ is a concave matrix functional, and using tools that are similar to the ones in [8], it is possible to show that the previous algorithm indeed converges to a local minimum.

Corollary 5 *If the first order differentiable matrix functional* $F(x)$ *is concave for all* x *in* Φ*, then Algorithm 1 converges to a local optimum of problem (23).*

3.1 Potential functionals for the structured linear control problem

For many control problems, feasible solutions might be available by using some suboptimal method. The objective of this section is to develop potential functionals so that Algorithm 1 might be used to improve the available solutions.

According to the discussion in Section 2, the solution of an SLC problem can be obtained if we are able to solve an optimization problem that is entirely convex, but for the presence of an equality constraint in one of the forms

$$X = \alpha^2 Y^{-1}, \qquad X = \alpha^{-1} Y Y. \tag{25}$$

These two constraints are similar enough so that they can be uniformly described by a nonconvex equality of the form

$$X = Z V^{-1} Z. \tag{26}$$

In fact, by replacing (Z,V) by

$$(Z,V) := (\alpha\mathbf{I},Y) \qquad \text{or} \qquad (Z,V) := (Y,\alpha\mathbf{I}) \tag{27}$$

the first and the second equalities in (25) are recovered. In this section we describe how to develop potential functions so that Algorithm 1 can be used to solve problems with the nonconvex constraints (25). If we substitute these equality constraints in the affine matrix inequalities, we obtain an equivalent problem where the matrix inequalities are no longer affine. For instance, the relations in (16) reduce to the nonaffine inequality

$$\begin{bmatrix} Y^{-1} & \mathcal{A}(K)^T \\ \mathcal{A}(K) & Y \end{bmatrix} > 0. \tag{28}$$

Indeed, using (27) it can be shown that all constraints defined in Section 2 reduce to matrix inequalities of the form

$$F(K,Z,V) := \bar{F}(K,Z,V) - \bar{U}^T Z V^{-1} Z \bar{U} < 0 \tag{29}$$

where $\bar{F}(K,Z,V)$ is an affine functional and $\bar{U}$ is a given constant matrix. Moreover, as $\bar{F}(Y,K)$ is affine and $\bar{U}$ is constant, the functional $F(K,Z,V)$ is concave. In expression (28), these values are defined as

$$\bar{F}(K,\mathbf{I},Y) := -\begin{bmatrix} \mathbf{0} & \mathcal{A}(K)^T \\ \mathcal{A}(K) & Y \end{bmatrix}, \quad \bar{U} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{30}$$

One convexifying potential for (29) is given by

$$\begin{aligned} &G(K,Z,V,\xi_K,\xi_Z,\xi_V) := \\ &\quad \bar{U}^T \left(\xi_Z\xi_V^{-1} - ZV^{-1}\right) V \left(\xi_Z\xi_V^{-1} - ZV^{-1}\right)^T \bar{U}. \end{aligned} \tag{31}$$

One can see that the combination of the above potential with (31) provides the affine functional

$$\begin{aligned} &F(K,Z,V) + G(K,Z,V,\xi_K,\xi_Z,\xi_V) = \bar{F}(K,Z,V) \\ &\quad + \bar{U}^T \left(\xi_Z\xi_V^{-1}V\xi_V^{-1}\xi_Z - \xi_Z\xi_V^{-1}Z - Z\xi_V^{-1}\xi_Z\right)\bar{U} \end{aligned} \tag{32}$$

This potential can be used with Algorithm 1 to try to solve an SLC problem. Furthermore, as (32) is affine, the constraints in problem (24) become LMI. Since (29) is a concave function, Corollary 5 guarantees that the obtained solutions will be local optima. If a feasible initial solution for a linear control problem is not available, it is also possible to introduce auxiliary slack variables and use Algorithm 1 to find one.

4 Illustrative examples

In the following examples, the proposed framework for linear control synthesis and Algorithm 1 will be used to improve available suboptimal solutions to a decentralized control problem [9] and a mixed H_2/H_∞ problem [5].

4.1 H_2 decentralized control

The plant given in [9] have been discretized with sampling period $h = 0.1$ and zero order hold on the inputs. The discrete-time system matrices are

$$A = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The problem discussed here is to stabilize the plant and minimize the H_2 norm of the transfer function $H_{wz}(z)$ using two decentralized and strictly proper dynamic output feedback controllers. The first controller feedbacks the first output to the first input and the second feedbacks the second output to the second input. As a guideline for the quality of the optimization we have calculated the optimal centralized H_2 controller which provides the lowerbound H_2 cost $\alpha^* = 0.36$. The algorithm [9] has been used to provide the feasible suboptimal decentralized controllers

$$\begin{aligned} K_1^0(z) &= -2.23\frac{z-0.68}{z^2-1.26z+0.44}, \\ K_2^0(z) &= -0.06\frac{z-0.76}{(z+0.84)(z-0.78)}. \end{aligned}$$

The value of $\|H_{wz}^0(z)\|_2$ of this controller is 3.19.

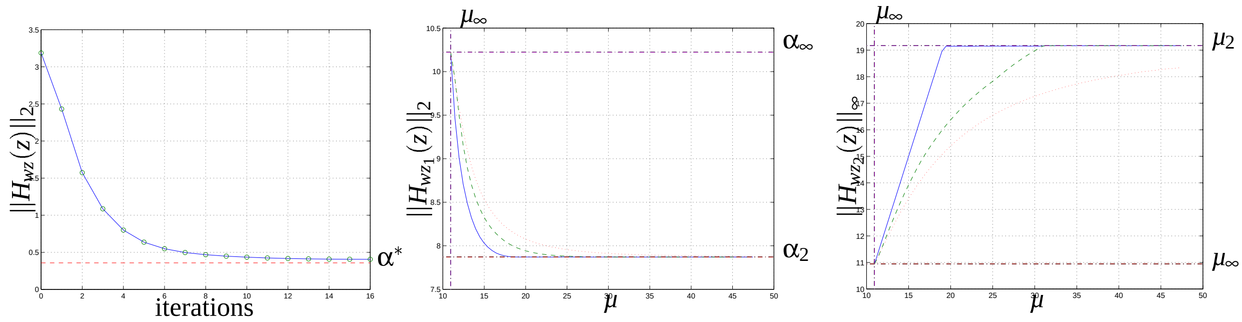
Using that controller as a starting point, we have run Algorithm 1 with the potential functional (31) to solve the minimization of μ subject to the constraints (17–18). The precision ε has been set to 10^{-3}. After 16 iterations we have obtained the decentralized controllers

$$\begin{aligned} K_1^{16}(z) &= -0.068\frac{z-0.37}{(z-0.37)(z-0.90)}, \\ K_2^{16}(z) &= -0.044\frac{z+0.23}{(z+0.23)(z-0.86)}, \end{aligned}$$

with cost $\|H_{wz}^{16}(z)\|_2 = 0.40$. Notice that this level of performance is only 11% worst than the optimal centralized performance and represents an improvement of about 800% with respect to the available suboptimal solution. Figure 1(a) shows the costs obtained by the algorithm versus the number of iterations.

4.2 Mixed H_2/H_∞ control

In this second example we will try to improve the suboptimal solution to a mixed H_2/H_∞ control problem. In this problem, we look for a unique controller that minimizes an H_2 performance cost while satisfying some H_∞ constraint. Suboptimal solutions to this problem are available using the LSP (Lyapunov Shaping Paradigm) or the extended approach introduced in [5]. In the first, a unique Lyapunov



(a) Decentralized control: H_2 performance $\times$ iterations

(b) Multiobjective problem: H_2 and H_∞ performances

matrix is used to provide upperbounds to the H_2 and H_∞ constraints. In the second, additional variables are introduced providing extended H_2 and H_∞ characterizations that allow the presence of one Lyapunov matrix per constraint. Some of these additional variables are constrained in a conservative way so that we can obtain a unique controller. Our objective in this example is to use the SLC synthesis framework and Algorithm 1 to improve the solutions given by the LSP and by [5]. So, let us introduce the following discrete-time unstable plant

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} w(k), \\ z_1(k) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \\ z_2(k) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(k). \end{aligned}$$

The objective is to design a strictly proper and full-order dynamic output feedback controller that minimizes $||H_{wz_1}(z)||_2$ while keeping $||H_{wz_2}(z)||_\infty$ below a certain level μ. By calculating the optimal H_2 and H_∞ controllers we obtain the minimum achievable values for these norms

$$\begin{aligned} \alpha_2 &:= \min ||H_{wz_1}(z)||_2 = 7.87, \\ \mu_\infty &:= \min ||H_{wz_2}(z)||_\infty = 10.94. \end{aligned}$$

We have also calculated $\mu_2 := ||H_{wz_2}(z)||_\infty = 19.16$ associated with the optimal H_2 controller and $\alpha_\infty := ||H_{wz_1}(z)||_2 = 10.22$ associated with the optimal H_∞ controller.

Then we have solved the suboptimal problems given by the LSP and [5] for values of μ ranging from μ_∞ to $3.5\mu_\infty$. The values of $||H_{wz_1}(z)||_2$ and $||H_{wz_2}(z)||_\infty$ for these controllers are shown in Figure 1(b) as, respectively, dotted and dashed lines. For each value of μ, using the solution provided by [5] as a starting point, we have run Algorithm 1 with the potential functional (31) to solve the mixed problem. The results are shown in Figure 1(a) as a solid line.

Figure 1(b) shows that the proposed algorithm provided a controller that satisfies the optimality conditions for this mixed problem, that is, the value of $||H_{wz_2}(z)||_\infty$ matches the upperbound μ. In this example, the algorithm was not sensitive to a change in the starting point. The differences in performance between the controllers shown in Figure 1(b) and the controllers obtained using the solutions provided by the LSP as a initial solution were below the specified precision $\varepsilon = 10^{-3}$.

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