Stability of Limit Cycles in Hybrid Systems using Discrete-Time Lyapunov Techniques [1]

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Abstract

This paper concerns stability analysis of limit cycles in hybrid systems. Continuous-time hybrid systems are modeled in a discrete-time affine framework. The discrete-time approach is shown to be appropriate in order to find a Lyapunov formulation for the stability of a hybrid limit cycle. Multiple Lyapunov functions are associated with the transitions in the hybrid system so that the trajectory is shown to converge to the switch points of the limit cycle. The results are formulated in Linear Matrix Inequalities (LMIs) which gives a constructive way to find the Lyapunov functions using efficient algorithms. The results are applied to a two-tank example with discrete valued actuators.

Keywords: hybrid systems, stability, limit cycles, Lyapunov, discrete time, linear matrix inequalities

1 Introduction

There is a wide range of systems that can be modeled in a hybrid context, e.g. multiple-model approaches, gain scheduling, systems with inherent switching as in relays and power electronics, switch schemes in adaptive control structures etc. These classes of systems can be described by a set of continuous-time sub-systems, activated and disabled with a discrete-event switch structure. Several of these systems may not have a unique equilibrium point. The switch behavior can end up in a periodic switch sequence while the continuous trajectory converge to a limit cycle. The most simple example of this is probably a thermodynamic system controlled by a thermostat.

From a control engineering perspective, the analysis of hybrid systems is dominated by extended Lyapunov techniques. A common approach is to use several Lyapunov functions for the different subsystems [7][2]. The obvious approach when analyzing continuous-time hybrid systems is to use continuous-time Lyapunov functions. With quadratic Lyapunov functions the stability conditions can be formulated with Linear Matrix Inequalities (LMIs) [1][8][5]. The search for the Lyapunov functions is then performed using efficient LMI-software, e.g. the LMI toolbox in Matlab. In [9] stability analysis and robustness properties were developed using discrete-time models for continuous-time hybrid systems. It was shown that the analysis of a wide range of hybrid systems benefits from such an approach. This paper uses a similar approach for the analysis of limit cycles. The discrete-time approach has similarities to the Poincaré map technique used for the analysis of nonlinear periodic systems [6]. Discrete-time Lyapunov theory for hybrid systems can also be found in [7][13].

The paper is organized as follows. Section 2 gives a description of hybrid systems and their switch-behavior with a discussion on periodic switch sequences and trajectories forming limit cycles. In Section 3 Lyapunov theory for discrete-time linear systems is recalled and adapted to the hybrid framework. Stability conditions for limit cycles are formulated using LMIs [1]. The Lyapunov function is parameterized as a multiple Lyapunov function where each parameterization is associated with a transition between two discrete states. In Section 4 a two-tank system is presented, from [10]. The system has a stable limit cycle to which the stability analysis is applied. Discussions on limit cycles and cycles in hybrid and switched systems can also be found in [2][4][3]. Some conclusions concludes the paper.

2 System Description

An autonomous hybrid system can be written on the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), q(t)), \qquad x(t_0) = x_0, \\ q(t) &= e(x(t), q(t^-)), \quad q(t_0) = i_0, \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the continuous state vector and $q(t) \in \mathcal{Q} = \{1, \ldots, n_Q\}$ indicates the discrete state. The hybrid state space is $\mathcal{H} = \Re^n \times \mathcal{Q}$ and the initial hybrid state is assumed to lie in a set of allowed initial conditions, $(x_0, i_0) \in \mathcal{H}_0 \subseteq \mathcal{H}$.

The function $e : \Re^n \times \mathcal{Q} \to \mathcal{Q}$ describes the change of the discrete state, where switching between state i and j occurs if $x(\cdot)$ reaches the switch set $\mathcal{S}_{i,j}$

$$\mathcal{S}_{i,j} = \{x : e(x, i) = j\}. \tag{2}$$

Each discrete state $i \in \mathcal{Q}$ is associated with the vector field $f(\cdot, i) : \Re^n \to \Re^n$ which will be referred to as the subsystem $f_i(\cdot)$. The vector field $f(\cdot, i)$ is assumed to be at least locally Lipschitz continuous.

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2.1 Solution Characteristics

Consider the solution to the hybrid system (1). From the initial conditions (x_0, i) the system evolves according to $\dot{x} = f(x, i)$. If $x(\cdot)$ reaches any $\mathcal{S}_{i,j}$ at time t_1 the discrete state becomes $q(t_1) = j$. The result is a switch sequence dependent on initial conditions

$$S(x_0, i_0) = (i_0, t_0), (i_1, t_1), \ldots, (i_N, t_N). \quad (3)$$

The notation $S(\mathcal{H}_0)$ is used for the set of all switch sequences originating from initial conditions in $\mathcal{H}_0$. Define the projections

$$\begin{aligned} \pi_i(S(x_0, i_0)) &= i_0, i_1, \ldots, i_N, \\ \pi_t(S(x_0, i_0)) &= t_0, t_1, \ldots, t_N. \end{aligned} \quad (4)$$

The switch to a new discrete state value is referred to as a transition $i, j \subseteq \pi_i(S(x_0, i_0))$. Denote elapsed time from most recent switch time with $\tau = t - t_k,\ t \in [t_k, t_{k+1})$ and define the *duty time* for the discrete state i_k as

$$\tau_k = t_{k+1} - t_k.$$

Possible duty times for a discrete state i is given by the set

$$\mathcal{T}_i = \{\tau_k : \ i_k = i \subseteq \pi_i(S(\mathcal{H}_0))\}. \quad (5)$$

The bounds on the duty times are given by, $\forall \tau \in \mathcal{T}_i,\ \inf(\mathcal{T}_i) < \tau < \sup(\mathcal{T}_i)$. In order to have a well defined solution to (1) it is required that only a finite number of switches occur in finite time, i.e. $\inf(\mathcal{T}_i) > \varepsilon > 0$. This puts restrictions on $e(x, q)$. For this work finite switches in finite time is assumed without further discussions on the implied properties on e. The continuous dynamics between the switch points $t_k \in \pi_t(S(x_0, i_0))$ is defined by

$$\dot{x}(t) = f_{i_k}(x(t)), \quad t \in [t_k, t_{k+1}). \quad (6)$$

An alternative notation similar to (1) is $\dot{x}(t) = f_{q(t)}(x(t))$.

2.2 Discrete-time Models and Affine Systems

A solution to hybrid system dynamics is uniquely defined by a sequence of switch points $x(t_k)$ and the active continuous dynamics $\dot{x} = f_{i_k}(x)$ between the switch times. The hybrid system can be viewed as a discrete-time system where the switch points form a state sequence $x[k] = x(t_k)$. The discrete time is the repeated time instances of switch occurences, not the result of uniform sampling.

For affine systems the continuous dynamics is defined by $\dot{x}(t) = A_{q(t)}x(t) + a_{q(t)}$. The state equation can be rewritten using the extended state vector $\tilde{x}$,

$$\dot{\tilde{x}}(t) = \tilde{A}_{q(t)}\tilde{x}(t), \quad \tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \ \tilde{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}. \quad (7)$$

The relation between consecutive switch points can be given in a discrete-time state equation

$$\tilde{x}[k+1] = F[k]\tilde{x}[k], \quad F[k] = F(i_k, \tau_k) = e^{\tilde{A}_{i_k}\tau_k}, \quad (8)$$

where $F[k]$ is a time-varying matrix depending on the continuous mode i_k and the duty time τ_k.

2.3 Switch Structure

The switch sets will be modeled as surfaces in the state space. A *switch function* $s_{i,j}(x) : \Re^n \to \Re$ is then used to define the switch set for a given transition

$$\mathcal{S}_{i,j} = \{x : \ s_{i,j}(x) = 0\}.$$

It will be assumed that the switch functions represent hyperplanes in the extended state space, i.e. $s_{i,j}(x) = c_{i,j}^T x + d_{i,j}$. Using the extended state vector

$$s_{i,j}(x) = \tilde{c}_{i,j}^T \tilde{x}, \quad \tilde{c}_{i,j} = [c_{i,j}^T \ d_{i,j}]^T. \quad (9)$$

The switch planes in the extended state description lie in $\Re^{n+1}$ and is spanned by orthonormal basis vectors $e_{i,j}^{(1)}, \ldots, e_{i,j}^{(n)}$ orthogonal to $\tilde{c}_{i,j}$, i.e.

$$E_{i,j} = \left[e_{i,j}^{(1)} \cdots e_{i,j}^{(n)} \right], \quad \tilde{c}_{i,j}^T E_{i,j} = 0, \ E_{i,j}^T E_{i,j} = I.$$

The switch sets can then be parameterized in $w \in \Re^n$

$$\tilde{\mathcal{S}}_{i,j} = \{\tilde{x} : \ \tilde{x} = E_{i,j} w, \ w \in \Re^n\}. \quad (10)$$

2.4 Fundamental Cycles and Closed Orbits

A fundamental cycle is a closed sequence of discrete states where each state occurs once,

$$i_1, \ldots, i_d, i_1 \quad i_k \neq i_l, \ \forall k \neq l. \quad (11)$$

A hybrid system's switch behavior may exhibit several fundamental cycles. In [3] it is pointed out that fundamental cycles is a means to describe the switch structure of a hybrid system; that in some cases it is possible to formulate all switch sequences $S(\mathcal{H}_0)$ with a limited set of fundamental cycles. In this paper the switch structure is implicitly formulated with $S(\mathcal{H}_0)$.

For a fundamental switch cycle defined according to (11) there exists a closed orbit if the corresponding state system maps a certain state onto itself

$$\begin{aligned} \tilde{x}_{i_d,i_1}^0 &= F(i_d, \tau_d^0) \cdots F(i_1, \tau_1^0)\tilde{x}_{i_d,i_1}^0 \\ &\vdots \\ \tilde{x}_{i_{d-1},i_d}^0 &= F(i_{d-1}, \tau_{d-1}^0) \cdots F(i_d, \tau_d^0)\tilde{x}_{i_{d-1},i_d}^0 \end{aligned} \quad (12)$$

where $\tau_k^0 \in \mathcal{T}_{i_k}$ and $\tilde{x}_{i,j}^0 \in \tilde{\mathcal{S}}_{i,j}$, see (2)(5). With the switch sets given as planes the switch points satisfy

$$\tilde{c}_{i,j}^T \tilde{x}_{i,j}^0 = 0, \quad i, j \subseteq \{i_1, \ldots, i_d, i_1\}. \quad (13)$$

Thus, if there exist $\tilde{x}_{i,j}^0$ and τ_k^0 that solve (12) subject to (13) then the $x_{i,j}^0$ define the switch points of a closed orbit. An isolated closed orbit is referred to as a limit cycle. In the following section we will establish stability conditions that show exponential convergence to these switch points. The exponential convergence to the switch points is assumed to be a measure on the stability of the limit cycle.

3 Lyapunov Stability

The objectives in this section are to develop stability conditions in discrete time for limit cycles in hybrid systems. The analysis will utilize discrete-time Lyapunov theory for linear systems [11]. The common approach to use several Lyapunov functions for the analysis of hybrid systems [7][2][8] will be used in a discrete-time framework where a unique functions is associated with the discrete state transitions.

3.1 Review of Discrete-Time Lyapunov Theory

The following notation will be used. For the symmetric matrix Q, $\eta I \leq Q \leq \rho I$ denotes its smallest and largest eigenvalue; $x^T Q x \leq \|Q\| \, \|x\|^2$ where $\|Q\| = \rho$. Consider a sequence of state points $x[k] = x(t_k)$. Let the sequence originate from the time varying state equation $x[k+1] = F[k]x[k]$. A time varying quadratic Lyapunov function is given on the form

$$V(k, x[k]) = x^T[k]P[k]x[k]. \tag{14}$$

The following theorem gives sufficient and necessary conditions for exponential stability.

Theorem 3.1 *The linear state equation $x[k+1] = F[k]x[k]$ is exponentially stable iff there exist a matrix sequence $P[k]$ that is symmetric such that $\forall k \geq 0$*

$$\begin{gathered} \eta\, I \leq P[k] \leq \rho\, I, \\ F^T[k]P[k+1]F[k] - P[k] \leq -\nu\, I, \end{gathered} \tag{15}$$

$\eta,\ \rho,\ \nu > 0$. Then $\|x[k]\| \leq \gamma \cdot \lambda^k \, \|x[0]\|$, where $\gamma^2 = \rho/\eta$ and $\lambda^2 = 1 - \nu/\rho$.

proof: can be found in [11]. □

3.2 Stability Conditions for Nominal Switch Points

Consider a state sequence $x[k]$ originating from a discrete-time state system. A "time-varying equilibrium point" $x^0[k]$ represents a sequence of nominal switch points, e.g. the switch points in a hybrid limit cycle. To show exponential stability of $x^0[k]$ consider the Lyapunov function

$$V(k, x[k]) = (x[k]-x^0[k])^T P[k](x[k]-x^0[k]) \tag{16}$$

If $V(k+1, x[k+1]) - V(k, x[k]) < -\nu\|x[k] - x^0[k]\|^2$, $\forall x[k] \neq x^0[k]$ then $x^0[k]$ is an exponentially stable "time-varying equilibrium point", cf. Theorem 3.1.

3.2.1 LMI Formulation

The stability conditions can be written as LMIs which gives a constructive way of finding a Lyapunov function. Introduce the transformation matrix

$$T_k = \begin{bmatrix} I & -x^0[k] \\ 0 & 1 \end{bmatrix}. \tag{17}$$

The $x[k] - x^0[k]$ quantity can be given using the extended state vector as $x[k] - x^0[k] = [I\ 0\,]\, T_k \tilde{x}[k]$. Assume that the discrete-time state system is given on an affine form $\tilde{x}[k+1] = F[k]\tilde{x}[k]$ where $\tilde{x}^0[k+1] = F[k]\tilde{x}^0[k]$. (Note that $F[k] = F(i_k, \tau_k)$ is dependent on the state value $\tilde{x}[k]$ since switch times are dependent on previous switch points $\tilde{x}[k]$.) Now $x[k+1] - x^0[k+1] = [I\ 0\,]\, T_{k+1} F[k]\tilde{x}[k]$. The conditions for stability using the Lyapunov function (16) are

$$\begin{gathered} \tilde{x}[k]^T \Big[(T_{k+1}F[k])^T \bar{P}[k+1]\,(T_{k+1}F[k]) - \\ -T_k^T (\bar{P}[k] - \nu \bar{I}\,)\, T_k \Big] \tilde{x}[k] < 0, \end{gathered} \tag{18}$$

$$\bar{P}[k] = \begin{bmatrix} P[k] & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{I} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

$\forall \tilde{x}[k] \neq \tilde{x}^0[k]$. This is a set of LMIs for each k.

3.2.2 Parameterization of Switch Regions

A solution to the LMI conditions in (18) implies that stability is shown for any switch points $\tilde{x}[k]$ in $\Re^{n+1} - \{\tilde{x}^0[k]\}$. This is a rather conservative result since the stability often is dependent on the actual switch regions. According to the definition of switch planes the switch-point $\tilde{x}[k]$ satisfy

$$\tilde{c}[k]^T \tilde{x}[k] = 0, \quad \tilde{c}[k] = \tilde{c}_{i_{k-1}, i_k}. \tag{19}$$

From the switch set parameterization in (10) we have

$$\tilde{x}[k] \in \{\tilde{x} :\ \tilde{x} = E_k w,\ w \in \Re^n\}, \quad E_k = E_{i_{k-1}, i_k}.$$

The parameterization is used in (18) which gives

$$\begin{gathered} w^T E_k^T \Big[(T_{k+1}F[k])^T \bar{P}[k+1]\,(T_{k+1}F[k]) - \\ -T_k^T (\bar{P}[k] - \nu \bar{I}\,)\, T_k \Big] E_k w < 0, \end{gathered} \tag{20}$$

$\forall E_k w \neq \tilde{x}^0[k]$. The conservativeness in the LMIs is reduced by effectively removing one state dimension.

3.2.3 Removing implicit nullspaces

Now it will be shown how to remove the constraint "$\neq \tilde{x}^0[k]$" in (20). The vector $\tilde{x}^0[k]$ lies in the switch set $\{E_k w,\ w \in \Re^n\}$. Let the columns in the matrix $W_k^\perp$ represent w vectors resulting in $\tilde{x}$ vectors orthogonal to $\tilde{x}^0[k]$

$$\tilde{x}^0[k]^T E_k W_k^\perp = 0.$$

Hence the constraint $\forall E_k w \neq \tilde{x}^0[k]$ can be replaced with $\forall E_k w :\ w = W_k^\perp v,\ v \in \Re^{n-1}$. We get

$$\begin{gathered} v^T (E_k W_k^\perp)^T \Big[(T_{k+1}F[k])^T \bar{P}[k+1]\,(T_{k+1}F[k]) - \\ -T_k^T (\bar{P}[k] - \nu \bar{I}\,)\, T_k \Big] (E_k W_k^\perp) v < 0, \end{gathered}$$

$\forall v \in \Re^{n-1}$, which can be rewritten as

$$\begin{gathered} Q_k^T P[k+1] Q_k - R_k^T (P[k] - \nu I\,) R_k < 0, \\ \begin{matrix} Q_k = [I\ \ 0\,]\, T_{k+1} F[k] E_k W_k^\perp \\ R_k = [I\ \ 0\,]\, T_k E_k W_k^\perp \end{matrix},\ Q_k, R_k \in \Re^{n \times n-1} \end{gathered} \tag{21}$$

This is an LMI in the variables $P[k], \eta, \rho, \nu$.

According to Theorem 3.1 the ν value indicates the convergence rate. In (21) ν is introduced as an inner product. It is, however, useful to obtain an explicit expression on ν.

For (21) to hold for some $P[k] > 0$ and ν it is necessary that R_k has independent columns. If not, the second term in the inequality would have an eigenvalue of zero resulting in $v^T Q_k^T P[k+1] Q_k v < 0$ for some v, which is infeasible due to the positive definiteness of $P[k+1]$. There is therefore no loss in generality to assume that R_k has independent columns. If the columns in R_k are independent, $R_k^T R_k$ is positive definite and there exists a non-singular matrix $\hat{R}_k \in \Re^{n-1\times n-1}$ such that $\hat{R}_k^T \hat{R}_k = R_k^T R_k$ which gives $\hat{R}_k^{-T}(R_k^T R_k)\hat{R}_k^{-1} = I$. Pre- and post-multiplication with $\hat{R}_k^{-T}$ and $\hat{R}_k^{-1}$ preserve positive definiteness (congruence transformation [12]). Applied on (21) we obtain

$$\hat{R}_k^{-T}\left[Q_k^T P[k+1] Q_k - R_k^T P[k] R_k\right]\hat{R}_k^{-1} + \nu I < 0. \quad (22)$$

The LMI problem has to be solved for each k which of course is not feasible when k gets larger. In the next section we parameterize the LMI problem for a cyclic switch sequence exhibiting a limit cycle. Note that there is a time-dependence in the LMI due to τ_k in $F[k] = F(i_k, \tau_k)$.

3.3 Theory Applied to Hybrid Limit Cycles

Assume that there exists a limit cycle for a given fundamental cycle $\{i_1, \ldots, i_d, i_1\}$, i.e.

$$\text{if } x[k] = x_{i,j}^0 \text{ then } x[k+1] = x_{j,k}^0$$

and $\tilde{x}_{j,k}^0 = F(j, \tau_j^0)\tilde{x}_{i,j}^0,\ i,j,k \subseteq \{i_1, \ldots, i_d, i_1, i_2\}$. The objectives are to establish stability conditions for this limit cycle. Stability of the limit cycle is defined as the exponential convergence to its switch points. Given that the switch points have been found, see Section 2.4, it is possible to parameterize the conditions in (21) for the transitions in the switch cycle. The time-dependence in the LMIs is dealt with in a two-step strategy. A nominal LMI formulation for a set of nominal duty times $\tau_i^0 \in \mathcal{T}_i$ and an eigenvalue condition for the remaining range of duty times. The result is formulated in the following theorem

Theorem 3.2 *Given a discrete-time state model on affine form* $\tilde{x}[k+1] = F(i_k, \tau_k)\tilde{x}[k]$ *with a set of nominal switch points* $\tilde{x}_{i,j}^0 \in \tilde{\mathcal{S}}_{i,j} = \{E_{i,j} w, w \in \Re^n\}$ *and duty times* τ_i^0 *for the fundamental switch cycle* $i_1, \ldots, i_d, i_1$ *such that*

$$\tilde{x}_{j,k}^0 = F(j, \tau_j^0)\tilde{x}_{i,j}^0, \quad i,j,k \in \{i_1, \ldots, i_d, i_1, i_2\}.$$

If there exists symmetric matrices $P_{i,j}$ *satisfying*

i)

$$Q_{i,j,k}^T(\tau_j^0) P_{j,k} Q_{i,j,k}(\tau_j^0) - R_{i,j}^T\left(P_{i,j} - \nu_0 I\right) R_{i,j} < 0$$
$$\eta I < P_{i,j} < \rho I,$$

$$Q_{i,j,k}(\tau) = [I \ \ 0]\, T_{j,k} F(j,\tau) E_{i,j} W_{i,j}^{\perp}, \quad (23)$$
$$R_{i,j} = [I \ \ 0]\, T_{i,j} E_{i,j} W_{i,j}^{\perp},$$

$\eta, \rho, \nu_0 > 0,\ \tilde{x}_{i,j}^{0T} E_{i,j} W_{i,j}^{\perp} = 0$ *and*

ii)

$$\nu = \inf_{\tau \in \mathcal{T}_j} \lambda_{\min}\Big[\hat{R}_{i,j}^{-T}\big(R_{i,j}^T P_{i,j} R_{i,j} - Q_{i,j,k}^T(\tau) P_{j,k} Q_{i,j,k}(\tau)\big)\hat{R}_{i,j}^{-1}\Big] > 0 \quad (24)$$

$\forall i,j,k \subseteq \{i_1, \ldots, i_d, i_1, i_2\}$. *Then* $x[k]$ *converges to* $x^0[k]$ *exponentially.*

proof: this is a parameterized version of (21) and (22) where variables X_k are associated with transitions, $X_k = X_{i_{k-1}, i_k}$, and (22) is formulated as an eigenvalue problem. Suppose there exists a solution to (23) for nominal duty times τ_j^0 which gives the convergence rate ν_0 and matrices $P_{i,j}$. Then the $R_{i,j}$ matrices have independent columns. The obtained $P_{i,j}$ matrices are used in (24) to validate the entire range of duty times $\mathcal{T}_j$. If (24) holds the Lyapunov function $V(x) = (x - x_{i,j}^0)^T P_{i,j}(x - x_{i,j}^0)$, $i,j \in \pi_i(S(x_0, i_0))$ decreases over the entire switch sequence and the system converges exponentially to the nominal switch points. □

If $x_{i,j}^0$ are the switch points for a hybrid limit cycle, Theorem 3.2 gives the exponential stability of the limit cycle.

The LMI problem can be easily implemented in available software, e.g. the LMI toolbox in Matlab. In the LMI software it is possible to specify an optimization criteria. The nominal convergence rate ν_0 can be maximized by the operation: *Minimize* $-\nu_0$ *subject to the LMI* (23).

Suppose that (24) does not hold for all $\tau_j \in \mathcal{T}_j$. It is then possible to parameterize $\mathcal{T}_j$ and use several $P_{j,k}$ matrices corresponding to different duty times $\tau_{j,1}^0, \ldots, \tau_{j,m}^0 \in \mathcal{T}_j$.

4 Example

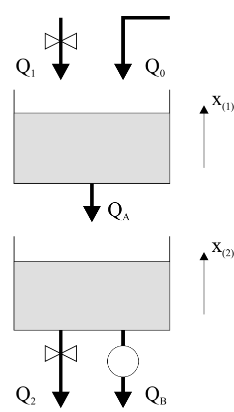

Figure 1: A two-tank setup.

This example was presented in [10] where the existence of a stable limit cycle was shown. The tank system shown in Fig. 1 consists of two tanks and two on/off valves. The first valve adds to the inflow in Tank 1 and the second valve is a drain valve from Tank 2. There is a constant outflow from Tank 2 caused by a pump. The state dynamics is obtained by a linearization about an operating point and the objectives are to keep water-levels in both tanks within some limits using an on/off switch strategy for the valves. The two discrete valve settings result in four discrete states with different continuous dynamics. The system is such that it is unstable at the operating point for each mode. Without changing the settings of the valves the tanks will either be flooded or drained. Switching is based on the water-levels $x_{(1)}$ and $x_{(2)}$ in the tanks. (The notation $x_{(i)}$ is used for the i:th element in the state vector.) For an initial continuous state there is an associated unique initial discrete state defining the allowed initial hybrid states $\mathcal{H}_0$,

$$\begin{array}{l} q(t)=1: \ \textit{off}\,/\,\textit{off} \\ q(t)=2: \ \textit{on}\,/\,\textit{off} \\ q(t)=3: \ \textit{off}\,/\,\textit{on} \\ q(t)=4: \ \textit{on}\,/\,\textit{on} \end{array} \qquad i_0=\begin{cases} 1: \ x_{(1)} \geq 0\ ,\ x_{(2)} < 0 \\ 2: \ x_{(1)} < 0\ ,\ x_{(2)} < 0 \\ 3: \ x_{(1)} \geq 0\ ,\ x_{(2)} \geq 0 \\ 4: \ x_{(1)} < 0\ ,\ x_{(2)} \geq 0 \end{cases}$$

The numerical values used for simulations are

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, & A_3 &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \\ a_1 &= \begin{bmatrix} -2 & 0 \end{bmatrix}^T, & a_3 &= \begin{bmatrix} -2 & -5 \end{bmatrix}^T, \\ A_2 &= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, & A_4 &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \\ a_2 &= \begin{bmatrix} 3 & 0 \end{bmatrix}^T, & a_4 &= \begin{bmatrix} 3 & -5 \end{bmatrix}^T. \end{aligned}$$

The defined switch planes corresponding to transitions are

$$\begin{aligned} \tilde{c}_{1,2} &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T/\sqrt{2}, & \tilde{c}_{1,3} &= \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T/\sqrt{2}, \\ \tilde{c}_{2,3} &= \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T/\sqrt{2}, & & \\ \tilde{c}_{3,1} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, & \tilde{c}_{3,4} &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T/\sqrt{2}, \\ \tilde{c}_{4,2} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, & \tilde{c}_{4,3} &= \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T/\sqrt{2}. \end{aligned}$$

The defined transitions can produce four possible fundamental switch cycles $i = 1,2,3,1;\ 1,3,1;\ \ 2,3,4,2;\ \ 3,4,3.$

4.1 Simulations

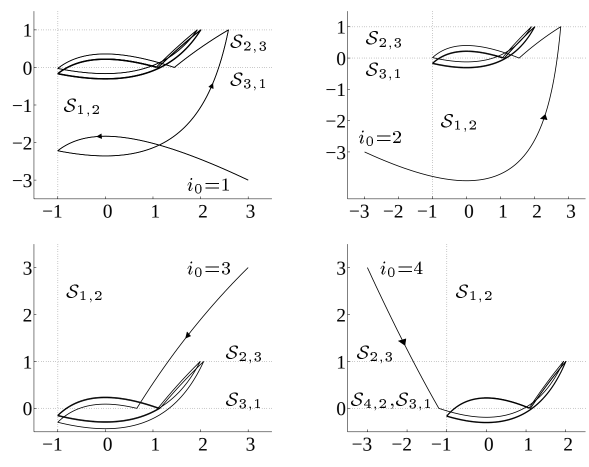

Figure 2: Simulations for different initial conditions.

In Fig. 2 simulations are shown for different initial conditions (x_0, i_0). The axes correspond to water-levels in the two tanks (Tank 1 level $x_{(1)}$ on the x-axis and Tank 2 level $x_{(2)}$ on the y-axis). The dashed lines correspond to switch sets. From the simulations it seems reasonable to assume that there exist a stable limit cycle to which the continuous trajectory converge. The fundamental switch sequence for the limit cycle is $i = 1,2,3$.

Note that choosing switch functions to obtain a stable behavior is not trivial. Fig. 4.1 shows a set of switch sets resulting in an unstable solution.

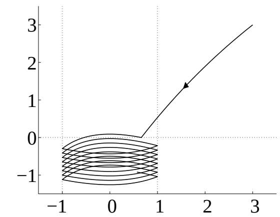

Figure 3: A set of switch functions resulting in an unstable switching strategy.

4.2 Stability of the Limit Cycle

Consider the switch cycle of the limit cycle. It is possible to solve (12) and (13) to find the switch points and the duty times for the limit cycle

$$x^0_{1,2} = \begin{bmatrix} -1.00 \\ -0.16 \end{bmatrix},\ x^0_{2,3} = \begin{bmatrix} -2.00 \\ 1.00 \end{bmatrix},\ x^0_{3,1} = \begin{bmatrix} 1.11 \\ 0.00 \end{bmatrix}$$

$$\tau^0_1 = 1.134,\ \tau^0_2 = 1.387,\ \tau^0_3 = 0.252.$$

The stability analysis is performed over a range of possible duty times. So far we have not come up with any constructive method to obtain these duty times in a general setting. For this example we consider regions about the nominal switch points which together with geometrical considerations are used to obtain a set of duty times for which the switch cycle is analyzed.

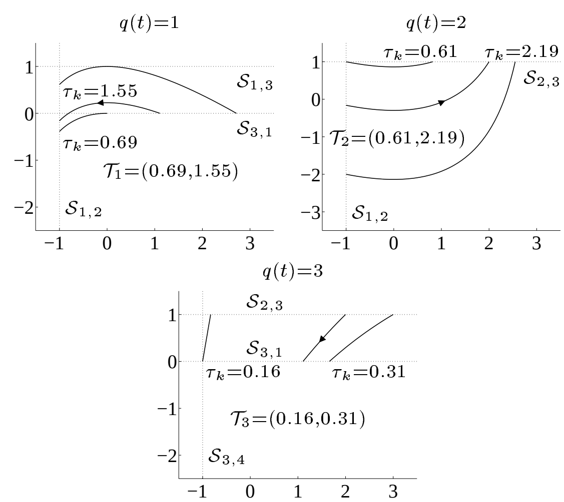

Figure 4: Range of duty times corresponding to state space regions about the limit cycle.

Starting with the discrete state 1 we can find all possible duty times such that the closed switch sequence is maintained. Note that too high duty times for this discrete state implies the transition 1,3 through $\mathcal{S}_{1,3}$. For the discrete state 2 we consider switch points in the region $x_{(2)} > -2$. Finally it can be shown that transitions to the discrete state 3 occur for $x_{(1)} < 3$; the sequence is maintained if switch set $\mathcal{S}_{3,4}$ is not reached. The findings are displayed in Figure 4.2 where the obtained duty times are $\mathcal{T}_1 = (0.693, 1.551)$, $\mathcal{T}_2 = (0.606, 2.190)$ and $\mathcal{T}_3 = (0.156, 0.311)$.

We will now examine the stability properties. There are three nominal system matrices $F(i, \tau_i^0) = e^{\tilde{A}_i \tau_i^0}$ and matrices $E_{i,j}$, $W_{i,j}^{\perp}$, $T_{i,j}$ for $i, j \subseteq \{1, 2, 3, 1, 2\}$ such that

$$\begin{cases} \tilde{c}_{i,j}^T E_{i,j} = 0 \\ \tilde{x}_{i,j}^{0T} E_{i,j} W_{i,j}^{\perp} = 0 \end{cases}, \quad T_{i,j} = \begin{bmatrix} I & -x_{i,j}^0 \\ 0 & 1 \end{bmatrix}.$$

Using Theorem 3.2, a solution to the nominal problem (23) can be found using the LMI toolbox in Matlab. A set of $P_{i,j}$ matrices satisfying (23) is

$$P_{1,2} = \begin{bmatrix} 0.348 & -0.473 \\ -0.473 & 0.652 \end{bmatrix}, \; P_{2,3} = \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 0.003 \end{bmatrix},$$

$$P_{3,1} = \begin{bmatrix} 0.778 & -0.414 \\ -0.414 & 0.223 \end{bmatrix},$$

with $\nu_0 = 0.648$ and $\eta = 1.86 \times 10^{-3}$ for the normalization $\rho = 1$. Hence a local Lyapunov function is found for the limit cycle.

For other switch points the duty times will vary. The second part of Theorem 3.2 is used to validate the stability conditions for the intervals of duty times given in Figure 4.2. Figure 5 shows the obtained convergence rates for different duty times, or rather the convergence rate that the nominal Lyapunov function implies. The dotted horizontal line is the nominal convergence rate $\nu_0 = 0.65$.

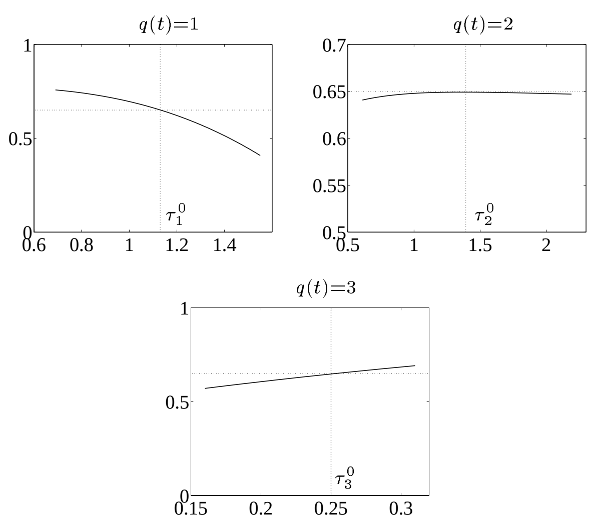

Figure 5: Convergence rates for different duty times.

Since $\nu > 0$ for the given intervals, the system has an exponentially stable limit cycle where stability is shown in the regions specified in Figure 4.2.

5 Conclusions

It has been shown how to find limit cycles in periodic switch sequences for hybrid systems. Using a discrete-time state description for the continuous-time system, Lyapunov theory for linear systems was utilized to formulate stability conditions. The conditions were given as a set of LMIs. The results were applied to an example.

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