# Robust Stability and Performance of Uncertain Delay Systems with Structured Uncertainties[1]

Yun-Ping Huang and Kemin Zhou
Department of Electrical and Computer Engineering
Louisiana State University
Baton Rouge, LA 70803
email: kemin@ee.lsu.edu
Tel: (225) 388-5533
Fax: (225) 388-5200

**Abstract**

This paper considers the robust stability and robust performance of uncertain delay systems subject to possibly structured uncertainties using structured singular values with phase information. Computationally efficient sufficient conditions for robust stability and performance problem are derived.

**Keywords:** Uncertain Delay Systems, Robust Stability, Robust Performance, Structured Singular Values

## 1 Introduction

It is well-known that many robust stability and performance problems in control lead to computing the structured singular values of some closed-loop system transfer matrices [4, 5, 17, 20, 21]. The standard definition of structured singular value assumes that the uncertainties are only norm bounded. However, in many applications, the uncertainties involved are constrained, for example, the uncertain parameters are nonnegative or have restricted phase angles. The structured singular value with phase information has been considered in a recent paper [19] where the block structured uncertainties are considered and a computable upper bound is derived. In this paper, we generalize the results in [19] to include repeated scalar blocks. We then apply these generalized results to the stability and performance problem of uncertain delay systems [10, 11]. The stability of uncertain delay systems has received much attention recently and many sufficient conditions have been derived, see e.g., [7, 12, 13, 14, 15, 18]. See also [16] for a survey and an extended list of references. Unfortunately, most of the existing conditions in the literature are extremely conservative (see [8] for some detailed analysis). Notable exceptions are the results in [3], where a complete solution is given for delay-independent stability using structured singular value characterization, and the results in [2], where a computable necessary and sufficient condition is given for the stability of uncertain delay systems. Complete solution has not been reported for testing the robust stability and performance of uncertain delay systems that are not stable independent of delay with structured uncertainties. Although the condition given in [18] is necessary and sufficient, it is not practically computable. In this paper, sufficient conditions are given for robust stability and performance of uncertainty delay systems with structured uncertainties. These conditions are derived using the structured singular value with phase information which are obviously less conservative than the standard structured singular value test. Numerical examples show that these conditions can be very effective. Due to space limitation some detailed development can be found in [9].

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## 2 Robust Stability and Performance of Uncertain Delay Systems

The structured singular value with phase information has been considered in [19] for matrices with block structured uncertainties. However, in many applications, such as the robust stability problem of uncertain delay systems considered here, the uncertain parameters appear as repeated scalar blocks. In order to deal with such problems, we need to generalize the results in [19] to include the repeated scalar blocks.

Let $r = k_1 + \cdots + k_{\ell+m+n} + n_1 + \cdots + n_p$ and $\ell,\ m,\ n,\ p\ \geq 0$. Define

$$\mathbf{\Gamma} := \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \delta_i \in \mathbf{R}, \gamma_i \in \mathbf{C}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\},$$

$$\mathbf{\Gamma}_{\mathbf{\Theta}_\ell} := \{\Gamma \in \mathbf{\Gamma} :\ |\angle\gamma_i| \leq \theta_i, i = 1, \ldots, \ell.\},$$

where $\Theta_\ell = (\theta_1, \ldots, \theta_\ell)$ with $\theta_i \in [0, \pi/2]$ for $i = 1, \ldots, \ell$.

The structured singular value of a matrix $M \in \mathbf{C}^{r\times r}$ with respect to a block structure $\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}$ is defined to be $\mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M) = 0$ if there is no $\Gamma \in \mathbf{\Gamma}_{\mathbf{\Theta}_\ell}$ such that $\det(I - \Gamma M) = 0$, and

$$\mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M) = \left(\min_{\Gamma\in\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}} \{\bar{\sigma}(\Gamma) :\ \det(I - \Gamma M) = 0\}\right)^{-1}$$

otherwise. A computable upper bound will be presented in the next section.

We shall now see how the structured singular value can be used for robust stability and performance analysis of uncertain time delay systems with possibly structured uncertainties.

Without loss of generality, we can assume that the uncertain delay system can be put in the standard linear fractional transformation form as shown in Figure 2. Assume that the system matrix $P(s)$ is a rational stable transfer matrix with suitable dimensions, and is denoted by

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

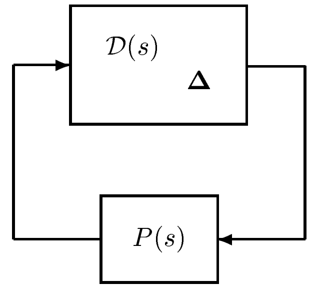


**Figure 1:** LFT Form of an Uncertain Delay System

We assume that

$$\mathcal{D}(s) = \mathrm{diag}\left\{e^{-\tau_1 s} I_{k_1}, \ldots, e^{-\tau_\ell s} I_{k_\ell}\right\}$$

includes all uncertain delays such that $\tau_i \in [0, h_i]$, $i = 1, \ldots, \ell$ and

$$h_1 < h_2 < \ldots < h_\ell,$$

and $\mathbf{\Delta}$ is a block structured uncertainty includes all real, complex scalar and full block uncertainties and those blocks associated with robust performance criteria. With an appropriate arrangement, uncertainty $\mathrm{diag}\{\mathcal{D}(s), \mathbf{\Delta}\}$ can be written as

$$\mathrm{diag}\{\mathcal{D}(s), \mathbf{\Delta}\} := \big\{\mathrm{diag}(e^{-\tau_1 s} I_{k_1}, \ldots, e^{-\tau_\ell s} I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \tau_i \in [0, h_i], \gamma_i \in \mathbf{C}, \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{C}^{n_i\times n_i}\big\},$$

where $\ell,\ m,\ n,\ p\ \geq 0$, and the corresponding uncertainty blocks is

$$\Gamma := \big\{\mathrm{diag}(e^{-\tau_1 s} I_{k_1}, \ldots, e^{-\tau_\ell s} I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \tau_i \in [0, h_i] \in \mathbf{R}, i = 1, \ldots, \ell, \gamma_j \in \mathbf{C}, j = \ell+1, \ldots, \ell+m, \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{C}^{n_i\times n_i}\big\}.$$

Note that

$$\angle e^{-j\tau_i\omega} = -\tau_i\omega \in [-h_i\omega, 0].$$

Then, $\Gamma(j\omega) \in \mathbf{\Gamma}_{\mathbf{\Theta}_\ell}$ with

$$\mathbf{\Gamma}_{\mathbf{\Theta}_\ell} := \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) :\ \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell+m; |\angle\gamma_j| \leq h_j\omega, j = 1, \ldots, \ell, \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{C}^{n_i\times n_i}\big\},$$

if $h_\ell\omega \leq \pi/2$ (since $h_1\omega < h_2\omega < \ldots < h_\ell\omega$).

**Theorem 1** *Suppose $A$ is stable. Then the uncertain delay system in Figure 2 is robustly stable if the following conditions hold*

*(a)* $\nu^P(\omega) = \mu_{\mathbf{\Gamma}_{\Theta_\ell}}(P(j\omega)) < 1$ *for* $0 \le \omega \le \pi/2h_\ell$ *with* $\theta_i = h_i\omega,\ i = 1, \ldots, \ell$;

*(b) For each* $q = 1, \ldots, \ell - 1$, *we have* $\nu^P(\omega) = \mu_{\mathbf{\Gamma}_{\mathbf{\Theta_q}}}(P(j\omega)) < 1$ *for* $\pi/2h_{q+1} < \omega \le \pi/2h_q$, *with* $\theta_j = h_j\omega,\ j = 1, \ldots, q$ *and*

$$\mathbf{\Gamma}_{\mathbf{\Theta_q}} = \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_q I_{k_q}, \ldots, \gamma_\ell I_{k_\ell}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \ \gamma_i \in \mathbf{C},\ |\angle\gamma_j| \le \theta_j, j = 1, \ldots, q, \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\};$$

*(c)* $\nu^P(\omega) = \mu_{\mathbf{\Gamma}}(M(j\omega)) < 1$ *for* $\omega > \pi/2h_1$ *with*

$$\mathbf{\Gamma} = \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell+m;\ \delta_i \in \mathbf{R},\ \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\}.$$

It is noted that $\nu^P(\omega)$ is computed with $-h_i\omega \le \angle\gamma_i \le h_i\omega$ but in fact $\angle\gamma_i = \angle e^{-j\tau_i\omega} \in [-h_i\omega, 0]$. Hence, the stability condition given in the above theorem may be conservative. To reduce the conservativeness, let us define

$$M(s) = P(s)\mathrm{diag}(e^{-h_1 s/2} I_{k_1}, e^{-h_2 s/2} I_{k_2}, \ldots, e^{-h_\ell s/2} I_{k_\ell}, I, \ldots, I)$$

and the corresponding uncertainty blocks as

$$\Gamma := \big\{\mathrm{diag}(e^{-\phi_1 s} I_{k_1}, \ldots, e^{-\phi_\ell s} I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \phi_i \in [-h_i/2, h_i/2] \in \mathbf{R}, i = 1, \ldots, \ell, \gamma_j \in \mathbf{C}, j = \ell+1, \ldots, \ell+m, \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\}.$$

Then

$$\angle e^{-j\phi_i\omega} = -\phi_i\omega \in [-h_i\omega/2, h_i\omega/2],$$

and $\Gamma(j\omega) \in \mathbf{\Gamma}_{\Theta_\ell}$ with

$$\mathbf{\Gamma}_{\Theta_\ell} := \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \ \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell+m; |\angle\gamma_j| \le h_j\omega/2, j = 1, \ldots, \ell, \delta_i \in \mathbf{R},\ \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\},$$

if $h_\ell\omega \le \pi$ (since $h_1\omega < h_2\omega < \ldots < h_\ell\omega$).

**Theorem 2** *Suppose $A$ is stable. Then the uncertain delay system in Figure 2 is robustly stable if the following conditions hold*

*(a)* $\nu^M(\omega) = \mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M(j\omega)) < 1$ *for* $0 \le \omega \le \pi/h_\ell$ *with* $\theta_i = h_i\omega/2,\ i = 1, \ldots, \ell$;

*(b) For each* $q = 1, \ldots, \ell - 1$, *we have* $\nu^M(\omega) = \mu_{\mathbf{\Gamma}_{\mathbf{\Theta_q}}}(M(j\omega)) < 1$ *for* $\pi/h_{q+1} < \omega \le \pi/h_q$, *with* $\theta_j = h_j\omega/2,\ j = 1, \ldots, q$ *and*

$$\mathbf{\Gamma}_{\mathbf{\Theta_q}} = \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_q I_{k_q}, \ldots, \gamma_\ell I_{k_\ell}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \ \gamma_i \in \mathbf{C},\ |\angle\gamma_j| \le \theta_j, j = 1, \ldots, q, \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\};$$

*(c)* $\nu^M(\omega) = \mu_{\mathbf{\Gamma}}(M(j\omega)) < 1$ *for* $\omega > \pi/h_1$ *with*

$$\mathbf{\Gamma} = \big\{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell+m;\ \delta_i \in \mathbf{R},\ \Delta_i \in \mathbf{C}^{n_i \times n_i}\big\}.$$

Let $\nu(\omega) = \min\big\{\nu^P(\omega),\ \nu^M(\omega)\big\}$, we have the following final result.

**Theorem 3** *Suppose $A$ is stable. Then the uncertain delay system in Figure 2 is robustly stable if*

$$\nu(\omega) = min\big\{\nu^P(\omega),\ \nu^M(\omega)\big\} < 1, \quad \forall\omega.$$

## 3 Computational Consideration and Examples

It is clear from the last section that the key to the robust stability test of an uncertain delay system is to compute $\mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M(j\omega))$, which is a very difficult problem in general. We now show a way to compute an upper bound of this value.

Define

$$\mathbf{T} := \{T: \quad T = \mathrm{diag}(T_1, \ldots, T_\ell, \ldots, T_{\ell+m}, \ldots,$$
$$T_{\ell+m+n}, d_1 I_{n_1}, \ldots, d_{p-1} I_{n_{p-1}}, I_{n_p}),$$
$$0 < T_i^* = T_i \in \mathbf{C}^{k_i \times k_i},\ d_i \in \mathbf{R},\ d_i > 0\},$$
$$\mathbf{S}_\ell := \{S: \quad S = \mathrm{diag}(S_1, \ldots, S_\ell, 0, \ldots, 0),$$
$$0 \le S_i^* = S_i \in \mathbf{C}^{k_i \times k_i}\},$$
$$\mathbf{B}_{\mathbf{\Theta}_\ell} := \{B: \quad B = \mathrm{diag}(\beta_1 I_{k_1}, \ldots, \beta_\ell I_{k_\ell}, 0, \ldots, 0):$$
$$\beta_i \in [-\cot\theta_i, \cot\theta_i]\ \},$$
$$\mathbf{G_n} := \{G: \quad G = \mathrm{diag}(0, \ldots, 0, G_{\ell+m+1}, \ldots,$$
$$G_{\ell+m+n}, 0, \ldots, 0), \quad G_i^* = G_i \in \mathbf{C}^{k_i \times k_i}\}.$$

We have the following result, which is a straightforward generalization of the corresponding result in [19] for block structured case.

**Theorem 4** *Let $M \in \mathbf{C}^{r \times r}$. Then $\mu_{\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}}(M) \le \hat{\mu}_{\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}}(M)$, where*

$$\hat{\mu}_{\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}}(M) = \inf\{\alpha: \ M^* R M - \alpha^2 R$$
$$+ (S(I + jB)M + M^*(I - jB)S) + j(GM - M^*G) < 0,$$
$$\alpha > 0, R \in \mathbf{T}, S \in \mathbf{S}_\ell, B \in \mathbf{B}_{\mathbf{\Theta}_\ell}, G \in \mathbf{G_n}\}.$$

It is noted that the optimization involved in Theorem 4 is in general non-convex. However, as pointed out in [19], if $k_j = 1, j = 1, \ldots, \ell$, then the optimization in Theorem 4 can be converted to a convex one by setting $\hat{B} = SB$ and computing

$$\inf\{\alpha: \ M^* R M - \alpha^2 R + SM + M^* S$$
$$+ j(\hat{B}M - M^*\hat{B}) + j(GM - M^*G) < 0\} \qquad (1)$$

with convex constraints $-S\bar{B} \le \hat{B} \le S\bar{B}$ where $\bar{B} = \mathrm{diag}(\cot\theta_1 I_{k_1}, \ldots, \cot\theta_\ell I_{k_\ell})$.

It is also possible to compute an upper bound of $\mu_{\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}}(M)$ by solving equation (1), even when it is not the case that $k_j = 1, j = 1, \ldots, \ell$. In this case, we will get more conservative result. In general, a suboptimal $\hat{\mu}_{\mathbf{\Gamma}_{\mathbf{\Theta}_\ell}}(M)$ can be obtained through an iterative algorithm: first solving $R$, $S$ and $G$ with a fixed $B$, then solving $B$ with $R$, $S$ and $G$ obtained in the previous step, repeat the process until a satisfactory solution is found.

We shall now illustrate these results with two examples.

**Example 1**: This example is taken from [14]. Consider an uncertain time delay system with

$$\dot{x}(t) = \begin{bmatrix} -2 + 1.6\delta_1 & 0 \\ 0 & -1 + 0.05\delta_2 \end{bmatrix} x(t)$$
$$+ \begin{bmatrix} -1 + 0.1\delta_3 & 0 \\ -1 & -1 + 0.3\delta_4 \end{bmatrix} x(t - h)$$

where uncertain parameters $\delta_i$, $i = 1, \cdots, 4$ are real constants and satisfy $|\delta_i| \le 1$. To find out the maximum of $h$ such that system stays stable, we first put the system in a general linear fractional form with

$$P(s) = \left[\begin{array}{cc|cccccc} -2 & 0 & -1 & 0 & 1.6 & 0 & 0.1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0.05 & 0 & 0.3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

and $\Gamma = \mathrm{diag}(e^{-hs} I_2, \delta_1, \delta_2, \delta_3, \delta_4)$. Define $M(s) = P(s)\mathrm{diag}(e^{-h/2s} I_2, 1, 1, 1, 1)$. Then an upper bound of $\nu(\omega) = \min\{\nu^P(\omega),\ \nu^M(\omega)\}$ as defined in Theorem 5 and Theorem 6 can be computed using Theorem 8 and is plotted in Figure 2.

By Theorem 3, the uncertain system is robustly stable if $h \le 2.57$ since the minimum of the upper bounds of $\nu(\omega)$ is less than 1 for all $\omega$. Note that the stability conclusion can be drawn by [8] if $h < 1.212$, and by [14], if $h < 0.689$.

**Example 2** This example is adopted from [7]. Consider an uncertain time delay system with *time-varying* uncertainties

$$\dot{x}(t) = \begin{bmatrix} -2 + \rho(t) & \rho(t) \\ \rho(t) & -0.9 + \rho(t) \end{bmatrix} x(t)$$
$$+ \begin{bmatrix} -1 + \rho(t) & 0 \\ -1 & -1 - \rho(t) \end{bmatrix} x(t - h)$$

where uncertain parameter $|\rho(t)| \le 0.1$.

First, rewrite the original system in the form of

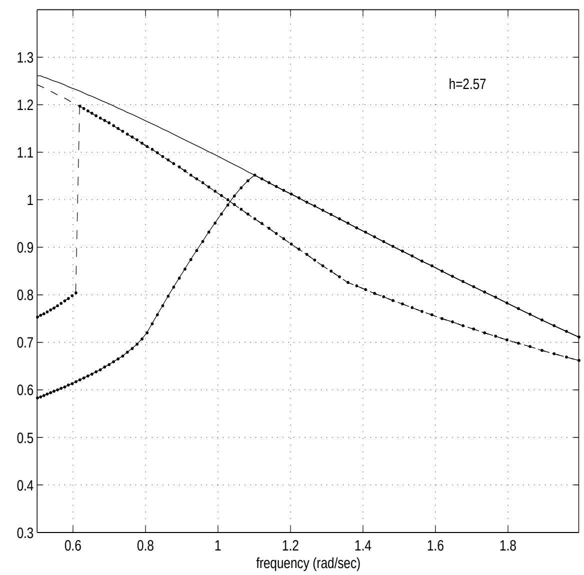


**Figure 2:** Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line), $\mu_\Gamma(P)$ (dashed line), and $\mu_{\Gamma_\Theta}(M)$ (dot-solid line) for $h = 2.57$

general linear fractional transformation with

$$P(s) = \left[\begin{array}{cc|cccc} -2 & 0 & -1 & 0 & 0.1 & 0 \\ 0 & -0.9 & -1 & -1 & 0 & 0.1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \end{array}\right]$$

and $\Gamma = \mathrm{diag}(e^{-hs}I_2, 10\rho(t)I_2)$, $\|\Gamma\| \leq 1$. Define $M(s) = P(s)\mathrm{diag}(e^{-h/2s}I_2, 1, 1)$.

Since the lower part of uncertainty is time-varying. The corresponding part of the scaling matrix has to be restricted to be a two by two constant matrix *over all frequencies*. We choose this part of the scaling matrix as follows: (1) Let the scaling matrix associated with time-varying part be an identity, $I_2$, and compute the upper bound of $\mu_{\Gamma_\theta}(M)$ for some time-delay $h$. The maximun $h$ such that $\mu_{\Gamma_\theta}(M) < 1$, $\forall\omega$, is found to be $h = 2.70$. (2) At the frequnecy where the peak of $\mu_{\Gamma_\theta}(M)$ occurs, we can get the solutions of $R \in \mathbf{T}$, $S \in \mathbf{S}_\ell$, $\hat{B}$, $-S\bar{B} \leq \hat{B} \leq S\bar{B}$ and $\alpha$ from Theorem 4. Next, given $R$, $S$, $\hat{B}$ and $\alpha$, solve the scaling matrix associated with time-varying part from Theorem 4 again.

Then upper bounds of $\nu(\omega)$ as defined in Theorem 1 and Theorem 2 can be computed by Theorem 4 at each frequency with the lower part of the scaling matrix being fixed at the value obtained above and the results are plotted in Figure 3.

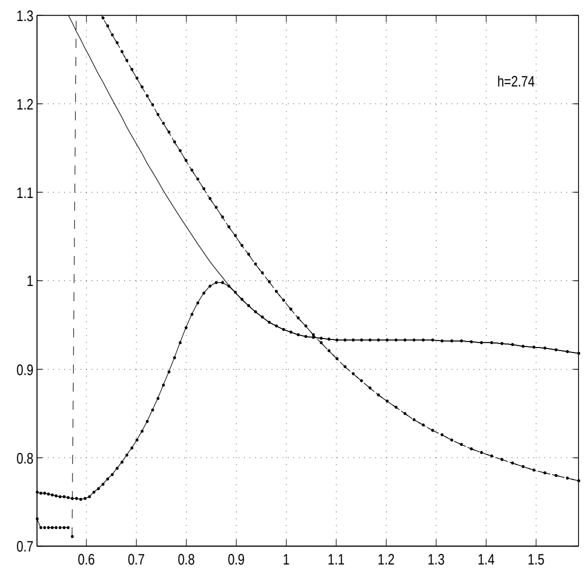


**Figure 3:** $\mu_\Gamma(M)$ (solid line), $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_\Gamma(P)$ (dashed line), and $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line) for h=2.74

By Theorem 3, the uncertain system is robustly stable if $h \leq 2.74$, since the minimum of the upper bounds of $\nu(\omega)$ is less than 1 for all $\omega$. Note that the stability is guaranteed by the results in [7] if $h < 2.61$, and by the results of [14] if $h < 0.72$.

## 4 Conclusion

This paper extends $\mu$ analysis for block structured uncertainties with phase information case in [19] into general cases. The result is then applied to the robust stability and performance problem of uncertain time-delay systems. Examples show that the conservativeness of stability test is largely reduced by taking the phase information into consideration.

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