

Adaptive λ -tracking for Nonlinear Systems with Higher Relative Degree

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The paper is organized as follows. After stating the system class and explaining the structure of the controller in Section 2, the theory and an outline of the proof are presented in Section 3.

Abstract

Previous results for adaptive λ -tracker have focused on nonlinear systems with relative degree one or on linear systems with higher relative degree. In this paper we extend the adaptive λ -tracker to λ -stabilize nonlinear system with higher relative degree. Only little structural information about the system to be controlled is needed. λ -stability and convergence of the adaptation is proven for tracking a large class of reference trajectories. The design of the controller is very simple and intuitive, only few parameters have to be tuned.

1 Introduction

Adaptive λ -tracking is a relatively recent controller design method. Nevertheless, there are already several successful applications. The concept behind adaptive λ -tracking is as follows. To increase the robustness, especially in the presence of output noise, a dead-zone (of width λ) in the gain adaptation has been introduced in [6]. This approach is usually called λ -stabilization or λ -tracking as the objective is to control the output or the tracking error no longer to zero but to a λ -neighborhood of zero. Thus, an output error smaller than the width of the dead-zone does not increase the adaptation parameter. In [5, 9] an adaptive λ -tracking controller is presented for systems of relative degree one.

In this paper we show that the adaptive λ -tracker proposed in [3] for linear systems can be extended to λ -stabilize a large class of nonlinear systems with an arbitrary, but known relative degree. In [11], a λ -tracking controller for nonlinear systems with higher relative degree is derived via backstepping. Both the controller design and the resulting controller are significantly more complicated.

2 Setup

System class

We consider nonlinear, single-input single-output systems that are affine in the input:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot u, \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad (1a)$$

$$y = h(\mathbf{x}), \quad (1b)$$

and satisfy the following assumptions.

A1) The relative degree r is known, i.e.

$$\begin{aligned} L_{\mathbf{g}} L_{\mathbf{f}}^i h(\mathbf{x}) &\equiv 0, \quad i = 0, \dots, r-2, \quad \forall \mathbf{x} \in \mathbb{R}^n \\ L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} h(\mathbf{x}) &\geq \underline{g}, \quad \forall \mathbf{x} \in \mathbb{R}^n \end{aligned} \quad (2)$$

and a positive lower bound of the high frequency gain, $\underline{g} > 0$, needs to be known.

A2) There exists a coordinate transformation $T : \mathbf{x} \mapsto [\boldsymbol{\xi}^T, \boldsymbol{\eta}^T]^T$ bringing (1) into input-normalized Byrnes-Isidori normal form (3):

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{J}\boldsymbol{\xi} + \mathbf{b}(\boldsymbol{\psi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{\phi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\eta} \\ &\quad + \mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\eta})u + \mathbf{v}(\boldsymbol{\xi}, \boldsymbol{\eta})) \end{aligned} \quad (3a)$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(\boldsymbol{\xi}, \boldsymbol{\eta})y + \mathbf{H}(\boldsymbol{\eta}) + \mathbf{w}(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad (3b)$$

$$y = \mathbf{c}^T \boldsymbol{\xi}, \quad (3c)$$

with $\boldsymbol{\xi}(t) \in \mathbb{R}^r$, $\boldsymbol{\eta}(t) \in \mathbb{R}^{n-r}$ and $(\mathbf{J}, \mathbf{b}, \mathbf{c}^T)$ an r -dimensional prime triple [7].

A3) The dynamics $\dot{\boldsymbol{\eta}} = \mathbf{H}(\boldsymbol{\eta})$ satisfy global exponential stability, i.e. that $\mathbf{H} \in C^2$, $\mathbf{H}(\mathbf{0}) = \mathbf{0}$ and there exist constants $\mu, \delta, \lambda > 0$ such that $\|\boldsymbol{\eta}(t)\| \leq \mu \|\boldsymbol{\eta}\| \exp(-\delta(\tau - t)) \forall \tau \geq t \geq 0, \forall \boldsymbol{\eta} \in \mathbb{R}^n, \frac{\partial}{\partial \boldsymbol{\eta}} \mathbf{H}(\boldsymbol{\eta}) \leq \mu \|\boldsymbol{\eta}\|$.

A4) The following functions are bounded.

$$\underline{g} \leq g(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)),$$

$$\begin{aligned} v &\in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^r), & \mathbf{w} &\in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n-r}), \\ H &\in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n}), & \boldsymbol{\chi} &\in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n-r}), \\ \phi &\in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n-r}), & \boldsymbol{\psi} &\in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^r). \end{aligned}$$

Remark 1 The zero-dynamics of (3) are

$$\dot{\boldsymbol{\eta}} = H(\boldsymbol{\eta}) + \mathbf{w}(\mathbf{0}, \boldsymbol{\eta}).$$

As $w(\cdot)$ is bounded, $\dot{\boldsymbol{\eta}} = H(\boldsymbol{\eta})$ is a kind of unperturbed zero-dynamics. Assumption A3 guarantees the existence of a C^2 Lyapunov function for the unperturbed zero dynamics, see [10, Section 5.7]. \diamond

Remark 2 The assumptions A1 to A4 are not very restrictive and essentially the same as for the relative degree one case. \diamond

Objective

The control objective is to asymptotically track a reference signal $y_{ref}(\cdot)$ while tolerating a tracking error smaller than a user-defined λ , i.e.

$$\lim_{t \rightarrow \infty} |y(t) - y_{ref}(t)| \leq \lambda.$$

All states should remain bounded, i.e. $\mathbf{x} \in \mathcal{L}_\infty([0, \infty))$. The reference signal $y_{ref}(\cdot)$ is considered to be in $W^{r, \infty}$, the set of all bounded functions that are absolutely continuous on compact subintervals and whose r first derivatives are essentially bounded. This set is very broad.

For this problem an adaptive output-feedback controller is designed in the state-space. It consists of an adaptive high-gain observer and an adaptive high-gain observer-state feedback controller, described in the following.

Observer

The observer is an adaptive version of the high-gain observer introduced by Nicosia and Tornambè [8] as described in [4]. The observer is given by

$$\dot{\hat{\mathbf{x}}} = \hat{A}_{\hat{\kappa}} \hat{\mathbf{x}} + \mathbf{p}_{\hat{\kappa}} e \quad (4a)$$

$$e = y - y_{ref} \quad (4b)$$

with $\hat{\mathbf{x}}(t) \in \mathbb{R}^r$ and

$$\hat{A}_{\hat{\kappa}} = \begin{bmatrix} -p_{r-1} \cdot \hat{\kappa} & 1 & 0 & & \\ -p_{r-2} \cdot \hat{\kappa}^2 & 0 & 1 & & \\ \vdots & & & \ddots & \\ -p_1 \cdot \hat{\kappa}^{r-1} & 0 & 0 & & 1 \\ -p_0 \cdot \hat{\kappa}^r & 0 & 0 & & 0 \end{bmatrix}, \quad \mathbf{p}_{\hat{\kappa}} = \begin{bmatrix} p_{r-1} \cdot \hat{\kappa} \\ p_{r-2} \cdot \hat{\kappa}^2 \\ \vdots \\ p_1 \cdot \hat{\kappa}^{r-1} \\ p_0 \cdot \hat{\kappa}^r \end{bmatrix},$$

Thus, $\hat{A}_{\hat{\kappa}} = J - \mathbf{p}_{\hat{\kappa}} \mathbf{c}^T$. The parameters p_i are chosen such that $p(s) = s^r + \sum_{i=0}^{r-1} p_i s^i$ is Hurwitz. For any positive value of the observer gain $\hat{\kappa}$, the spectrum of

$\hat{A}_{\hat{\kappa}}$ lies in the open left half plane ($\sigma(\hat{A}_{\hat{\kappa}}) \subset \mathbb{C}_-$) and the observer dynamics are stable. No further knowledge of the system besides that of the relative degree is needed for the observer design. The observer gain $\hat{\kappa}$ is adapted according to the adaptation law described below.

Controller

The controller is an observer-state feedback

$$u = -\mathbf{q}_{\kappa} \hat{\mathbf{x}}, \quad (5)$$

where $\mathbf{q}_{\kappa} = [q_0 \cdot \kappa^r, \dots, q_{r-1} \cdot \kappa]$. The parameters q_i are chosen such that $q_{\underline{g}}(s) = s^r + g \sum_{i=0}^{r-1} q_i s^i$ is Hurwitz for all $g \geq \underline{g}$ which is always solvable, see [2]. Then, for any positive values of the controller gain κ and any $g \geq \underline{g}$, the spectrum of $J - g\mathbf{b}\mathbf{q}_{\kappa}$ lies in the open left half plane. Only the relative degree r and a lower bound of the high-frequency gain \underline{g} are needed for the controller design. The adaptation law for the controller gain κ is described below.

Gain Adaptation

The adaptation for the observer gain $\hat{\kappa}$ and the controller gain κ is chosen such that the gains are increased as long as the amplitude of the tracking error e is larger than the user-defined bound λ (the control objective).

The observer and controller gains are defined by

$$\hat{\kappa} = k^\alpha, \quad \kappa = k^\beta \quad (6)$$

where the parameters α and β have to satisfy that

$$\alpha > \beta > 0. \quad (7)$$

With $\lambda > 0$, $\gamma > 0$ and $k(0) = k_0 > 0$ the adaptation parameter k is computed according to

$$\dot{k} = \frac{\gamma^2}{k^{2\gamma}} \begin{cases} (|e| - \lambda)^2 & \text{for } |e| > \lambda, \\ 0 & \text{for } |e| \leq \lambda, \end{cases} \quad (8)$$

where $\tilde{\gamma}$ has to satisfy

$$\tilde{\gamma} > 2\alpha\epsilon + 2(\alpha - \beta)(r - 1), \quad \text{for } r > 1, \quad (9a)$$

$$\tilde{\gamma} \geq 2\alpha\epsilon \quad \text{for } r = 1. \quad (9b)$$

The parameter ϵ is a positive constant and will be defined in Section 3.

Remark 3 The parameters α and β can be used to tune individually the “gains” of the observer and the controller, respectively. \diamond

Remark 4 This adaptation law ensures a monotonic increase of the observer and controller gains and that the observer gain $\hat{\kappa}$ grows faster than the controller gain κ for large k 's. \diamond

Remark 5 Systems with negative high frequency gain, i.e. $\mathbf{c}A^{r-1}\mathbf{b} = g \leq \underline{g} < 0$ instead of (2), can easily be treated by changing the control law (5) to

$$u = +\mathbf{q}_{\kappa} \hat{\mathbf{x}}. \quad \diamond$$

Remark 6 The adaptive λ -tracker consisting of the observer (4), the adaptive controller (5) and the adaptation law (8) with (6), (7) and (9) is exactly the same as the adaptive λ -tracker for linear systems with higher relative degree [3]. \diamond

3 Results

The main result of this paper is the proof that combining the adaptive observer (4) with the adaptive controller (5) and using the adaptation law (8) with (6), (7) and (9) to close the loop for an arbitrary system of class (1), satisfying Assumptions A1 to A4 yields that the tracking error asymptotically converges to the λ -strip. Furthermore, the adaption converges, no finite escape time can occur and all states remain bounded. This will be stated in Theorem 9.

First, we need the following definition of a subset of the Hurwitz polynomials.

Definition 1 A polynomial $p(s) = s^r + g \sum_{i=0}^{r-1} s^i p_i$ is in the set $H(\epsilon, \mu)$ if there exists a symmetric, positive definite matrix P such that the companion matrix

$$A_c = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -gp_0 & \dots & \dots & -gp_{r-1} \end{bmatrix}$$

satisfies for $\Delta = \text{diag}\{0, 1, \dots, r-1\}$ the inequalities

$$A_c^T \cdot P + P \cdot A_c \leq -2\mu P \quad (10a)$$

$$(\Delta + \epsilon I) \cdot P + P \cdot (\Delta + \epsilon I) \geq 0. \quad (10b)$$

Remark 7 By (10a) $H(\epsilon, \mu)$ is a subset of the Hurwitz polynomials. It can easily be shown that if $p(s) \in H(\underline{\epsilon}, \bar{\mu})$ then

$$p(s) \in H(\epsilon, \mu) \text{ for all } \epsilon \geq \underline{\epsilon} \text{ and for all } \mu \leq \bar{\mu}. \quad \diamond$$

Remark 8 For systems with relative degree one, $\Delta = 0$. In that case, $\epsilon = 0$ and $\tilde{\gamma} = 0$ are valid choices. \diamond

We now state the main theorem.

Theorem 9 The application of the λ -tracker (4)–(9) and polynomials $p(s)$ and $q_g(s)$ in $H(\epsilon, \mu)$ for all $g \geq \underline{g}$, to any stabilizable system of the class (1) satisfying Assumptions A1 to A4 with any reference signal $y_{ref}(\cdot) \in W^{r, \infty}$ results in a closed-loop system which, independently of the initial values $\mathbf{x}(0) \in \mathbb{R}^n$, $\hat{\mathbf{x}}(0) \in \mathbb{R}^r$ and $k(0) > 0$ has a unique solution which exists on the whole half axis $t \in [0, \infty)$ and, moreover,

- a) $(\mathbf{x}(\cdot), \hat{\mathbf{x}}(\cdot), k(\cdot)) \in \mathcal{L}_\infty([0, \omega))$,
- b) $\lim_{t \rightarrow \infty} |y(t) - y_{ref}(t)| \leq \lambda$.

Remark 10 Theorem 9 states that for any nonlinear system satisfying Assumptions A1 to A4 with known relative degree r and lower bound \underline{g} of the high-frequency gain, a λ -tracking controller can be designed that guarantees that all states and adaptation parameters remain bounded and that the tracking error $y - y_{ref}$ asymptotically converges to the λ -strip. The width of this strip is a parameter which can be chosen by the user and does usually depend on the specifications, on model uncertainties and on the quality of the measurement. \diamond

Remark 11 The motivation for such an observer-based controller comes from the fact that for fixed adaptation parameters κ and $\hat{\kappa}$ the transfer function from y to $\hat{\mathbf{x}}_i$, the i -th state of the observer, is at “low” frequencies, i.e. for small $(\frac{s}{\kappa})$, approximately a series of differentiators, $\hat{\mathbf{x}}_i(s) \approx s^{i-1} y(s)$ with a band-width proportional to the observer gain. Thus, the observer states approximate the derivatives of the output y . Therefore, the controller can be seen as an approximation of a PD...D $^{r-1}$ controller:

$$u(s) \approx -\kappa^r \sum_{i=0}^{r-1} q_i \left(\frac{s}{\kappa}\right)^i y(s). \quad \diamond$$

Proof (of Theorem 9)

Outline of the Proof. We will prove Theorem 9 in five steps. First, it is shown that k remains bounded on the maximal time interval where all states remain bounded. Then it is shown in the second part that the observer states $\hat{\mathbf{x}}$ and thus the plant input u are bounded. In part three boundedness of the plant states \mathbf{x} is shown. In step four, it is proven that the solution of the differential equations exists for all times and thus that the maximal time interval of existence of the solution is infinite. A consequence of the first and fourth step is that the adaptation parameter k and, by that, also κ and $\hat{\kappa}$ converge. The proof concludes by showing that the tracking error converges to the λ -strip.

1) Boundedness of the adaption parameters.

Since this part of the proof is rather tedious, a short summary is given. First, the closed loop is transformed into a coordinate system with states for the tracking and the observer error and their time-derivatives ($\bar{\mathbf{x}}$ -coordinates). Then a sort of k -dependent time scaling is applied ($\bar{\bar{\mathbf{x}}}$ -coordinates). In the resulting coordinates it is possible to define a Lyapunov-like function V in such a way that it can be used to bound \dot{k} : $\dot{k} \leq \bar{M}V(\bar{\mathbf{x}}, k)$ for some $\bar{M} > 0$. The boundedness of k is then shown by contradiction: We assume that k grows to infinity. By upper bounding the derivative of V along closed loop trajectories, we get that $\dot{V} \leq -2\tilde{\mu}V = \dot{V}_{max}$ for some $\tilde{\mu} > 0$. An upper bound for V can be derived by integrating \dot{V}_{max} . This bound is then used together with $\dot{k} \leq \bar{M}V(\bar{\mathbf{x}}, k)$ to show that k cannot grow to infinity. Therefore, k has to remain bounded.

Due to lack of space several details of the proof are skipped. The complete proof can be found in [2].

The nonlinear closed-loop system is given by (3), (4), (5), (8) and takes the form

$$\dot{\boldsymbol{\xi}} = (J + \mathbf{b}\boldsymbol{\psi}^T(\cdot))\boldsymbol{\xi} + \mathbf{b}(\boldsymbol{\phi}^T(\cdot)\boldsymbol{\eta} + g(\cdot)u + v(\cdot)) \quad (11a)$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(\boldsymbol{\xi}, \boldsymbol{\eta})y + H(\boldsymbol{\eta}) + \mathbf{w}(\boldsymbol{\xi}, \boldsymbol{\eta}), \quad (11b)$$

$$\dot{\hat{\mathbf{x}}} = \hat{A}_{\hat{\kappa}}\hat{\mathbf{x}} + \mathbf{p}_{\hat{\kappa}}e, \quad (11c)$$

$$\dot{k} = d_\lambda(e, k)^2, \quad (11d)$$

$$e = \mathbf{c}^T\boldsymbol{\xi} - y_{ref} \quad (11e)$$

with

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \in \mathbb{R}^r, \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0 \in \mathbb{R}^{n-r}$$

$$\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \in \mathbb{R}^r, k(0) = k_0 > 0.$$

For the reference signal and its derivatives, the notation $\hat{\mathbf{y}}_{ref}^T = [y_{ref}, \dot{y}_{ref}, \dots, y_{ref}^{(r-1)}]$ is used.

We introduce the coordinates

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \boldsymbol{\eta} \\ \hat{e} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi} - \hat{\mathbf{y}}_{ref} \\ \boldsymbol{\eta} \\ \bar{x}_1 - \hat{\mathbf{x}} \end{bmatrix},$$

where $\hat{\boldsymbol{\xi}}^T = [e \ \dot{e} \ \dots \ e^{(r-1)}]^T$ denotes the tracking error and its derivatives, and \hat{e} is the observer error.

From now on we write $\hat{\kappa} = k^\alpha$ and $\kappa = k^\beta$. The matrices $J - g(\cdot)\mathbf{b}\mathbf{q}_\kappa^T$ and $\hat{A}_{\hat{\kappa}} = J - \mathbf{c}^T\hat{\mathbf{b}}_{\hat{\kappa}}$ have a special form. They can both be factored with the help of the matrix

$$K_r = \text{diag}\{1, k, \dots, k^{r-1}\}$$

as

$$J - g(\cdot)\mathbf{b}\mathbf{q}_\kappa^T = k^\beta K_r^\beta \bar{A}_{11} K_r^{-\beta}$$

$$\hat{A}_{\hat{\kappa}} = J - \mathbf{p}_{\hat{\kappa}}\mathbf{c}^T = k^\alpha K_r^\alpha \bar{A}_{33} K_r^{-\alpha}$$

with

$$\bar{A}_{11} = J - g(\cdot)\mathbf{b}\mathbf{q}^T \text{ with } \mathbf{q} = \mathbf{q}_\kappa|_{\kappa=1},$$

$$\bar{A}_{33} = J - \mathbf{p}\mathbf{c}^T \text{ with } \mathbf{p} = \hat{\mathbf{b}}_{\hat{\kappa}}|_{\kappa=1}.$$

Define new coordinates $\bar{\bar{\mathbf{x}}}$ via a gain-dependent transformation

$$\bar{\bar{\mathbf{x}}} = \bar{C}\bar{K}^{-1}\bar{\mathbf{x}}, \quad (12)$$

where

$$\bar{C} = \text{diag}\{c_1 I_r, c_2 I_m, c_3 I_r\},$$

$$\bar{K} = \text{diag}\{K_r^\beta, I_m, K_r^\alpha\}.$$

The matrix \bar{K}^{-1} can be seen as a k-dependent time scaling. The coefficients c_1, c_2 and c_3 are chosen as $c_i = k^{-\tilde{c}_i}$ for $i = 1, 2, 3$ where

$$\tilde{c}_1 = 2(r-1)(\alpha - \beta) + 2\alpha\epsilon,$$

$$\tilde{c}_2 = 2(r-1)(\alpha - \beta) + \beta + 2\alpha\epsilon,$$

$$\tilde{c}_3 = (r-1)(\alpha - \beta) + 2\alpha\epsilon.$$

As $r \geq 1, \alpha > \beta > 0$ and $\epsilon \geq 0$, the parameters \tilde{c}_1, \tilde{c}_2 and \tilde{c}_3 are non-negative.

The time derivative of the coordinate transformation matrices \bar{C} and \bar{K}^{-1} is

$$\frac{d}{dt}\bar{C} = \text{diag}\left\{-\frac{\dot{k}}{k}\tilde{c}_i\right\}, \quad \frac{d}{dt}\bar{K}^{-1} = -\frac{\dot{k}}{k}\Delta$$

with $\Delta = \text{diag}\{0, 1, \dots, r-1\}$.

In the coordinates $\bar{\bar{\mathbf{x}}}$, the closed-loop differential equations are

$$\dot{\bar{\bar{\mathbf{x}}}} = \begin{bmatrix} k^\beta \bar{A}_{11} \bar{\bar{\mathbf{x}}}_1 \\ \bar{H}(\bar{\bar{\mathbf{x}}}_2) \\ k^\alpha \bar{A}_{33} \bar{\bar{\mathbf{x}}}_3 \end{bmatrix} + \bar{E}\bar{\bar{\mathbf{x}}} + \bar{V} - \frac{\dot{k}}{k}\boldsymbol{\Psi}\bar{\bar{\mathbf{x}}}, \quad (13a)$$

$$e = c_1^{-1}\mathbf{c}^T\bar{\bar{\mathbf{x}}}_1, \quad (13b)$$

where \bar{E} and \bar{V} are bounded functions in $\mathbb{R}^{n+r \times n+r}$ and \mathbb{R}^{n+r} , respectively. By assumption, the matrices \bar{A}_{11} and \bar{A}_{33} are such that their characteristic polynomials are of class $H(\epsilon, \mu)$. Therefore, there exist symmetric, positive definite solutions P_1 and P_3 of the Lyapunov inequalities

$$\bar{A}_{ii}^T P_i + P_i \bar{A}_{ii} \leq -\mu P_i, \quad i = 1, 2, 3, \quad (14a)$$

$$P_i(\Delta + \epsilon I) + (\Delta + \epsilon I)P_i \geq 0, \quad i = 1, 3. \quad (14b)$$

The functions $\bar{\bar{\mathbf{x}}}_1 \mapsto V_1(\bar{\bar{\mathbf{x}}}_1) = \bar{\bar{\mathbf{x}}}_1^T P_1 \bar{\bar{\mathbf{x}}}_1$ and $\bar{\bar{\mathbf{x}}}_3 \mapsto V_3(\bar{\bar{\mathbf{x}}}_3) = \bar{\bar{\mathbf{x}}}_3^T P_3 \bar{\bar{\mathbf{x}}}_3$ will be used as a sort of Lyapunov function candidates for $\bar{\bar{\mathbf{x}}}_1$ and $\bar{\bar{\mathbf{x}}}_3$, respectively. The zero-dynamics are globally exponentially stable by Assumption A3. Thus, there exists a Lyapunov function candidate $V_2(\bar{\bar{\mathbf{x}}}_2)$ satisfying for some positive constants m_1, \dots, m_4 .

$$m_1 \|\bar{\bar{\mathbf{x}}}_2\|^2 \leq V_2(\bar{\bar{\mathbf{x}}}_2) \leq m_2 \|\bar{\bar{\mathbf{x}}}_2\|^2 \quad (15a)$$

$$\frac{\partial}{\partial \bar{\bar{\mathbf{x}}}_2} V_2(\bar{\bar{\mathbf{x}}}_2) \bar{H}_2(\bar{\bar{\mathbf{x}}}_2) \leq -m_3 \|\bar{\bar{\mathbf{x}}}_2\|^2 \quad (15b)$$

$$\left\| \frac{\partial}{\partial \bar{\bar{\mathbf{x}}}_2} V_2(\bar{\bar{\mathbf{x}}}_2) \right\| \leq m_4 \|\bar{\bar{\mathbf{x}}}_2\|. \quad (15c)$$

Now, the Lyapunov function candidates $V_i(\cdot)$ are combined to a single one, $V(\cdot)$.

$$V(\bar{\bar{\mathbf{x}}}, k) = \frac{1}{2} D(\bar{\bar{\mathbf{x}}}, k)^2 \quad (16a)$$

$$D(\bar{\bar{\mathbf{x}}}, k) = \begin{cases} \nu(\bar{\bar{\mathbf{x}}}) - \rho(k), & \text{if } \nu(\bar{\bar{\mathbf{x}}}) \geq \rho(k) \\ 0, & \text{if } \nu(\bar{\bar{\mathbf{x}}}) < \rho(k) \end{cases} \quad (16b)$$

(16c)

where

$$\begin{aligned} \nu(\bar{\mathbf{x}}) &= \sqrt{V_1(\bar{\mathbf{x}}_1) + V_2(\bar{\mathbf{x}}_2) + V_3(\bar{\mathbf{x}}_3)} \\ \rho(k) &= \frac{\lambda}{2} \frac{c_1(k)}{\sqrt{\|P_1^{-1}\|}}. \end{aligned} \quad (17)$$

The k -dependent parameter ρ has been chosen in such a way that

$$\nu(\bar{\mathbf{x}}) \leq 2\rho(k) \Rightarrow |e| \leq \lambda \Rightarrow \dot{k} = 0. \quad (18)$$

To see this, combine (13b), (12) and (16) to

$$|e| \leq \frac{\sqrt{\|P_1^{-1}\|}}{c_1(k)} \sqrt{2V(\bar{\mathbf{x}}, k)} + \frac{\lambda}{2}. \quad (19)$$

Since

$$\nu(\bar{\mathbf{x}}) \leq 2\rho(k) \Leftrightarrow V(\bar{\mathbf{x}}, k) \leq \frac{1}{2}\rho(k)^2, \quad (20)$$

(17) and (19) yield $\nu(\bar{\mathbf{x}}) \leq 2\rho(k) \Rightarrow |e| \leq \lambda$, which is in the dead-zone of the gain adaptation, implying that $\nu(\bar{\mathbf{x}}) \leq 2\rho(k) \Rightarrow \dot{k} = 0$.

The function $V(\bar{\mathbf{x}}, k)$ will be used to upper bound \dot{k} . From (8), the definition of the adaptation, it holds that

$$\dot{k} \leq \frac{\gamma^2}{k^{2\bar{\gamma}}} (|e| - \lambda)^2. \quad (21)$$

From (19) it follows that

$$(|e| - \lambda)^2 \leq \frac{\|P_1^{-1}\|}{c_1^2(k)} 2V(\bar{\mathbf{x}}, k).$$

Combining this with (21) yields that

$$\dot{k} \leq \frac{\gamma^2}{k^{2\bar{\gamma}}} \frac{\|P_1^{-1}\|}{c_1^2(k)} 2V(\bar{\mathbf{x}}, k).$$

Using (9) yields for some $\bar{M} > 0$

$$\dot{k} \leq \bar{M}V(\bar{\mathbf{x}}, k). \quad (22)$$

(22) is the first key inequality of Step 1 of the Proof.

From now on the k -dependency of $V(\cdot)$, $D(\cdot)$ and $\rho(\cdot)$, will be skipped to increase the readability. From the theory of ordinary differential equations it follows that the initial value problem (11) possesses an absolutely continuous solution $(\bar{\mathbf{x}}(\cdot), k(\cdot)) : [0, \omega) \rightarrow \mathbb{R}^{n+1}$, maximally extended over $[0, \omega)$ for some $\omega \in (0, \infty)$.

The derivative of V along the trajectory of the system (11) (denoted for ease of exposition by $\frac{d}{dt}$) is for all $t \in [0, \omega)$ and for all values of $\bar{\mathbf{x}}$

$$\frac{d}{dt}V(\bar{\mathbf{x}}) = D(\bar{\mathbf{x}})\frac{d}{dt}D(\bar{\mathbf{x}}) = D(\bar{\mathbf{x}}) \left(\frac{d}{dt}\nu(\bar{\mathbf{x}}) - \frac{d}{dt}\rho(k) \right).$$

Straightforward calculation using completion of squares yield that there exist constants $t_1 \geq 0$ and $\bar{\mu} > 0$ such that for all $\bar{\mathbf{x}}$ and for almost all $t \in [t_1, \omega)$ that

$$\frac{d}{dt}V(\bar{\mathbf{x}}) \leq -\bar{\mu}D(\bar{\mathbf{x}})^2 = -2\bar{\mu}V(\bar{\mathbf{x}})^2.$$

Therefore, for all $t \in [t_1, \omega)$,

$$V(\bar{\mathbf{x}}(t), k(t)) \leq e^{-2\bar{\mu}(t-t_1)} \cdot V(\bar{\mathbf{x}}(t_1), k(t_1)). \quad (23)$$

Inequality (23) is the second key inequality of this part of the Proof. If $\omega < \infty$, then (22) and (23) yield that $k(\cdot) \in \mathcal{L}_\infty([0, \omega))$. If $\omega = \infty$, then by (23), V enters in finite time the interval $[0, \frac{\lambda^2}{2}]$ which by (18) and (20) implies that $|e| \leq \lambda$. Whence, the dead-zone in the gain adaptation (8) yields that $k(\cdot) \in \mathcal{L}_\infty([0, \omega))$. This contradicts unboundedness of $k(\cdot)$, thus proving boundedness of $k(\cdot)$.

2) Boundedness of the observer states. As $k(\cdot)$ is bounded, $d_\lambda(\cdot) \in \mathcal{L}_2([0, \omega))$. From this, (8) and the Hölder inequality, it follows that

$$\gamma^{-1}k^{\bar{\gamma}}(\cdot)d_\lambda(\cdot) \in \mathcal{L}_2([0, \omega)). \quad (24)$$

From (8) it follows that

$$\gamma^{-1}k^{\bar{\gamma}}d_\lambda(e, k) = \begin{cases} |e| - \lambda & \text{for } |e| \geq \lambda, \\ 0 & \text{for } |e| \leq \lambda. \end{cases}$$

Therefore,

$$|e(\cdot)| - \gamma^{-1}k^{\bar{\gamma}}(\cdot)d_\lambda(e(\cdot), k(\cdot)) \in \mathcal{L}_\infty([0, \omega)).$$

Combining this with (24) yields

$$|e(\cdot)| = \underbrace{|e(\cdot)| - \gamma^{-1}k^{\bar{\gamma}}(\cdot)d_\lambda(\cdot)}_{\in \mathcal{L}_\infty([0, \omega))} + \underbrace{\gamma^{-1}k^{\bar{\gamma}}(\cdot)d_\lambda(\cdot)}_{\in \mathcal{L}_2([0, \omega))}.$$

Thus, $e(\cdot) \in \mathcal{L}_\infty([0, \omega))$ and $y(\cdot) \in \mathcal{L}_\infty([0, \omega))$.

By boundedness of k there exist a k_ω such that

$$k(t) \leq k_\omega \text{ for all } t \in [0, \omega).$$

Defining $\hat{A} = \hat{A}_{\hat{k}=k_\omega}$, $\hat{A}_1 = \hat{A}_{\hat{k}} - \hat{A}$, (11c) is equivalent to

$$\dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{A}_1\hat{\mathbf{x}} + \hat{\mathbf{b}}e.$$

\hat{A} is Hurwitz, $\|\hat{A}_1\|$ decreases monotonically to zero and $\|\hat{\mathbf{b}}\|$ is bounded. Therefore, it follows from variation of constants that $\hat{\mathbf{x}}$ is bounded, i.e.

$$\hat{\mathbf{x}}(\cdot) \in \mathcal{L}_\infty^r([0, \omega)).$$

As $u = -\mathbf{q}_k\hat{\mathbf{x}}$, it follows that also $u(\cdot) \in \mathcal{L}_\infty([0, \omega))$.

3) Boundedness of the states of the plant. We first take a closer look at the internal dynamics of (3). Using variation of constants, it follows that

$$\eta(\cdot) \in \mathcal{L}_{\infty}^{n-r}([0, \omega]).$$

In a second step we look at the remaining states, i.e. at ξ , which satisfy $\dot{\xi} = (J + b\psi^T(\cdot))\xi + b\tilde{v}$, where $\tilde{v} \in \mathcal{L}_{\infty}([0, \omega])$. By induction, boundedness of ξ is shown.

$$i = 1: y = \xi_1, \text{ therefore } \xi_1 \in \mathcal{L}_{\infty}([0, \infty));$$

$$i \leq 1: \dot{\xi}_i = \xi_{i+1}, \text{ therefore } \xi_{i+1} \in \mathcal{L}_{\infty}([0, \infty)), \text{ for all } 1 \leq i \leq r.$$

Thus, $\xi(\cdot) \in \mathcal{L}_{\infty}^r([0, \omega])$.

4) Global existence of a unique solution. As k , x and \hat{x} are bounded on $[0, \omega)$, it follows by maximality of ω that $\omega = \infty$.

5) Convergence of the tracking error. It remains to show b). For this we prove that $\lim_{t \rightarrow \infty} d_{\lambda}(e(t), k(t)) = 0$. Since $e(\cdot)$ and $k(\cdot)$ are bounded, $\dot{k}(\cdot) = d_{\lambda}(\cdot)^2 \in \mathcal{L}_{\infty}([0, \omega])$. From $\dot{e} = c[Ax - b\mathbf{q}_k \hat{x}] - \dot{y}_{ref}$ and using the boundedness of $x(\cdot)$ and $\hat{x}(\cdot)$ we conclude that $\dot{e}(\cdot) \in \mathcal{L}_{\infty}([0, \omega])$. As

$$\frac{d}{dt}d_{\lambda}^2 = 2d_{\lambda} \left(\frac{e\dot{e}(\cdot)}{|e|} - \tilde{\gamma} \frac{\dot{k}}{k} d_{\lambda} \right) \in \mathcal{L}_{\infty}([0, \omega]),$$

$d_{\lambda}(\cdot)^2$ is uniformly continuous. This, together with $d_{\lambda}(\cdot)^2 \in \mathcal{L}_1([0, \infty))$, yields, by Barbălat's Lemma [1] that $\lim_{t \rightarrow \infty} d_{\lambda}(t)^2 = 0$.

This completes the proof. ■

4 Conclusions

The paper extends adaptive λ -tracking to nonlinear systems with higher relative degree. Previous results for adaptive λ -tracker have focused on nonlinear systems with relative degree one and on linear systems with higher relative degree. By including dynamics in the controller and slightly modifying the adaptation law, the same properties can be achieved as in the relative degree one case. Stability and convergence of the adaptation is proven for tracking a large class of reference trajectories. The design of the controller is very simple and intuitive, only few parameters have to be tuned.

An important drawback of the proposed controller is that the adaptation only increases the observer and controller gains. It is possible to include a gain decrease in the adaptation, for example by adding a sigma-modification or by decreasing the gain whenever the

tracking error is inside the λ -strip [2]. The disadvantage of these modifications is that it is not possible to guarantee the convergence of the tracking error any more.

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