On Geometry of Control Systems Equivalent to Canonical Contact Systems: Regular Points, Singular Points, and Flatness

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Abstract

We give necessary and sufficient geometric conditions for a nonlinear control system to be feedback equivalent to a canonical contact system for curves. We study the geometry of that class of systems, in particular, the existence of involutive subdistributions of corank one. We also distinguish regular points, at which the system is equivalent to the canonical contact system, and singular points. We show how the geometry of the studied class of systems is reflected by their flatness.

1 Introduction

Consider a nonholonomic control system, with $m+1$ controls, of the following form:

$$\Sigma: \quad \dot{x} = \sum_{i=0}^{m} f_i(x)\, u_i, \tag{1}$$

where $x(\cdot)$ belongs to $\mathbb{R}^N$, the controls u_i, for $0 \le i \le m$, take values in $\mathbb{R}$, and $f_0, \ldots, f_m$ are smooth vector fields on $\mathbb{R}^N$. We call the control system (1) a *canonical contact system for curves* or, equivalentely, a *canonical contact system on* $J^n(\mathbb{R}, \mathbb{R}^m)$ if we have

$$\begin{array}{rcl} f_1 & = & \frac{\partial}{\partial x_1^n}\,, \ldots,\ f_m \;=\; \frac{\partial}{\partial x_m^n} \\ f_0 & = & \frac{\partial}{\partial x_0^0} + \sum\limits_{j=1}^{m} \sum\limits_{i=0}^{n-1} x_j^{i+1} \frac{\partial}{\partial x_j^i}, \end{array} \tag{2}$$

where $x = (x_0^0, x_1^0, \ldots, x_1^n, \ldots, x_m^0, \ldots, x_m^n)$ are coordinates on $\mathbb{R}^N$ and $N = (n+1)m + 1$. The name follows from the fact that the distribution spanned by the canonical contact system (2) appears naturaly (see [3] and [24]) when describing those curves in $J^n(\mathbb{R}, \mathbb{R}^m)$, which denotes the space of n-jets of maps from $\mathbb{R}$ into $\mathbb{R}^m$ (that is, the space of n-jets of curves in $\mathbb{R}^m$), that are n-graphs of curves in $\mathbb{R}^m$. In [24] the canonical contact system is called the *Cartan distribution*.

In other words, the system (1) reads as

$$\begin{array}{lll} \dot{x}_0^0 = u_0 \quad & \dot{x}_1^0 = x_1^1 u_0 \quad \cdots & \dot{x}_m^0 = x_m^1 u_0 \\ & \quad\vdots & \quad\vdots \\ & \dot{x}_1^{n-1} = x_1^n u_0 & \dot{x}_m^{n-1} = x_m^n u_0 \\ & \dot{x}_1^n = u_1 & \dot{x}_m^n = u_m. \end{array}$$

If $m = 1$, the canonical contact system for curves is just Goursat normal form, called also chain form [3], [9], [10], [11], [12], [14], and [15]. For an arbitrary m, canonical contact systems for curves constitute a subclass of nonholonomic control system in extended Goursat normal form [13]; for the latter we admit the chains of integrators to be, in general, of different lenghts $n_1, \ldots, n_m$. For systems that are in extended Goursat normal form, or, more generaly, in a multi-chained form [14], many problems like motion planning, path tracking, and point stabilization have been succesfully solved (see e.g. [5], [8], [10], [14], [18], [20], [21], [22], [23], and the references given there).

Consider two control systems

$$\Sigma: \quad \dot{x} = \sum_{i=0}^{m} f_i(x)\, u_i = f(x)\, u \quad \text{and} \tag{3}$$

$$\tilde{\Sigma}: \quad \dot{\tilde{x}} = \sum_{i=0}^{m} \tilde{f}_i(\tilde{x})\, \tilde{u}_i = \tilde{f}(\tilde{x})\, \tilde{u}, \tag{4}$$

evolving respectively on two open sets X and $\tilde{X}$ of $\mathbb{R}^N$, where $u = (u_0, \ldots, u_m)^t$, $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_m)^t$, $f = (f_0, \ldots, f_m)^t$, and $\tilde{f} = (\tilde{f}_0, \ldots, \tilde{f}_m)^t$. We say that the systems Σ and $\tilde{\Sigma}$ are *feedback equivalent* if there exits a smooth diffeomorphism $\varphi : X \to \tilde{X}$ and a feedback $u = \beta \tilde{u}$, where the matrix β, whose entries β_j^i are smooth functions, is invertible at any $x \in X$, such that

$$\varphi_*(f\beta) = \tilde{f},$$

where any vector field g on X is transformed by a diffeomorphism φ into a vector field $\varphi_* g$ on $\tilde{X}$ according to

$$(\varphi_* g)(\tilde{x}) = \frac{\partial \varphi}{\partial x}(\varphi^{-1}(\tilde{x})) \cdot g(\varphi^{-1}(\tilde{x})).$$

We say that the systems Σ and $\tilde{\Sigma}$ are *locally feedback equivalent* if above φ is a local smooth diffeomorphism and β is defined locally.

To the system Σ we associate the distribution $\mathcal{D}$ spanned by the vector fields $f_0, \ldots, f_m$ which will be denoted by

$$\mathcal{D} = (f_0, \ldots, f_m).$$

Since the distribution $\mathcal{D}$ remains invariant under any invertible transformation $u = \beta \tilde{u}$, all objects that we construct with the help of $\mathcal{D}$ can be considered as feedback invariant objects attached to Σ.

In this paper we give geometric necessary and sufficient conditions for a nonlinear control system to be feedback equivalent to a canonical contact system for curves. For other approaches to a slightly more general problem of equivalence to extended Goursat normal form, we send the reader to [1], [6], [13], [17], [23].

In Section 2 we introduce geometric tools which allow us to give our necessary and sufficient conditions: derived flags, Lie flags, charactersitic distibutions, and, the most important, Engel rank. Theorem 2.1, characterizes control systems equivalent to canonical contact systems for curves via the two following conditions: each distribution $\mathcal{D}^{(i)}$ of the derived flag of (1) contains an involutive subdistribution of corank 1 and the elements $\mathcal{D}^{(i)}$ of the derived flag coincide with the corresponding elements $\mathcal{D}_i$ of the Lie flag. The first condition encodes the geometry of canonical contact systems for curves while the second one is a regularity condition. A natural problem which thus arises is whether a given distribution contains an involutive subdistribution of corank 1. We give a constructive solution to that problem in Section 3.

In Section 4 we study the property of flatness of control systems that are equivalent to canonical contact systems for curves. The concept of flatness was introduced by Fliess, Lévine, Martin, and Rouchon [4], [5] (see also [7] and [19]) in order to characterize control systems that are linearizable by endogenous feedback. Is is easy to observe that canonical contact systems for curves are flat (and thus so are their static feedaback equivalents). We argue that the geometry of canonical contact systems provides additional information concerning their flatness: it describes their flat outputs, defines a precompensator which dynamically linearizes the system, and indicates the singular control value at which the system ceases to be flat.

Finally, in Section 5 we illustrate our results by a simple nonholonomic system: a five dimensional model of a car for which both the front and rear wheels are steerable (see [23]). We describe regular points at which the model is equivalent to a canonical contact system and briefly discuss the nature of different singular configurations.

2 The Canonical Contact System for Curves

A rank k *distribution* $\mathcal{D}$ on a smooth manifold M is a map that assigns smoothly to each point p in M a linear subspace $\mathcal{D}(p) \subset T_pM$ of dimension k. Such a field of tangent k-planes is spanned locally by k pointwise linearly independent smooth vector fields $f_1, \ldots, f_k$ on M, which will be denoted by $\mathcal{D} = (f_1, \ldots, f_k)$. Two distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ defined on two manifolds M and $\tilde{M}$, respectively, are *equivalent* if there exists a smooth diffeomorphism φ between M and $\tilde{M}$ such that $(\varphi_* \mathcal{D})(\tilde{p}) = \tilde{\mathcal{D}}(\tilde{p})$, for each point $\tilde{p}$ in $\tilde{M}$. They are called *locally equivalent* if φ is a local diffeomorphism. Clearly, two distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are locally equivalent if and only if the corresponding control systems Σ and $\tilde{\Sigma}$, given respectively by (3) and (4) are locally feedback equivalent

The *derived flag* of a distribution $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \cdots$ defined inductively by

$$\mathcal{D}^{(0)} = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}], \quad \text{for } i \geq 0.$$

The *Lie flag* is the sequence of modules of vector fields $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by

$$\mathcal{D}_0 = \mathcal{D} \quad \text{and} \quad \mathcal{D}_{i+1} = \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad \text{for } i \geq 0.$$

In general, the derived and Lie flags are different; though for any point p in the underlying manifold the inclusion $\mathcal{D}_i(p) \subset \mathcal{D}^{(i)}(p)$ clearly holds, for $i \geq 0$. A distribution $\mathcal{D}$, defined on a manifold M, (or the corresponding control system) is said to be *completely nonholonomic* if, for each point p in M, there exists an integer $N(p)$ such that $\mathcal{D}_{N(p)}(p) = T_pM$. A distribution $\mathcal{D}$ is *involutive* if its first derived system satisfies $\mathcal{D}^{(1)} = \mathcal{D}^{(0)}$.

We give now our solution to the problem: "Which control systems are locally feedback equivalent to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$?". The answer will be given in two steps. The first step is the following result [16].

Theorem 2.1 *A control system of the form (1), defined on an open subset X of $\mathbb{R}^{m(n+1)+1}$, is feedback equivalent, in a small enough neighborhood of any point p in X, to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$ if and only if the distribution $\mathcal{D} = (f_0, \ldots, f_m)$ satisfies the two following conditions for $0 \le i \le n$.*

(i) *Each element $\mathcal{D}^{(i)}$ of the derived flag has constant rank $(i+1)m+1$ and contains an involutive subdistribution $\mathcal{L}_i \subset \mathcal{D}^{(i)}$ that has constant corank one in $\mathcal{D}^{(i)}$.*
(ii) *Each element $\mathcal{D}_i$ of the Lie flag has constant rank $(i+1)m+1$.*

A proof of this theorem is given in [16]. This result yields a constructive test for the local feedback equivalence to the canonical contact system for curves, provided that we know how to check whether or not a given distribution admits a corank one involutive subdistribution. The second step of our characterization of control systems equivalent to canonical contact systems for curves is to give, in the next Section, a checkable necessary and sufficient condition for the existence of such an involutive subdistribution.

3 Corank One Involutive Subdistributions

The aim of this Section is to give an answer to the following question: "When does a given constant rank distribution $\mathcal{D}$ contain an involutive subdistribution $\mathcal{L} \subset \mathcal{D}$ that has constant corank one in $\mathcal{D}$?". In fact, the answer to this question, together with a construction of $\mathcal{L}$, if it exists, is an immediate consequence of a result contained in Bryant's Ph.D. thesis [2]. Links between Bryant's result and the characterization of the canonical contact system for curves have also been observed by Aranda-Bricaire and Pomet [1].

A *characteristic vector field* of a distribution $\mathcal{D}$ is a vector field f that belongs to $\mathcal{D}$ and satisfies $[f, \mathcal{D}] \subset \mathcal{D}$. The *characteristic distribution* of $\mathcal{D}$, which will be denoted by $\mathcal{C}$, is the module spanned by all its characteristic vector fields. It follows directly from the Jacobi identity that the characteristic distribution is always involutive. We will denote $c_0(p) = \dim \mathcal{C}(p)$.

The *Engel rank* [3] of a distribution $\mathcal{D}$, at a point p, is the biggest integer ρ such that there exists a 1-form ω in $\mathcal{D}^\perp$ for which we have

$$(d\omega)^\rho(p) \neq 0 \operatorname{mod} \mathcal{D}^\perp,$$

where $\mathcal{D}^\perp$ denotes the Pfaffian system that annihilates the distribution $\mathcal{D}$. Obviously, the Engel rank ρ equals zero at each point if and only if the distribution is involutive. In the particular case when $\rho = 1$, which will be important in the sequel, we send the reader to [16] for an equivalent definition of the Engel rank, in the language of vector fields.

For a distribution $\mathcal{D}$ such that $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ have constant ranks d_0 and d_1, respectively, we denote $r_0 = d_1 - d_0$. The following result is a direct consequence of Bryant's algebraic lemma [2] (see also [16]). In order to avoid the trivial case $r_0 = 0$, for which the existence of a corank one involutive subdistribution is obvious, we will assume that $r_0 \ge 1$.

Proposition 3.1 *Let $\mathcal{D}$ be a distribution such that $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ have constant ranks d_0 and d_1, respectively. Assume that $r_0 \ge 1$. Then the distribution $\mathcal{D}$ contains an involutive subdistribution $\mathcal{L} \subset \mathcal{D}$ that has constant corank one in $\mathcal{D}$ if and only if the three following conditions hold:*

(i) *The characteristic distribution $\mathcal{C}$ of $\mathcal{D}$ has constant rank $c_0 = d_0 - r_0 - 1$;*
(ii) *The Engel rank ρ of $\mathcal{D}$ is constant and equals 1;*
(iii) *If $r_0 = 2$ then, additionally, the unique corank one subdistribution $\mathcal{B} \subset \mathcal{D}$ such that $[\mathcal{B}, \mathcal{B}] \subset \mathcal{D}$ must be involutive.*

Moreover, if an involutive subdistribution $\mathcal{L} \subset \mathcal{D}$ of corank one in $\mathcal{D}$ exists and $r_0 \ge 2$ then it is unique.

We would like to emphasize that the above conditions are easy to verify. Indeed, for any distribution we can compute the characteristic distribution $\mathcal{C}$ and check whether or not the Engel rank equals 1 (see [2] and [16]). Moreover, as shown by Bryant [2] (see also [16]), both: the unique involutive subdistribution $\mathcal{L}_0$ of corank one in $\mathcal{D}$ and the unique distribution $\mathcal{B}$ satisfying $[\mathcal{B}, \mathcal{B}] \subset \mathcal{D}$, needed to check (iii) of Proposition 3.1, can be easily calculated (if they exist).

4 Flatness of contact systems

The concept of flatness was introduced by Fliess, Lévine, Martin, and Rouchon [4], [5] in order to characterize control systems that are linearizable by endogenous feedback. It is easy to observe that contact systems are flat. In this section we explain how the geometry of contact systems, described in Sections 2 and 3, provides additional information on various aspects of the flatness of contact systems.

Consider a nonlinear control system Σ of the form (1). We say that Σ is *x-flat* at $(x^*, u^*) \in \mathbb{R}^N \times \mathbb{R}^{m+1}$ (see [5], [7], [19]) if there exits a neighborhood $V_0 = X_0 \times U_0$ of (x^*, u^*) and smooth functions φ_i on X_0, for $0 \le i \le m$, called *x-flat outputs*, having the following property: there exits an integer k and smooth functions γ_i, $1 \le$

$i \le N$, and δ_i, $0 \le i \le m$, such that we have

$$\begin{array}{rcl} x_i & = & \gamma_i(\varphi, \dot\varphi, \ldots, \varphi^{(k)}), \quad 1 \le i \le N \\ u_i & = & \delta_i(\varphi, \dot\varphi, \ldots, \varphi^{(k)}), \quad 0 \le i \le m, \end{array}$$

where $\varphi = (\varphi_0, \ldots, \varphi_m)^t$, along any trajectory $x(t)$ of Σ generated by a control $u(t)$ such that $(x(t), u(t)) \in V_0$.

Below we suppose $m \ge 2$; in the case $m = 1$, canonical contact systems for curves are Goursat structures and their flatness is well understood, see e.g. [10].

Assume that the system Σ is feedback equivalent to a canonical contact system for curves. Then there exists an invertible feedback $u = \beta\tilde u$ such that the unique involutive subdistribution $\mathcal{L}_0$ of corank 1 contained in $\mathcal{D}^{(0)} = (f_0, \ldots, f_m)$ is spanned by the m last columns of the matrix $f\beta$. Any matrix β defining a feedback having this property will be denoted by β_{inv}.

By $U^0_{inv}(x)$ we denote the codimension one subspace of $\mathbb{R}^{m+1}$ such that $f(x)u \in \mathcal{L}_0(x)$ for any $u \in U^0_{inv}(x)$. It is clear that a control u belongs to $U^0_{inv}(x)$ if and only if the zeroth-component $(u\beta^{-1}_{inv}(x))^0$ of the feedback modified control $\tilde u = u\beta^{-1}_{inv}(x)$ is identically zero. Put $u^0_{inv}(x) = (u\beta^{-1}_{inv}(x))^0$. Although β_{inv} is not unique, the expression for u^0_{inv} is uniquely defined up to multiplication by a nonvanishing function and its interpretation is clear: if we apply to the system Σ any control u such that $u^0_{inv} \equiv 0$, then the system is governed by $\mathbb{R}^m$-valued contol and all its trajectories remain tangent to the involutive sudistribution $\mathcal{L}_0$. It is u^0_{inv} which is crucial in dynamic linearization of Σ as shows the following result.

Proposition 4.1 *Consider the control system Σ given by (1) and assume that it is feedback equivalent, in a neighborhood of $x^* \in X$, to a canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$. Then we have.*

(i) *Σ is x-flat at any (x^*, u^*) such that $(u^*)^0_{inv}(x^*) \ne 0$.*

(ii) *The dynamic precompensator*

$$\begin{array}{rcl} \dot z_1 & = & z_2 \\ & \vdots & \\ \dot z_{n-1} & = & z_n \\ \dot z_n & = & v_0 \end{array} \tag{5}$$

linked with the system via the dynamic feedback law

$$u = \beta_{inv}\begin{pmatrix} z_1/\gamma(x) \\ v^{\bar m} \end{pmatrix},$$

where $v^{\bar m} = (v_1, \ldots, v_m)^t$ and γ is a suitably chosen nonvanishing function, renders the overall system, controlled by $(v_0, v_1, \ldots, v_m)^t$, static feedback linearizable at any point (x^*, z^*) such that $z^*_1 \ne 0$.

(iii) *As x-flat outputs we can choose any $m+1$ smooth functions $\varphi_0, \varphi_1, \ldots, \varphi_m$ such that $d\varphi_i \perp \mathcal{L}_{n-1}$, for $0 \le i \le m$, and their differentials $d\varphi_i$, for $0 \le i \le m$, are linearly independent at x^*.*

This result shows how the geometry of contact systems is reflected by their flatness. Firstly, as we said in Section 3, the involutive subdistribution $\mathcal{L}_0$ of corank one in $\mathcal{D} = \mathcal{D}^{(0)}$ can be explicitely calculated (see [2] and [16]). Therefore we can calculate, up to multiplication by a nonvanishing function of the state, the single control which needs to be preintegrated: it is the control u^0_{inv}, which is the zeroth-component of the new control $\tilde u = \beta^{-1}_{inv}u$. Secondly, it is also the involutive distribution $\mathcal{L}_0$ that determines the controls at which the system ceases to be flat. Thirdly, the one before the last involutive distributions $\mathcal{L}_{n-1}$ in the sequence $\mathcal{L}_0 \subset \cdots \subset \mathcal{L}_{n-1} \subset \mathcal{L}_n$ determines flat outputs.

We would like to stress that functions whose differentials aninihilate $\mathcal{L}_{n-1}$ are not the only flat outputs of a system equivalent to the canonical systems for curves. They have, however, a certain minimality property. Indeed, when determining — with their help — the state and the control, we need to take the minimal possible number of derivatives. Equivalently, among all choices of $(m+1)$ smooth functions which, treated as pseudo-outputs, define a dynamically input-ouput linearizable system without zero-dynamics, a linearizing precompensator of the minimal dimension n is needed in the case of pseudo-outputs whose differentials anihilate $\mathcal{L}_{n-1}$.

Proof Put $x^j = (x^j_1, \ldots, x^j_m)^t$, for $0 \le j \le n$. Assume that the system Σ is locally feedback equivalent to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$. Apply to Σ only the diffeomorphism establishing that equivalence but do not modify the controls. Then Σ reads

$$\begin{array}{rcl} \dot x^0_0 & = & \tilde\beta^0(x)u \\ \dot x^1 & = & x^2\tilde\beta^0(x)u \\ & \vdots & \\ \dot x^{n-1} & = & x^n\tilde\beta^0(x)u \\ \dot x^n & = & \tilde\beta^{\bar m}(x)u, \end{array} \tag{6}$$

where $\tilde\beta = (\tilde\beta^0, \tilde\beta^{\bar m})$ is a matrix invertible at any x, and $\tilde\beta^0$ and $\tilde\beta^{\bar m}$ are, respectively, $(m+1)\times 1$ and $(m+1)\times m$ matrices. It is clear that $u^0_{inv}(x) = \tilde\beta^0(x)u$ and as β_{inv} we can take, for instance, $\beta_{inv} = \tilde\beta^{-1}$.

(i) To prove that the system is flat, take as flat outputs

$$\varphi_0 = x^0_0,\ \varphi_1 = x^0_1,\ \ldots,\ \varphi_m = x^0_m.$$

Differentiating $\varphi_0 = x^0_0$ we get $\tilde\beta^0(x)u$. Recall that we suppose that $(u^*)^0_{inv}(x^*) \ne 0$. This and $u^0_{inv}(x) =$

$\tilde{\beta}^0(x)u$ imply that in a neighborhood of (x^*, u^*) we have $\tilde{\beta}^0(x)u \neq 0$. Therefore differentiating the functions x_i^0, for $1 \leq i \leq m$, and dividing by $\tilde{\beta}^0(x)u$ we get the components of x^1. We keep differentiating and dividing to get successively $x^2, \ldots, x^n$ and hence the whole state x. Differentiating x^n gives all components of u since we know the state depending matrix $\tilde{\beta}^{\bar{m}}(x)$ and the function $\tilde{\beta}^0(x)u$.

(ii) Assume that the system is given by the form (6). For any feedback matrix β_{inv} there exists a nonvanishing function $\gamma(x)$ such that the zeroth componant of $\beta_{inv}^{-1}(x)u$ equals $\gamma(x)\tilde{\beta}^0(x)u$. Apply the feedback

$$u = \beta_{inv} \begin{pmatrix} z_1/\gamma(x) \\ v^{\bar{m}} \end{pmatrix},$$

together with the precompensator (5) to the system. For any $0 \leq j \leq n-1$ we have $\dot{x}^j = x^{j+1} z_1$. Put $\tilde{x}^0 = x^0$ and introduce new coordinates $\tilde{x}^1 = x^1 z_1$. We have $\dot{\tilde{x}}^1 = x^2 (z_1)^2 + x^1 z_2$. Put

$$\tilde{x}^2 = x^2 (z_1)^2 + x^1 z_2.$$

We keep doing that in order to obtain $\dot{\tilde{x}}^j = \tilde{x}^{j+1}$, for $0 \leq n-1$. Notice that this transformation is a local coordinate change provided that $z_1 \neq 0$. The transformed system is clearly static feedback linearizable.

(iii) Assume that the system is given in the form (6). We proved in (i) that x_0^0, $x_1^0 \ldots, x_m^0$ are flat outputs. If the functions $\varphi_0, \ldots, \varphi_m$ are such that their differentials are independent and anihilate $\mathcal{L}_{n-1}$, which is given by $\mathcal{L}_{n-1} = (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$, it is clear that x_0^0, $x_1^0 \ldots, x_m^0$ can be expressed as smooth functions of $\varphi_0, \ldots, \varphi_m$. It thus follows that $\varphi_0, \ldots, \varphi_m$ are flat outputs. $\square$

5 An Example

In this Section we will consider a simple example given by a kinematical model of a car for which both the front and rear wheels are steerable, see [23]. We denote by (x_0, y_0) (resp. by (x_1, y_1)) the coordinates of the midpoint between the two front (resp. rear) wheels and by α_0 (resp. by α_1) the angle of the front (resp. rear) axle with respect to the horizontal. Finally, by ψ we denote the angle of the bar connecting the two axles with respect to the horizontal.

The two nonholonomic constraints reflecting the fact that the front and rear wheels roll without sleeping are given, respectively, by differential 1-forms

$$\begin{aligned} \Omega_0 &= \sin\alpha_0 dx_0 - \cos\alpha_0 dy_0 \\ \Omega_1 &= \sin\alpha_1 dx_1 - \cos\alpha_1 dy_1. \end{aligned}$$

The holonomic constraints

$$\begin{aligned} x_1 &= x_0 + \cos\psi \\ y_1 &= y_0 + \sin\psi \end{aligned}$$

define a map φ from $(S^1)^3 \times \mathbb{R}^2$, equipped with the coordinates $(\psi, \alpha_0, \alpha_1, x_0, y_0)$ into $(S^1)^3 \times \mathbb{R}^4$, equipped with the coordinates $(\psi, \alpha_0, \alpha_1, x_0, , y_0, x_1, y_1)$. The pull-backs $\omega_0 = \varphi^*\Omega_0$ and $\omega_1 = \varphi^*\Omega_1$ given, respectively, by

$$\begin{aligned} \omega_0 &= \sin\alpha_0 dx_0 - \cos\alpha_0 dy_0 \\ \omega_1 &= \sin\alpha_1 dx_0 - \cos\alpha_0 dy_1 - \cos(\alpha_1 - \psi) d\psi, \end{aligned}$$

define nonholonomic constraints imposed on the car evolving on the five dimensional state space $X = (S^1)^3 \times \mathbb{R}^2$.

Denote

$$S_1 = \{\alpha_1 - \psi = \pi/2 + k\pi,\ k \in \mathbb{Z}\} \subset X \quad \text{and}$$

$$S_2 = \{\alpha_1 - \alpha_0 = k\pi,\ k \in \mathbb{Z}\} \subset X.$$

The differential 1-forms ω_0 and ω_1 are linearly independent at $p \in X$ if and only if $p \notin S_1 \cap S_2$. Therefore outside $S_1 \cap S_2$, the nonholonomic constraints define a control system of the form

$$\dot{\xi} = f_0(\xi)u_0 + f_1(\xi)u_1 + f_2(\xi)u_2,$$

where $\xi = (\psi, \alpha_0, \alpha_1, x_0, y_0)$ are coordinates on $X = (S^1)^3 \times \mathbb{R}^2$, and the distribution $\mathcal{D} = (f_0, f_1, f_2)$, where $f_0 = \frac{\partial}{\partial \alpha_0}$, $f_1 = \frac{\partial}{\partial \alpha_1}$, and

$$\begin{aligned} f_2 &= \cos(\alpha_1 - \psi)\left(\cos(\alpha_0)\frac{\partial}{\partial x} + \sin(\alpha_0)\frac{\partial}{\partial y}\right) \\ &+ \sin(\alpha_1 - \alpha_0)\frac{\partial}{\partial \psi}. \end{aligned}$$

It is interesting to notice that at points of $S_1 \cap S_2$, integral curves of the distribution $\mathcal{D} = (\omega_0, \omega_1)^\perp$ do not represent all trajectories of the system subject to the nonholonomic constraints (ω_0, ω_1). Indeed, through any point $p \in S_1 \cap S_2$ there pass trajectories of the system whose velocities do not belong to $\mathcal{D}(p)$.

We thus consider the model of a car outside $S_1 \cap S_2$. Clearly, the distribution $\mathcal{D}$ contains the involutive subdistribution

$$\mathcal{L} = \left(\frac{\partial}{\partial \alpha_0}\ ,\ \frac{\partial}{\partial \alpha_1} \right).$$

of corank one in $\mathcal{D}$. By a direct calculation we check that $\mathcal{D}^{(1)}(p)$ is of rank five if and only if $p \notin S_1 \cup S_3$, where

$$S_3 = \{\alpha^0 - \psi = \pi/2 + k\pi,\ k \in \mathbb{Z}\} \subset X.$$

Therefore at any configuration point outside $S_1 \cup S_3$, the model is feedback equivalent to the canonical contact system for curves on $J^1(\mathbb{R}, \mathbb{R}^2)$

It is interesting to observe that at points of S_2 that are outside $S_1 \cup S_3$, the system, although not being equivalent to a cononical contact system, is flat. It can, moreover, be transformed via a local feedback to the following normal form

$$\begin{aligned} \dot{x}_1 &= x_5 u_0 \\ \dot{x}_2 &= x_1 u_1 \\ \dot{x}_3 &= u_0 \\ \dot{x}_4 &= u_1 \\ \dot{x}_5 &= u_2, \end{aligned}$$

which generates a nilpotent Lie algebra. As flat outputs we can choose x_2, x_3, and x_4. Notice that the form of the set of singular controls at which the system ceases to be flat changes: it is formed by the union of the planes $u_0 = 0$ and $u_1 = 0$ in $\mathbb{R}^3$.

References

[1] E. Aranda-Bricaire and J.-B. Pomet. Some explicit conditions for a control system to be feedback equivalent to extended Goursat normal form. In *Proceedings of the IFAC Nonlinear Control Systems Design Symposium*, Tahoe City (California), 1995.

[2] R. Bryant. *Some aspects of the local and global theory of Pfaffian systems*. Ph.D. thesis, University of North Carolina, Chapel Hill, 1979.

[3] R. Bryant, S-S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths. *Exterior Differential Systems*. Mathematical Sciences Research Institute Publications. Springer-Verlag, New York, 1991.

[4] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *Comptes Rendus de l'Académie des Sciences*, 315:619–624, 1992.

[5] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: Introductory theory and examples. *International Journal of Control*, 61(6):1327–1361, 1995.

[6] R. Gardner and W. Shadwick. The GS algorithm for exact linearization to Brunovsky normal form. *IEEE Transactions on Automatic Control*, 37(2):224–230, 1992.

[7] B. Jakubczyk. Invariants of dynamic feedback and free systems. In *Proceedings of the European Control Conference*, Groningen, 1993.

[8] Z-P. Jiang and H. Nijmeijer. A recursive technique for tracking control of nonholonomic systems in chained form. *IEEE Transactions on Automatic Control*, 44(2):265–279, 1999.

[9] A. Kumpera and C. Ruiz. Sur l'équivalence locale des systèmes de Pfaff en drapeau. In F. Gherardelli, editor, *Monge-Ampère equations and related topics*, pages 201–247. Instituto Nazionale di Alta Matematica Francesco Severi, Rome, 1982.

[10] P. Martin and P. Rouchon. Feedback linearization and driftless systems. *Mathematics of Control, Signals, and Systems*, 7:235–254, 1994.

[11] R. Montgomery and M. Zhitomirskiĭ. Geometric approach to Goursat flags. Preprint, University of California, Santa Cruz, 1999.

[12] P. Mormul. Goursat flags: classification of codimension-one singularities. Preprint, Warsaw University, 1999.

[13] R. Murray. Nilpotent bases for a class of non-integrable distributions with applications to trajectory generation for nonholonomic systems. *Mathematics of Control, Signals, and Systems*, 7:58–75, 1994.

[14] R. Murray and S. Sastry. Nonholonomic motion planning: Steering using sinusoids. *IEEE Transactions on Automatic Control*, 38(5):700–716, 1993.

[15] W. Pasillas-Lépine and W. Respondek. On the geometry of Goursat structures. Submitted to ESAIM Control, Optimisation, and Calculus of Variations, 1999.

[16] W. Pasillas-Lépine and W. Respondek. Contact systems and corank one involutive subdistributions. Submitted to Acta Applicandae Mathematicae, 2000.

[17] W. Respondek and W. Pasillas-Lépine. Extended Goursat normal form: a geometric characterization. To be published in A. Isidori, F. Lamnabhi-Lagarrigue, and W. Repondek, editors, *Nonlinear Control in the Year 2000*, Springer-Verlag, London, 2000.

[18] J.-B. Pomet. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems and Control Letters*, 18:147–158, 1992.

[19] J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In B. Jakubczyk, W. Respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, volume 32, pages 319–339. Banach Center Publications, Warszawa, 1995.

[20] C. Samson. Control of chained systems: Application to path following and time-varying point-stabilization of mobile robots. *IEEE Transactions on Automatic Control*, 40(1):64–77, 1995.

[21] O. Sørdalen and O. Egeland. Exponential stabilization of nonholonomic chained systems. *IEEE Transactions on Automatic Control*, 40(1):35–49, 1995.

[22] A. Teel, R. Murray, and G. Walsh. Nonholonomic control systems: From steering to stabilization with sinusoids. *International Journal of Control*, 62(4):849–870, 1995.

[23] D. Tilbury and S. Sastry. The multi-steering n-trailer system: A case study of Goursat normal forms and prolongations. *International Journal of Robust and Nonlinear Control*, 5(4):343–364, 1995.

[24] A. Vinogradov, I. Krasil'shchik, and V. Lychagin. *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*. Gordon and Breach, New York, 1986.