# New Integral Representations and Algorithms for Computing nth Roots and the Matrix Sector Function of Nonsingular Complex Matrices

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## Abstract

*It is known that sector switching is a problem of many locally convergent methods for computing the matrix sector function such as Newton's and Halley's methods. In this paper, fast convergent and stable algorithms for approximating the matrix sector function and the principal $n$th root of complex matrices which avoid these problems are presented. These methods are based on new integral representations of the matrix sector function and the principal $n$th root of a complex matrix. The new representations are based on Cauchy integral formula and the residue theorem in analytic function theory. The generalized Householder method for computing the matrix sector function and the principal $n$th root of a complex matrix are introduced. Finally, a new matrix decomposition called "sector factorization" is defined.*

## 1 Introduction

There are several applications in signal processing and control and systems theory where only a subset of the eigenpairs of a given matrix are of interest. For example, high resolution methods in sinusoidal frequency estimation and array signal processing require only the dominant or the minor subspace. The MUSIC and ESPRIT methods for bearing or sinusoidal frequency estimation utilizes the minor and dominant subspaces, respectively, of the data [1-2]. The interest in the matrix sector function stems from the fact that it can be used to compute invariant subapaces. However the computation of the matrix sector function requires the computation of the principal $n$th root of a matrix [3]-[5]. Consequently, the goal of this work is to develop efficient techniques for computing both the matrix sector function and the principal $n$th root of a complex matrix. The matrix sector function has several applications in stability analysis of control systems. The special case $n = 2$ yield the matrix sign function which can be applied to solve the matrix Lyapunov and Riccati equations and to approximate some matrix-valued functions [6].

There are many techniques in the literature for the computation of $n$th roots and the matrix sector function of complex matrices. In [7], the matrix sign algorithm is developed. A matrix continued fraction method for computing square roots was presented in [8]. Numerically efficient methods for computing square roots of nonsingular complex matrices can be found in [9]-[11]. Methods for computing the matrix sector function were given in [12]-[13]. In [14] the matrix sign function is used for the separation of eigenvalues within a sector of a circle. It can also be used for block diagonalization of complex matrices [15]. A comprehensive treatment of the matrix sign function can be found in [6].

Let $A \in \mathcal{C}^{m\times m}$ be a nonsingular complex matrix with no negative eigenvalue, where $\mathcal{C}$ is the field of complex numbers. The matrix sector function of a matrix $A$, denoted by $S_n(A)$, is defined as

$$S_n(A) = A(\sqrt[n]{A^n})^{-1}, \tag{1}$$

where $\sqrt[n]{A^n}$ is the principal $n$th root of $A^n$. The principal $n$th root of the complex matrix $A$ is defined to be any matrix $B \in \mathcal{C}^{m\times m}$ such that $B^n = A$ and for every $\gamma_r \in \sigma(B)$, $\sigma(A)$ denotes the set of eigenvalues of $A$, we have $\gamma_r = |\gamma_r|e^{i\theta_r}$, where $\theta_r \in (-\pi/n, \pi/n)$, for $r = 1, \cdots, m$ and $i$ is the complex number $\sqrt{-1}$.

Generally, an $n$th root of a complex matrix $A \in \mathcal{C}^{m\times m}$ is defined to be any matrix $X \in \mathcal{C}^{m\times m}$ such that $X^n = A$. When all eigenvalues of a non singular matrix $A$ are distinct there are $n^m$ distinct $n$th roots of $A$ since if $A = P^{-1}\text{diag}(\lambda_1, \cdots, \lambda_m)P$, then each of the matrices $P^{-1}\text{diag}(w^{j_1}\sqrt[n]{\lambda_1}, \cdots, w^{j_m}\sqrt[n]{\lambda_m})P$ is also an $n$th root of $A$, for every set of integers $\{j_1, j_2, \cdots, j_m\} \subset \{0, 1, \cdots, n-1\}$, where $\omega$ is a primitive $n$th root of unity.

From the above introduction we observe that the matrix sector function is an $n$th root of the identity matrix $I_m$ which commutes with $A$, i.e., $S_n(A)^n = I_m$, and $AS_n(A)^{-1} = S_n(A)^{-1}A$ has all of its eigenvalues in the sector $-\frac{\pi}{n} < \theta < \frac{\pi}{n}$. These two conditions can be viewed as a characterization of the matrix sector function. Note that for the matrix sign function this implies that $S_2(A)^2 = I_m$, and that $S_2(A)A$ has all its eigenvalues in the right half plane.

The matrix sector function provides an elegant way of splitting $\mathcal{C}^n$ into many complementary subspaces without actually computing any eigenvalues. The matrix $S_n(A)$ is diagonalizable and has the same invariant subspaces as $A$; its eigenvalues are $n$th roots of unity corresponding to eigenvectors of $A$ whose eigenvalues are in the sectors $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$, $k = 0, 1, \cdots, n-1$. The region $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$ of the complex plane will be called the "$k$th sector". The sector $\frac{-\pi}{n} < \theta < \frac{\pi}{n}$ will be called the principal sector. Throughout this paper, the notation $\lambda(A)$ will be used to denote an eigenvalue of $A$.

## 2 Properties of the Matrix Sector Function

From the definition (1) we can state several important properties of the matrix sector function.

**Theorem 1.** *the matrix sector function $S_n(A)$ satisfies the following properties:*

(a) $S_n(A^T) = S_n(A)^T$ *and* $S_n(A^*) = S_n(A)^*$.

(b) $S_n(\alpha A) = S_n(\alpha) S_n(A)$, *where $\alpha$ is any nonzero complex number such that* $\frac{-\pi}{n} \neq arg(\alpha) \neq \frac{\pi}{n}$.

(c) $S_n(A)A = AS_n(A)$.

(d) *The eigenvalues of $S_n(A)$ are $n$th roots of 1, i.e.,* $S_n(A)^n = I_m$.

(e) $AS_n(A)^{-1} = S_n(A)^{-1}A$ *and all eigenvalues of* $AS_n(A)^{-1}$ *are in the sector* $(\frac{-\pi}{n}, \frac{\pi}{n})$.

(f) *If $V$ and $Z$ are nonsingular matrices of same order, then* $S_n(V^{-1}ZV) = V^{-1}S_n(Z)V$, *provided that $S_n(Z)$ is defined.*

(g) $S_n(w^r A) = w^r S_n(A)$, *where $w$ is any primitive $n$th root of 1.*

(h) $S_n(A^{-1}) = S_n(A)^{-1}$.

For convenience, the theory described in this paper is carried out sometimes for the scalar case . However, it must be emphasized that the scalar theory carries over directly to the matrix case. In this paper two interesting aspects of the integral representations are considered. The first aspect is expressing the matrix sector function as a linear combinations of projections $P_k$ corresponding to eigenvectors of $A$ whose eigenvalues are in the sectors $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$, $k = 0, 1, \cdots, n-1$. The second goal deals with computing the matrix sector function and $n$th roots using the generalized Householder method.

## 3 Preliminaries

In this section, we present few results from the complex function theory. From Cauchy integral formula it follows that for any simple closed piecewise smooth contour $C$

$$\frac{1}{2\pi i}\int_C \frac{dz}{z-a} = \begin{cases} 1 \text{ if } a \in D \\ 0 \text{ if } a \in D, \end{cases} \tag{2}$$

where $D$ is the interior of the closed contour $C$. One can generalize this formula to the matrix case so that

$$\frac{1}{2\pi i}\int_C (zI_m - A)^{-1} dz = P_D, \tag{3}$$

where $P_D$ is the projection into the subspace spanned by the eigenvectors of $A$ whose corresponding eigenvalues are in $D$.

If $C$ is chosen to be the boundary of the disk $|z| \leq R$, then (3) simplifies to

$$\frac{1}{2\pi}\int_0^{2\pi} (R^2 I_m - 2RA\cos(\theta) + A^2)^{-1}(RI_m - A\cos(\theta))d\theta = \sum_{r\in K} q_r p_r^*, \tag{4}$$

where $p_r$ and $q_r$ is the $r$th right and left eigenvectors normalized so that $||q_r|| = ||p_r|| = 1$ and $q_r^* p_s = \delta(r-s)$. The set $K$ is defined as $K = \{1 \leq r \leq m, |\lambda_r| \leq R\}$.

Several integral formulas for the principal $n$th root and the matrix sector function can be derived using the following result:

$$\frac{1}{2\pi j}\int_C \frac{f(z)dz}{g(z)} = \sum_{z_i \in D} \frac{f(z_i)}{g'(z_i)}, \tag{5}$$

where $D$ is the interior of the closed contour $C$, and $z_i$ are zeros of $g(z)$. Note that (5) follows directly from the Residue Theorem [16].

From (5) and if $C$ is the boundary of the principal sector, it follows that

$$\frac{1}{2\pi j}\int_C \frac{z^m dz}{z^n - a^n} = \frac{a^m}{na^{n-1}} = \frac{a^{m-n+1}}{n},$$

where $(a^n)^{\frac{1}{n}}$ is the principal $n$th root of $a^n$.

The projections $P_k$ onto the $k$th sector can be represented in integral form as follows.

**Theorem 2.** *Let $A$ be a nonsingular $m \times m$ matrix such that none of its eigenvalues are on the boundary of the sectors* $\{\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}\}_{k=0}^{n-1}$. *Let $P_k$ be the projection into the subspace spanned by the eigenvectors of $A$ with eigenvalues inside the $k$th sector* $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$. *Then an integral formula for $P_k$ can be given as*

$$P_k = \frac{\theta_2 - \theta_1}{\pi} I_m + \frac{A(e^{j\theta_2} - e^{j\theta_1})}{2\pi j} \times \int_0^\infty (r^2 e^{j(\theta_1+\theta_2)} I_m - 2rA(e^{j\theta_1} + e^{j\theta_2}) + A^2)^{-1} dr. \tag{6a}$$

*where* $\theta_1 = \frac{(2k-1)\pi}{n}$ *and* $\theta_2 = \frac{(2k+1)\pi}{n}$.

If $D$ is chosen to be the principal sector $-\frac{\pi}{n} < \theta < \frac{\pi}{n}$, then $P_0$ can be simplified to

$$P_0 = \frac{n\sin(\frac{\pi}{n})A}{\pi}\int_0^\infty (y^2 I_m - 2Ay\cos(\frac{\pi}{n}) + A^2)^{-1} dy + \frac{1}{n} I_m. \tag{6b}$$

The projections $\{P_k\}_{k=0}^{n-1}$ onto the $n$ sectors can be used to compute the matrix sector function as follows.

**Theorem 3.** *Let $A$ be a nonsingular $m \times m$ matrix such that none of its eigenvalues are on the boundary of the sectors $\{\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}\}_{k=0}^{n-1}$. Let $P_k$ be the projection onto the subspace spanned by the eigenvectors of $A$ with eigenvalues inside the $k$th sector $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$. Then,*

$$S_n(A) = \sum_{k=0}^{n-1} w^k P_k. \qquad (7a)$$

*Additionally, for any integer $l$*

$$S_n(A)^l = \sum_{k=0}^{n-1} w^{kl} P_k. \qquad (7b)$$

**Proof:** The proof of the first assertion of this result follows from the observation that $P_k P_l = P_k \delta(k-l)$ and therefore

$$S_n(A)^{-1} = \sum_{k=0}^{n-1} w^{n-k} P_k. \qquad (8)$$

Let $U_i$ be a matrix whose columns are the eigenvectors of $A$ in $i$th sector such that $U_k^* U_l = I\delta(k-l)$, where $I$ is an identity matrix of appropriate dimension, then $P_k = U_k U_k^*$ and

$$A = \sum_{i=0}^{n-1} U_i \Lambda_i U_i^*$$

where $\Lambda_i$ is a matrix in Jordan form such that all eigenvalues of $A$ in the $i$th sector are eigenvalues of $\Lambda_i$. Clearly, $S_n(A)^n = I_m$ and $S_n(A)^{-1}A = \sum_{i=0}^{n-1} w^{n-i} U_i \Lambda_i U_i^*$. One can easily verify that the eigenvalues of $w^{n-i}\Lambda_i$ are the same eigenvalues of $S_n(A)^{-1}A$ and that all these eigenvalues lie in the principal sector. To prove the theorem, we have $w^{n-i} = \exp(\frac{2(n-i)\pi}{n})$ and if $\lambda$ is an eigenvalue of $\Lambda_i$ then, $\lambda = |\lambda|\exp(\theta)$, where $\frac{(2i-1)\pi}{n} < \theta < \frac{(2i+1)\pi}{n}$. Now $w^{n-i}\lambda = |\lambda|\exp(\frac{2(n-i)\pi}{n} + \theta)$. Obviously, the angle $\frac{2(n-i)\pi}{n} + \theta$ is in the principal sector. This shows that $S_n(A)$ as defined in (7a) is actually the matrix sector function of order $n$. Q.E.D.

## 4 Integral Representation of the Matrix Sector Function

We can incorporate Theorems 2 and 3 to derive an integral formula for computing the matrix sector function by simplifying the formulas (7a) and (6a) as shown in the following result.

**Corollary 4.** *Let $A$ be a nonsingular $m \times m$ matrix such that none of its eigenvalues are on the boundary of the sectors $\{\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}\}_{k=0}^{n-1}$. Then the matrix sector function has the following integral representations:*

$$S_n(A)^{-1} = \frac{n\sin(\frac{\pi}{n})}{\pi} \int_0^\infty (y^n I_m + A^n)^{-1} A^{n-1} dy, \qquad (9a)$$

*and*

$$S_n(A) = \frac{n\sin(\frac{\pi}{n})}{\pi} \int_0^\infty (y^n A^n + I_m)^{-1} A dy. \qquad (9b)$$

Part (9b) of this result follows directly from (9a) and the relation $S_n(A^{-1}) = S_n(A)^{-1}$.

Clearly, (9) is a generalization of Robert's integral representation of the matrix sign function. Specifically, Robert's formula [7] is

$$S_2(A) = S_2(A)^{-1} = \frac{2}{\pi} \int_0^\infty (y^2 I_m + A^2)^{-1} A dy,$$

or with appropriate change of variables,

$$S_2(A) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\sin^2(\theta) I_m + A^2 \cos^2(\theta))^{-1} A d\theta.$$

The integrals (9) can be computed over the finite interval $[0, \frac{\pi}{2}]$ by using the change of variable $y = \tan(\theta)$ in which case (9) transcribed to

$$S_n(A)^{-1} = \frac{n\sin(\frac{\pi}{n})}{\pi} \times \int_0^{\frac{\pi}{2}} (\sin^n(\theta) I_m + A^n \cos^n(\theta))^{-1} A^{n-1} \cos^{n-2}(\theta) d\theta, \qquad (10a)$$

and

$$S_n(A) = \frac{n\sin(\frac{\pi}{n})}{\pi} \times \int_0^{\frac{\pi}{2}} (A^n \sin^n(\theta) + \cos^n(\theta) I_m)^{-1} A \cos^{n-2}(\theta) d\theta. \qquad (10b)$$

## 5 Integral Representation of the Principal $n$th Root

Several integral representation of fractional powers of matrices can be obtained by computing the contour integral $\frac{1}{2\pi j}\int_C \frac{z^m dz}{z^n - a^n}$. This integral can be evaluated over the sector $\frac{-\pi}{n} < \theta < \frac{\pi}{n}$ by setting $z = re^{j\theta}$, where $\theta = \pm\frac{\pi}{n}$. This combines with (4) lead to the following results.

**Corollary 5.** *Let $A$ be a nonsingular $m \times m$ matrix such that none of its eigenvalues are on the boundary of the sectors $\{\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}\}_{k=0}^{n-1}$. Then the principal $n$th root of $A$ can be represented in integral form as*

$$A^{\frac{1}{n}} = \frac{n\sin(\frac{\pi}{n})}{\pi} \int_0^\infty (y^n I_m + A)^{-1} A dy. \qquad (11)$$

Note that (9) can be obtained from the relation $S_n(A^{-1}) = S_n(A)^{-1}$ and (1). Fractional powers of the matrix $A$ or powers of the principal $n$th root of $A$ can be represented in integral form as shown in the following result.

**Corollary 6.** *Let $A^{\frac{1}{n}}$ be the principal $n$th root of $A$ and define $A^{\frac{m}{n}} = (A^{\frac{1}{n}})^m$, then*

$$A^{\frac{m}{n}} = \frac{n\sin(\frac{m\pi}{n})}{\pi} \int_0^\infty (y^n I_m + A)^{-1} A y^{m-1} dy, \qquad (12a)$$

*or*

$$A^{\frac{m}{n}} = \frac{n\sin(\frac{m\pi}{n})}{\pi}\times$$
$$\int_0^\infty (\sin^n(\theta)I_m + A\cos^n(\theta))^{-1}A\sin^{m-1}(\theta)\cos^{n-m-1}(\theta)d\theta, \qquad (12b)$$

To compute the integrals in the previous sections, one can use the Gaussian quadrature with 16 points. The matrix sector function was computed using (10) for different matrices, however, these computations are not reported here due to space limitation.

# 6 Computing the Principal $n$th Root Using the Generalized Householder Method

The results developed in this section are primarily inspired by the algorithm developed in [17] for computing the dominant zeros of polynomials. In this work, we place no restriction on the matrix or its eigenvalues except it has to be nonsingular and that none of its eigenvalues are on the boundary of the sectors $\{\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}\}_{k=0}^{n-1}$. We will generalize the algorithm in [17] , to compute the $n$th roots of nonsingular complex matrices and their matrix sector function. The proposed algorithm is based on shifting the eigenvalues of $A^{\frac{1}{n}}$ by a number $a$. The main feature of this type of algorithms is that all $n$th roots with eigenvalues in a given sector can be computed. In the next theorem, an $r$th order convergent algorithm for computing an $n$th root of a square matrix is presented.

**Theorem 7.** *Let $A$ be a nonsingular $m\times m$ matrix and $n\geq 2$ be an integer and let $a>0$. Let*

$$H = \begin{bmatrix} aI_m & I_m & \cdots & 0_m & 0_m \\ 0_m & aI_m & \cdots & 0_m & 0_m \\ 0_m & 0_m & \cdots & 0_m & 0_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A & 0_m & \cdots & 0_m & aI_m \end{bmatrix}, \qquad (13)$$

*where the matrices $I_m$ and $0_m$ denote the $m\times m$ identity and the null matrix respectively. For each positive integer $k$, let*

$$H^k = \begin{bmatrix} H_{11}(k) & H_{12}(k) & \cdots & H_{1n}(k) \\ H_{21}(k) & H_{22}(k) & \cdots & H_{2n}(k) \\ \vdots & \vdots & \vdots & \vdots \\ H_{n1}(k) & H_{n2}(k) & \cdots & H_{nn}(k) \end{bmatrix}, \qquad (14)$$

*where the $H_{ij}(k)$'s are $m\times m$ matrices. Then for $1\leq i,j,l\leq n$,*

$$\lim_{k\to\infty} H_{ij}(k)^{-1}H_{il}(k) = I_m, \qquad (15a)$$

$$\lim_{k\to\infty} H_{ij}(k)^{-1}H_{ij}(k+1) = W_1 = aI_m + X, \qquad (15b)$$

$$\lim_{k\to\infty} H_{ij}(k)^{-1}H_{i+l,j}(k) = X^l. \qquad (15c)$$

*where $X$ be the principal $n$th root of $A$.*

**Proof.** The main point of the proof is the observation that if $X$ be the principal $n$th root of $A$, then $|\frac{\lambda_i(aI+w^jX)}{\lambda_i(aI+X)}|<1$, for $i=1,\cdots,m$, $j=1,\cdots,n-1$, where $w$ is a primitive $n$th root of unity. The matrix $H$ has the following block eigendecomposition

$$H = \begin{bmatrix} I_m & I_m & \cdots & I_m \\ X_1 & X_2 & \cdots & X_n \\ X_1^2 & X_2^2 & \cdots & X_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ X_1^{n-1} & X_2^{n-1} & \cdots & X_n^{n-1} \end{bmatrix} \begin{bmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & W_m \end{bmatrix}$$
$$\begin{bmatrix} I_m & I_m & \cdots & I_m \\ X_1 & X_2 & \cdots & X_n \\ X_1^2 & X_2^2 & \cdots & X_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ X_1^{n-1} & X_2^{n-1} & \cdots & X_n^{n-1} \end{bmatrix}^{-1},$$

where $W_i = aI_m + X_i$, for $1\leq i\leq n$ and $X_i = w^{i-1}X$. Hence

$$H^k = [H_{ij}(k)]$$
$$= \begin{bmatrix} I_m & I_m & \cdots & I_m \\ X_1 & X_2 & \cdots & X_n \\ X_1^2 & X_2^2 & \cdots & X_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ X_1^{n-1} & X_2^{n-1} & \cdots & X_n^{n-1} \end{bmatrix} \begin{bmatrix} W_1^k & 0 & \cdots & 0 \\ 0 & W_2^k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & W_m^k \end{bmatrix}$$
$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}, \qquad (16)$$

where

$$V^{-1} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

It follows from (16) that

$$H_{ij}(k) = \sum_{r=1}^m X_r^{i-1}W_r^kC_{rj},$$

Since $W_iW_j = W_jW_i$ for all $i,j$ and $|\lambda_i(W_1)| > |\lambda_i(W_j)|$, $i=1,\cdots,m$, $j=2,\cdots,n$, it follows $\lambda(W_1^{-1}W_j) = |\frac{\lambda(W_j)}{\lambda(W_j)}| < 1$ and therefore $(W_1^{-1}W_j)^{r^k}\to 0$ as $k\to\infty$. It can be shown that $C_{r1}=C_{s1}$, for $1\leq r,s\leq n$, and therefore

$$\lim_{k\to\infty} H_{rj}(k)^{-1}H_{sj}(k) =$$
$$\lim_{k\to\infty} X^{s-r}\{\sum_{j=1}^n C_{rj}W_j^{r^k}\}^{-1}\{\sum_{j=1}^n C_{sj}W_j)^{r^k}\} = X^{s-r}.$$

Q.E.D.

Theorem 7 remains valid if the matrix $aI_m$ is replaced with $X_0$ for which $X_0A = AX_0$. In this case the method converges to an $n$th root of $I_m$ which has the same eigenvalue distribution as those of $X_0$, i.e., if $X_0$ has $k$ eigenvalues in the $i$th sector then so is $X$. Thus when $a$ is chosen to fall in the $i$th sector, then the method converges to an $n$th root whose eigenvalues fall in the $i$th sector. This implies that the choice of $a$ determines the $n$th root of $A$ which the sequence $H_{i,j}^{-1}(k)H_{i,j+r}(k)$ converges to. If we wish to compute the principal $n$th root of $A$ we must choose $a = |a|e^{j\theta}$ in the principal sector so that $\frac{-\pi}{n} < \theta < \frac{\pi}{n}$, e.g. $a > 0$. Similarly if an $n$th root of $A$ with all eigenvalues in the $i$th sector, then choose $a = |a|e^{j\theta}$ with $\frac{i-1}{n}\pi < \theta < \frac{i+1}{n}\pi$.

As can be observed from the last result, approximating the principal $n$th root requires only powers of $H$. Note that since $H$ is block Toeplitz, $H^k$ is also block Toeplitz. This property can be utilized to compute $H^k$ more efficiently. A simple method of achieving this goal is the squaring procedure in which $H, H^2, H^4, \cdots, H^{2^{p_0}}$ are computed, where $p_0$ is a sufficiently large integer. The only problem in computing these power matrices is that $H^k$ becomes large (overflow) if $|\lambda_1| > 1$ or small (underflow) if $|\lambda_1| < 1$, where $\lambda_1$ is the largest (in magnitude) eigenvalue of $H$. To alleviate this numerical problem, scaling may be applied. A stable method of generating scaled powers of complex matrices can be described as follows:

$$
\begin{aligned}
B_0 &= \rho \frac{H}{Trace(H)} \\
B_{k+1} &= \rho \frac{B_k^2}{Trace(B_k)}, \; k = 0, 1, \cdots, p_0,
\end{aligned}
\tag{17}
$$

where $\rho$ is a positive number slightly less than unity and $p_0$ is a sufficiently large integer. By a suitable choice of $\rho$ one can be sure of staying within the range from $-1$ to 1 even with round-off error. Assuming that $B_k = [B_{ij}^{(k)}]$, then for $1 \leq i, j, l \leq n$,

$$\lim_{k\to\infty} B_{ij}(k)^{-1}B_{il}(k) = I_m, \tag{18a}$$

$$\lim_{k\to\infty} B_{ij}(k)^{-1}B_{ij}(k+1) = W_1 = aI_m + X, \tag{18b}$$

$$\lim_{k\to\infty} B_{ij}(k)^{-1}B_{i+l,j}(k) = X^l. \tag{18c}$$

Here $X$ is the principal $n$th root of $A$ provided that $a$ is chosen to be any nonzero number in the principal sector.

## 7 The Matrix Sector Function

Let $A$ be a nonsingular matrix, then the matrix sector function is defined as $A(\sqrt[n]{A^n})^{-1}$, where $\sqrt[n]{A^n}$ is the principal $n$th root of $A^n$. As noted earlier, $S_n(A)$ is an $n$th root of the identity matrix which commutes with $A$. In the following result, we introduce a power-like method for computing $S_n(A)$.

**Theorem 8.** *Let $A$ be a nonsingular matrix and let $\alpha$ be any nonzero complex number in the principal sector*

$$
H = \begin{bmatrix}
\alpha A & I_m & \cdots & 0_m & 0_m \\
0_m & \alpha A & \cdots & 0_m & 0_m \\
0_m & 0_m & \cdots & 0_m & 0_m \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
I_m & 0_m & \cdots & 0_m & \alpha A
\end{bmatrix}, \tag{19}
$$

*and let $H^k = [H_{ij}(k)]$, then $H^k$ is block Toeplitz and $H_{ij}(k)^{-1}H_{i+r,j}(k) \to S_n(A)^r$ as $k \to \infty$, where $S_n(A)$ is the matrix $n$th sector function of $A$.*

**Proof.** Let $X = S_n(A)$ be the matrix sector function of $A$. Then $X, wX, \cdots, w^{n-1}X$ are solutions of $X^n - I_m = 0$. The conclusion follows from the observation that $\lambda_i(A + X) > \lambda_i(A + w^jX)$ for $i = 1, \cdots, m$, $j = 1, \cdots, n-1$. This can be seen from the observation that the eigenvalues of $A$ and $X$ are in the same sector. Q.E.D.

It can be shown that $H(k)$ is block Toeplitz with the first row is given as

$$H_{1l}(k) = \sum_{r=0}^{[\frac{k}{n}]+1} \binom{k}{l-1+r(n-1)} (\alpha A)^{k-l+1-r(n-1)}.$$

Given the matrix sector function, the number of eigenvalues of a given matrix in a a specific sector can be determined by solving the system

$$\sum_{i=1}^{n} (w^r)^{i-1} P_i = S_n(A)^r$$

for $r = 0, \cdots, n-1$. Hence the $P_i$ can be obtained by solving the following Vandermonde system:

$$
\begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_{n-1} \end{bmatrix}
=
\begin{bmatrix}
I_m & wI_m & \cdots & w^{n-1}I_m \\
I_m & w^2 I_m & \cdots & (w^{n-1})^2 I_m \\
\cdots & \cdots & \cdots & \cdots \\
I_m & w^{n-1}I_m & \cdots & (w^{n-1})^{n-1}I_m
\end{bmatrix}^{-1}
\begin{bmatrix} I_m \\ S_n(A) \\ \vdots \\ S_n(A)^{n-1} \end{bmatrix},
$$

from which we obtain that

$$P_k = \frac{1}{n}\{I_m + S_n(A)w^k + (S_n(A)w^k)^2 + \cdots + (S_n(A)w^k)^{n-1}\} \tag{20}$$

is a projection matrix on the sector that contains the root $w^k$.

Thus one can compute all $n$th roots of a given matrix by considering the principal $n$th root $X$ and all possible distinct matrices $XS$ with $S = \sum_{i=1}^{r} w^{j_i} P_i$. These projections into different sectors can also be used to locate eigenvalues of a given matrix. To examine the number of eigenvalues in $i$th sector, one may calculate $rank(P_i) = trace(P_i)$. Hence stability test for polynomials and matrices inside given sectors can be established numerically by computing the matrix sector function $S_n(A)$.

## 8 Sector Factorization

It will be shown that if $A$ is a nonsingular matrix which does not have eigenvalues on the boundary of the sectors $\frac{(2k-1)\pi}{n} < \theta < \frac{(2k+1)\pi}{n}$, for $k = 0, 2, \cdots, n-1$, then $A$ can be factored as

$$A = A_n W_n(A), \tag{21}$$

where all eigenvalues of $A_n$ are in one sector and $W_n$ is an $n$th root of the identity matrix such that $AW_n = W_nA$.

We will next present several identities which can be utilized to solve for $A_n$ and $S_n$.

$$S_n(\begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & I_m \\ A^n & 0 & 0 & \cdots & 0 \end{bmatrix}) = \begin{bmatrix} 0 & A_n^{-1} & 0 & \cdots & 0 \\ 0 & 0 & A_n^{-1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A_n^{-1} \\ A_n^{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{22}$$

where $AA_n = A_nA$, and $A_n^n = A^n$. Note also that $A^n - A_n^n = \Pi_{i=1}^n (A - w^{i-1}A_n) = 0$ for some $w$, an $n$th root of 1. We also discovered the following new link between $W_n$ and the sector function

$$S_n(\begin{bmatrix} 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A \\ A & 0 & 0 & \cdots & 0 \end{bmatrix}) = \begin{bmatrix} 0 & W_n^{-1} & 0 & \cdots & 0 \\ 0 & 0 & W_n^{-1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & W_n^{-1} \\ W_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{23}$$

where $W_nA = AW_n$, $W_n^n = I_m$, $W_n^{n-1} = S_n(A)$, and all eigenvalues of $AW_n$ lie in the sector $\frac{-\pi}{n} < \theta < \frac{\pi}{n}$. The importance of this identity is that any method for computing $W_n$ will automatically translates into a method of computing sector factorization of $A$ and the principal $n$th root of $A$.

It is interesting to note that the block Toeplitz matrix of the left hand side of (23) has the following factorization in terms of block Vandermonde and block diagonal matrices:

$$\begin{bmatrix} 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A \\ A & 0 & 0 & \cdots & 0 \end{bmatrix} =$$
$$\begin{bmatrix} I_m & I_m & \cdots & I_m \\ S_n & wS_n & \cdots & w^{n-1}S_n \\ \cdots & \cdots & \cdots & \cdots \\ S_n^{n-1} & w^{n-1}S_n^{n-1} & \cdots & wS_n^{n-1} \end{bmatrix} \begin{bmatrix} AS_n & 0 & \cdots & 0 \\ 0 & wAS_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w^{n-1}AS_n \end{bmatrix}$$
$$\begin{bmatrix} I_m & I_m & \cdots & I_m \\ S_n & wS_n & \cdots & w^{n-1}S_n \\ \cdots & \cdots & \cdots & \cdots \\ S_n^{n-1} & w^{n-1}S_n^{n-1} & \cdots & wS_n^{n-1} \end{bmatrix}^{-1}. \tag{24}$$

Once the matrix $S_n(A)$ is obtained, then projections onto each sector can be computed. The columns of the orthogonal factor of the QR decomposition of these projections form bases for the corresponding eigen-spaces. This factorization can be considered as generalization of the polar and the sign decomposition of matrices.

## 9 Conclusion

Integral formulas and algorithms for computing $n$th roots and matrix sector function are developed. The integral forms can be computed using several numerical integration techniques. The proof of some of these results are not shown here but can be provided by the first author upon request. A paper that includes this work is being under preparation and will be submitted for publication soon. The paper will include rigorous numerical stability of these algorithms which we have not dealt with in this paper. Simulations and numerical evaluation of all results will also be established.

## References


[1] R. O. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. 34, pp. 276-280, 1986.

[2] R. H. Roy and T. Kailath, "ESPRIT-Estimation of Signal Parameters via Rotational Invariance Techniques," *IEEE Trans. Acoustic, Speech and Signal Processing,* Vol. ASSP-37, No 7, pp. 984-995, July 1989.

[3] W. D. Hoskins and D. J. Walton, "A Fast, More Stable Method for Computing the $p$th Root of Positive Definite Matrices, "*Linear Algebra Appl.*, Vol. 26, pp. 139-163,1979.

[4] W. D. Hoskins and D. J. Walton, "A Fast Method of Computing the Square Root of a Matrix," *IEEE Trans. Automatic. Contr.*, Vol. AC-23, No. 3, pp. 494-495, 1978.

[5] L. S. Shieh, Y. T. Tsay and R. E. Yates, "Computation of the Principal $n$th Root of Complex Matrices," *IEEE Trans. Automatic Control*, Vol. AC-30, No. 6, pp. 606-608, 1985.

[6] C. S. Kenney and A. J. Laub, The Matrix Sign Function, *IEEE Trans. Automatic Control*, Vol. 40, No. 8, pp. 1330-1348, August 1995.

[7] J. D. Roberts, "Linear Model Reduction and Solution of the Algebraic Riccati Equation by Use of the Sign Function," Internat. J. of Control, 32, pp. 677-687, 1980.

[8] L. S. Shieh and N. Chahin, "A computer-aided method for the factorization of matrix polynomials," *Appl. Math. Comput.*, Vol. 2, pp. 63-94, 1976.

[9] M. A. Hasan, "Higher Order Convergent Algorithms for Computing the Square Roots of Complex Matrices", J. Math. Anal. with Applications, 213, 393-405 (1997).

[10] A. Bjorck and S. Hammarling, "A Schur Method for the Square Root of a Matrix," *Linear Algebra Appl.*, Vol. 52/53, 1983, pp. 127-140.

[11] N. J. Higham, "Stable Iterations for the Matrix Square Root," Numerical Algorithms,15(2):227-242, 1997.

[12] Y. T. Tsay, L. S. Shieh, and R. E. Yates, "Fast and Stable Algorithms for Computing the Principal $n$th Root of Complex Matrix and the Matrix Sector Function," *Computers & Mathematics with Applications*, Vol. 15, No. 11, pp. 903-913, 1988.

[13] C. K. Koc and B. Bakkaloglu, Halley's Method for the Matrix Sector Function, *IEEE Trans. Automatic Control*, Vol. 40, No. 5, pp. 944-949, 1995.

[14] E. U. Stickel, Separating Eigenvalues Using the Matrix Sign Function, *Linear Algebra Appl.*, 148, pp. 75-88, 1991.

[15] E. D. Denman and A. N. Beavers, "The Matrix Sign Function and Computation of Systems," *Appl. Math. Comput.*, Vol. 2, pp. 63-94, 1976.

[16] P. Henrici, Applied and Computational Complex Analysis, Vol. 1, John Wiley & Sons, New York, 1974.

[17] A. S. Householder, Generalization of an Algorithm of Sebastiao e Silva, Numer. Math., 16, 1971, pp. 375-382.