

Global stability of periodic orbits in relay feedback systems¹

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Abstract

In this paper, we provide a sufficient condition for the global stability of a periodic orbit using the contraction mapping theorem. The condition is obtained by identifying an invariant set of the system dynamics in which the Poincaré map is continuous and contractive. An upper bound on the norm of the derivative of the map is obtained by exploiting its geometric structure.

1 Introduction

We consider relay feedback systems consisting of a single input, single output linear component and a relay hysteresis element connected in feedback. The typical configuration is shown in Fig. 1 where the linear component is denoted by \mathbf{P} and the *on-off* switching points and the corresponding outputs of the relay are scaled to ± 1 . Such systems are encountered in many industrial control applications and are also used as models for relaxation oscillations (see [1–5]). In the context of relay feedback systems, we are concerned with the existence and stability of periodic orbits. Of particular interest are the *symmetric unimodal orbits* which involve exactly two relay switchings per period and have equal time intervals between the positive and negative switchings. Conditions for the existence and local stability of such orbits have been obtained by Aström [2] using Poincaré maps. A more complete analysis identifying degenerate and asymmetric orbits has been presented in [5]. However, these conditions do not rule out the existence of complex attractors and hence a more thorough analysis is needed to determine whether there is a globally stable periodic orbit.

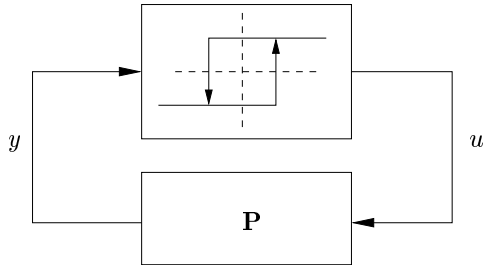


Figure 1: Feedback connection of relay hysteresis and a linear system

Characterizing the global stability in relay feedback systems is a difficult problem due to the fact that the Poincaré map is highly nonlinear. The results in this direction have been sparse. Kolesov [6] defined the notion of *strongly stable* transfer functions that will guarantee a unique, globally stable periodic orbit. Using time domain methods, global stability of some second order systems has been shown by Megretski [3]. More recently, using quadratic Lyapunov functions for the Poincaré map, Goncalves and Megretski [4] proposed sufficient conditions that require the existence of a positive definite matrix satisfying a linear matrix inequality.

In what follows, we consider relay feedback systems whose linear component, \mathbf{P} is described by a controllable and observable state space model

$$\dot{x} = Ax + bu \quad (1)$$

$$y = cx. \quad (2)$$

Furthermore, we assume that the matrix A is Hurwitz and $cA^{-1}b > 1$ so that the asymptotic solution is oscillatory. In order to obtain a sufficient condition for global stability, we pursue an approach that involves identifying a compact invariant set of the system along the lines of Bliman and Krasnoselskii [7] and requires that the Poincaré map be contractive over this set. To this end, we first seek an invariant set that does not contain any discontinuities of the

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map and then exploit the geometric structure of the derivative of the map to obtain an upper bound on its norm. When this bound is less than unity, the differentiability of the map in the invariant set implies the contractiveness of the map.

In Section 2, we briefly review the existence and local stability results for symmetric unimodal periodic orbits. A more detailed account can be found in [2, 5]. In Section 3 we present our main results on global stability. In the sequel, we use a prime ($'$) to denote the transpose of a matrix and $\mathcal{N}(\cdot)$ to denote the null space.

2 Existence and Local Stability of Unimodal Orbits

To analyze the periodic orbits of the relay feedback system, it is convenient to map the switching points of the relay $y = \pm 1$ to the switching planes in state space $S_+ = \{x : cx = 1\}$ and $S_- = \{x : cx = -1\}$ respectively. The relay output u switches from -1 to $+1$ when x hits S_+ and *vice versa*. Let $\xi \in S_+$ and $\eta \in S_-$ denote the states when switchings occur on a trajectory. We can define the Poincaré maps $P_+ : S_+ \rightarrow S_-$ and $P_- : S_- \rightarrow S_+$ that map from one switching point to the immediate next one *i.e.*, $\eta_k = P_+(\xi_k)$ and $\xi_{k+1} = P_-(\eta_k)$ in terms of the appropriate state transition map. Notice that both P_+ and P_- involve the corresponding switching time as an implicit parameter which is determined by the condition that the image point lies on the other switching plane. By symmetry, we have $P_-(\eta) = -P_+(-\eta)$. The fixed points of the composition map $P := P_- \circ P_+$ provide the candidates for unimodal periodic orbits. Equivalently, one could consider the map $P_+ \circ P_-$. When A is stable, the fixed points are isolated and if we consider symmetric orbits *i.e.*, orbits that take equal time to go from ξ to η and back, they always satisfy $\xi + \eta = 0$ [5]. Thus, the necessary and sufficient conditions for the existence of a symmetric unimodal periodic orbit can be stated as follows:

Proposition 1 [2, 5] *The relay feedback system in Fig. 1 where the linear system \mathbf{P} is described by (1)–(2) has a symmetric unimodal periodic orbit of half-period T iff $\xi := -(I + e^{AT})^{-1}A^{-1}(e^{AT} - I)b$ satisfies $c\xi = 1$ and the inequality $1 + ce^{At}\xi + cA^{-1}(e^{At} - I)b > 0$ holds for $0 < t < T$. Moreover, ξ is the point on the orbit where the relay output u switches to $+1$.*

This result allows us to identify all the symmetric unimodal orbits by observing the roots of a scalar equation. The Jacobian derivative of the map $\eta = P_+(\xi)$ at ξ is given by

$$D_{P_+} = \left[I - \frac{vc}{cv} \right] e^{At} \quad (3)$$

where $v = e^{At}(A\xi + b) = A\eta + b$ and t is the switching time from ξ . Symmetry implies that $D_{P_-}|_{\eta=-\xi} = D_{P_+}|_{\xi}$. The derivative of the return map $P = P_- \circ P_+$ is given by $D_P = D_{P_-}|_{\eta} D_{P_+}|_{\xi} = D_{P_+}|_{-\eta} D_{P_+}|_{\xi}$. Thus, it suffices to study the behavior of one map, P_+ .

The local stability of a fixed point ξ of P is characterized by the eigenvalues of D_{P_+} (which is the same as D_{P_-} at the fixed point). There is always a zero eigenvalue due to the fact that the variations of ξ are restricted to a hyperplane, S_+ . When the eigenvalues are all within the unit disk, the fixed point and hence the unimodal orbit is locally stable. In the following section, we show that a useful upper bound can be obtained on $\|D_{P_+}\|$ by separating the two factors appearing in (3).

3 Global Stability

Henceforth, we assume that there is a unique, locally stable symmetric periodic orbit satisfying the conditions in Proposition 1 and focus on the global stability of such an orbit. Notice that D_{P_+} given by (3) is composed of two factors, $I - \Pi$ and e^{At} where Π denotes the parallel projection operator $\frac{vc}{cv}$. Such parallel projection operators are encountered in control and their properties have been studied in detail [8, 9]. The property that is of main interest to us is that $\|I - \Pi\| = \|\Pi\|$. We get a much tighter bound on $\|I - \Pi\|$ using this property than by the triangular inequality, especially since $\|\Pi\| \geq 1$. In the linear case, $\|\Pi\|$ relates to the sine of the angle between the underlying subspaces. When $\Pi = \frac{vc}{cv}$, we have $\|\Pi\| = \frac{\|c\| \cdot \|v\|}{|cv|}$. Since

$$\|D_{P_+}\| \leq \|\Pi\| \cdot \|e^{At}\|, \quad (4)$$

the derivative is contractive when the right hand side is less than unity. Since $\|\Pi\| \geq 1$, this condition is useful only when $\|e^{At}\| < 1$ for t in the range of interest. Although this condition can be used as a sufficient condition for the local stability of a fixed point, a more exact result can be easily obtained by directly checking the eigenvalues of D_{P_+} at the fixed

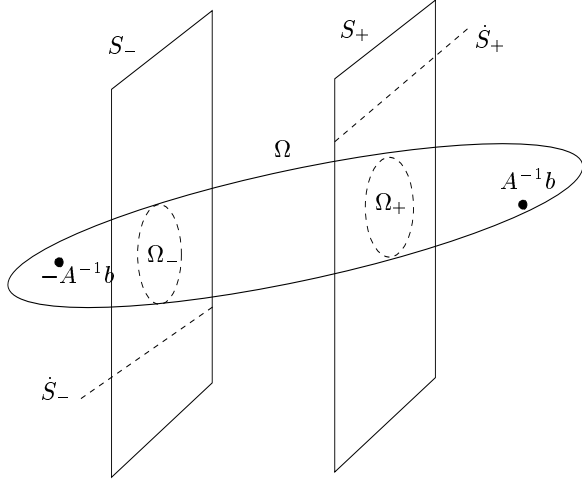


Figure 2: The invariant set Ω and the sets of anomalous points \dot{S}_+ and \dot{S}_-

point. However, (4) proves to be more useful in characterizing the global stability.

Since A is stable and u is bounded, all the system trajectories irrespective of the initial condition, ultimately enter a compact invariant set Ω . For any symmetric positive definite matrix W , it can be shown, along the lines of [4, Proposition 6.1] that

$$\Omega := \{x : \|Wx\| \leq \|We^{At}b\|_1\} \quad (5)$$

is an invariant set. Let Ω_- and Ω_+ denote $\Omega \cap S_-$ and $\Omega \cap S_+$ respectively. Our goal is to seek an Ω such that Ω_+ does not contain any discontinuities of the Poincaré map, $P_+(\xi)$. This enables us to obtain a Lipschitz condition on $P_+(\xi)$ with the Lipschitz constant, ρ , being the bound given by (4). By symmetry, the same result holds for $P_-(\eta)$ and hence it holds for $P = P_- \circ P_+$ also with ρ^2 . If $\rho < 1$, then P is contractive over the invariant set Ω_+ and hence it has unique fixed point.

A discontinuity of $P_+(\xi)$ occurs when the trajectory starting at $\xi \in S_+$ hits S_- tangentially at η . The set of points on S_- where this is possible is

$$\dot{S}_- := \{\eta : c\eta = -1 \ \& \ c(A\eta + b) = 0\}.$$

We use the dot notation to emphasize that this is the set where $\dot{y} = 0$ with $u = +1$. Borrowing the terminology of Bliman and Krasnoselskii [7], this set will be referred to as the *set of anomalous points*. The corresponding set for P_- is

$$\dot{S}_+ := \{\xi : c\xi = 1 \ \& \ c(A\xi - b) = 0\}.$$

The invariant set Ω and the anomalous sets \dot{S}_+ and \dot{S}_- are sketched in Fig. 2 for a three dimensional system. The sets \dot{S}_- and \dot{S}_+ are linear manifolds and can also be represented as

$$\begin{aligned} \dot{S}_- &= -\dot{s} + \mathcal{N}\left(\begin{bmatrix} c \\ cA \end{bmatrix}\right) \\ \dot{S}_+ &= \dot{s} + \mathcal{N}\left(\begin{bmatrix} c \\ cA \end{bmatrix}\right) \end{aligned}$$

where \dot{s} is the particular solution of

$$\begin{bmatrix} c \\ cA \end{bmatrix} \xi = \begin{bmatrix} 1 \\ cb \end{bmatrix}.$$

The weighting matrix W in the definition of Ω is to be chosen such that the anomalous sets \dot{S}_- and \dot{S}_+ are outside Ω . Let the scaled distance from the origin to \dot{S}_- or \dot{S}_+ be $\delta := \|W\dot{s}\|$ and the radius of Ω be $\mu := \|We^{At}b\|_1$. We then require $\delta > \mu$. In general, this could be very conservative since not all points on \dot{S}_- and \dot{S}_+ are discontinuities. Selecting Ω in such a way allows us to obtain a bound on the norm of $\Pi = \frac{(A\eta+b)c}{c(A\eta+b)}$ without actually computing the exact image $\eta \in \Omega_-$ of the given $\xi \in \Omega_+$ under P_+ . The bound is obtained by taking the supremum of $\|\Pi\|$ over $\eta \in \Omega_-$ which contains the range of P_+ .

The next step is to obtain a bound on $\|e^{At}\|$. It has been shown in [4] that we can obtain upper and lower bounds on the asymptotic switching time. Thus, $t \in [t_-, t_+]$ when $\xi \in \Omega_+$ and we need to find $\max \|e^{At}\|$ over this interval. Even if A is stable, it is not straight forward to provide such a bound. If A has non-trivial Jordan structure, the supremum norm may be achieved at some intermediate t in $(0, \infty)$ and it can be very high. We can use a similarity transformation on the system to make A block-diagonal with the off-diagonal elements as small as possible. This can reduce $\|e^{At}\|$ below unity so that the global stability test can be applied. This reduction in $\|e^{At}\|$ comes at an expense of increasing $\|\Pi\|$. This is acceptable since we have some control over $\|\Pi\|$ via the scaling matrix W . We can now state the global stability result as below:

Proposition 2 *Suppose the system in Fig. 1 with P described by (1)–(2) has a unique, locally stable symmetric unimodal periodic orbit. Suppose further that there exists a symmetric positive definite matrix W such that $\delta > \mu$ is satisfied i.e., $\|W\dot{s}\| > \|We^{At}b\|_1$ and the inequality $\rho < 1$ holds where*

$$\rho := \max_{\eta \in \Omega_-} \frac{\|A\eta + b\| \cdot \|c\|}{|c(A\eta + b)|} \cdot \max_{t \in [t_-, t_+]} \|e^{At}\|$$

and $\Omega_- = \Omega \cap S_-$ with Ω given by (5). Then the unimodal orbit is globally stable.

For two-dimensional systems, a suitable W can be selected, in most cases, manually. For higher dimensional systems, a numerical search procedure that minimizes the maximum norm of Π is required. The following example illustrates the application of the above result for a two-dimensional system.

Example 1 Consider the system given by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix}'.$$

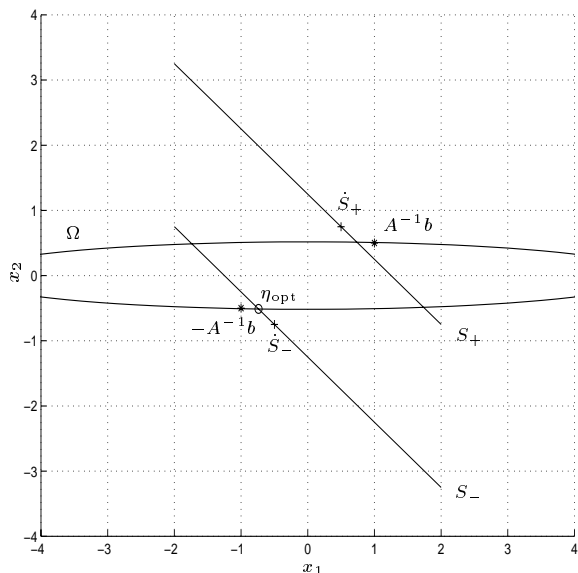


Figure 3: An invariant set Ω satisfying $\|D_{P_+}\| < 1$ for Example 1. The maximum of $\|\Pi\|$ occurs at η_{opt} marked by a circle.

The set of anomalous points on S_- for this system is $\dot{S}_- = \left\{ \begin{bmatrix} -0.5 \\ -0.75 \end{bmatrix} \right\}$ and with $W = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$, this set lies outside Ω_- . The lower bound on the switching time can be obtained to be 0.625 and $\max_{t \in [t_-, \infty)} \|e^{At}\| = e^{-t_-} = 0.5333$. The maximum norm of Π is 1.566 and it occurs at $\eta_{\text{opt}} = \begin{bmatrix} -0.7378 & -0.5122 \end{bmatrix}$. Thus we have $\|D_{P_+}\| \leq \rho = 0.8382 < 1$. Hence the orbit is globally stable. Fig. 3 shows the switching planes, the invariant set Ω , and the anomalous points \dot{S}_- and \dot{S}_+ . The steady states corresponding to $u = \pm 1$ are marked by asterisks and the point η_{opt} , by a circle. \square

The sufficient condition for global stability in Proposition 2 involves computations that are much easier than actually solving for the switching time for each state in Ω_+ . However the disadvantage is that for higher dimensional systems, it is not easy to find W satisfying $\delta > \mu$. When this condition is not satisfied, the maximization of $\|\Pi\|$ over Ω_- is ill-posed because the norm blows up on \dot{S}_- . Even when $\delta > \mu$ is satisfied, the bound, ρ , could be much larger than unity. However, when the linear system is strongly stable in the sense that $\max_{t \in [t_-, t_+]} \|e^{At}\| \ll 1$, then Proposition 2 could be very useful. We are currently working on an efficient search algorithm for W for higher dimensional systems.

4 Summary

In this paper we present a sufficient condition for the global stability of unimodal periodic orbits in relay feedback systems. The condition requires the existence of a positive definitive matrix that defines a compact invariant set of the system over which the Poincaré map is contractive. We illustrated by an example that the search for such a matrix can be performed easily for two-dimensional systems. For higher dimensional systems, a numerical optimization is required.

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