An Unified Framework for LPV System Analysis and Control Synthesis

Fen Wu
Dept. of Mechanical and Aerospace Engineering
North Carolina State University
Raleigh, NC 27695
e-mail: `fwu@eos.ncsu.edu`

Abstract

A new approach for LPV system analysis and control synthesis was proposed in this paper. This unified framework combined two seemingly diversified methods in systematic gain-scheduling, LPV control theory, and extended the appliability of full block S-procedure to general LPV systems.

1 Introduction

The study of LPV systems is motivated by the gain scheduling control design methodology [23, 17]. The classical approach to gain-scheduling involves the design of several LTI controllers for a parameterized family of linearized models of a system and the interpolation of the controller gains. Although it seems working well in practice, this heuristic design procedure does not take the parameter variations into account. In particular, it can not provide stability and performance guarantees except for slow varying parameters [24, 25].

The notion of LPV systems was first introduced in [23]. This class of systems is different from standard linear time-varying counterpart due to the causal dependence of its controller gains on the variations of the plant dynamics. The implications of parameter-dependent systems theory has for gain-scheduling is obvious, since gain-scheduling conceptually involves a linear, parameter-dependent plant. LPV control theory has been proven useful to simplify the interpolation and realization problems associated with conventional gain-scheduling. Specifically, it allows us to treat gain-scheduled controllers as a single entity, with the gain-scheduling achieved entirely by the parameter-dependent controller.

Using scaled small-gain theorem, a systematic gain-scheduling control design technique has been developed in [15, 2]. When the parameter dependency in both plant and controller is linear fractional, the existence of such a gain-scheduled controller is fully characterized in terms of linear matrix inequalities (LMIs). The underlying synthesis problem is therefore a convex problem for which efficient optimization techniques are available [5, 12]. This new and promising control structure is applicable whenever the value of parameter is measured in real-time. The resulting controller is time-varying and smoothly scheduled by the measurements of parameter. It was also demonstrated that the original gain-scheduling problem can be re-formulated as one of robust performance with special plant/uncertainty structure [2]. Recently, a general form of multipliers was recognized in the context of mixed μ analysis [13], which is capable to translate the robust performance condition to its equivalent analysis test. It is has been applied to LPV gain-scheduling problem to reduce the conservatism associated with block diagonal scaling [19]. However, full block multipliers necessitate modification on the structure of the scheduling functions.

Parallel to the first approach, the use of a single or parameter-dependent Lyapunov function in the analysis and control design for parameter-dependent plants has been studied in robust control framework [4, 27, 3, 28]. The LPV system is allowed to have general parameter-dependence other than trivial continuity requirement. Whereas the analysis test in [4] introduces a potential conservatism by measuring performance against arbitrarily fast variations in scheduling parameters, known bounds on the rate of parameter variation are incorporated into the analysis conditions in [27, 28, 1]. In contrast to scaled small gain approach, the solution to the LPV control synthesis problem was formulated as a parameter-depepdent LMI optimization problem. A brutal force griding method is typically used to divide the parameter space and renders the semi-infinite optimization problem to be finite, but resulting LMI constraints grow rapidly as the parameter numbers increases. Alternative solution was provided for affine parameter-dependent LMIs in [11, 8].

In this paper, a general framework was developed for LPV system analysis and synthesis problems. The motivation of this research is to combine the power of existing LPV control design techniques, and develop a computationally efficient gain-scheduling strategy applicable to sophisticated real systems. For a large-scale gain-scheduling problem involving lots of parameters, it is conceivable that existing LPV approaches will fail either because of performance inadequacy or computational complexity. Inspired by the paper [19], this research work expands the applicability of full block S-procedure to general LPV systems, and provides alternative solution approach for LPV control synthesis problem. Improved from the constant Lyapunov function, its parameter dependency is introduced to exploit parameter variation rates information and reduce the conservatism in LPV analysis test. The proposed LPV framework provides a useful gain-scheduling design technique which unifies two prevailing approaches from scaled small-gain and control Lyapunov function perspective.

2 Unified Framework for LPV system Analysis

Consider an LPV system

$$\begin{bmatrix} \dot{x} \\ q \\ e \end{bmatrix} = \begin{bmatrix} A(\theta,\dot{\theta}) & B_0(\theta,\dot{\theta}) & B_1(\theta,\dot{\theta}) \\ C_0(\theta,\dot{\theta}) & D_{00}(\theta,\dot{\theta}) & D_{01}(\theta,\dot{\theta}) \\ C_1(\theta,\dot{\theta}) & D_{10}(\theta,\dot{\theta}) & D_{11}(\theta,\dot{\theta}) \end{bmatrix} \begin{bmatrix} x \\ p \\ d \end{bmatrix} \tag{1}$$

$$p(t) = \Delta(t)q(t) \tag{2}$$

where $x, \dot{x} \in \mathbf{R}^n$, $d \in \mathbf{R}^{n_d}$, $q, p \in \mathbf{R}^{n_p}$, $e \in \mathbf{R}^{n_e}$, $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$. All the matrices have compatible dimensions. The state-space data are continuous functions of the scheduling parameter θ, and in linear fractional dependency of the structured scheduling parameter Δ.

It is assumed that the vector-valued parameter θ evolves continuously over time and its range is limited to a compact subset $\Theta \subset \mathbf{R}^s$. In addition, its time derivative is bounded and satisfies the constraint $\underline{\nu}_i \le \dot{\theta}_i \le \bar{\nu}_i, i = 1, 2, \cdots, s$. For notational purpose, denote $\mathcal{V} = \{v : \ \underline{\nu}_i \le v \le \bar{\nu}_i, \ i = 1, 2, \cdots, s\}$, where $\mathcal{V}$ is a given convex polytope in $\mathbf{R}^s$ that contains the origin. Given the sets Θ and $\mathcal{V}$, we define the parameter ν-variation set as

$$\mathcal{F}_\Theta^\nu = \left\{ \theta \in \mathcal{C}^1(\mathbf{R}_+, \mathbf{R}^s) : \ \theta(t) \in \Theta, \ \dot{\theta} \in \mathcal{V}, \ \forall t \ge 0 \right\}$$

Moreover, the time-varying parameter Δ obeys the following structure

$$\begin{aligned} \mathit{\Delta} = \{ \Delta = \operatorname{diag}\{ & \delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_f I_{r_f} \} : \\ & \delta_i \in \mathcal{C}(\mathbf{R}_+, \mathbf{R}), \ |\delta_i| \le 1, \ i = 1, 2, \cdots, f \} \end{aligned}$$

where $\sum_{i=1}^f r_i = n_p$. Then the parameter varying set $\mathcal{F}_{\mathit{\Delta}}$ can be defined correspondingly. At each time instant t, the parameter vector $(\theta(t), \dot{\theta}(t), \Delta(t))$ is assumed measurable in real-time. So $\mathcal{F}_\Theta^\nu \times \mathcal{F}_{\mathit{\Delta}}$ specifies the set of all allowable parameter trajectories.

Clearly, the state-space matrices of the LPV system can be written as

$$\begin{aligned} \begin{bmatrix} \mathcal{A}(\theta,\dot{\theta},\Delta) & \mathcal{B}_1(\theta,\dot{\theta},\Delta) \\ \mathcal{C}_1(\theta,\dot{\theta},\Delta) & \mathcal{D}_{11}(\theta,\dot{\theta},\Delta) \end{bmatrix} &= \begin{bmatrix} A(\theta,\dot{\theta}) & B_1(\theta,\dot{\theta}) \\ C_1(\theta,\dot{\theta}) & D_{11}(\theta,\dot{\theta}) \end{bmatrix} \\ &+ \begin{bmatrix} B_0(\theta,\dot{\theta}) \\ D_{10}(\theta,\dot{\theta}) \end{bmatrix} (I - \Delta D_{00}(\theta,\dot{\theta}))^{-1} \Delta \begin{bmatrix} C_0(\theta,\dot{\theta}) & D_{01}(\theta,\dot{\theta}) \end{bmatrix} \end{aligned}$$

The LPV system (1)–(2) is said to be parameter-dependent stable if there exists a continuously differentiable matrix function $X(\theta) = X^T(\theta) > 0$, such that for $(\theta, v, \Delta) \in \Theta \times \mathcal{V} \times \mathit{\Delta}$

$$\mathcal{A}^T(\theta, v, \Delta) X(\theta) + X(\theta) \mathcal{A}(\theta, v, \Delta) + \sum_{i=1}^s v_i \frac{\partial X}{\partial \theta_i} < 0 \tag{3}$$

By compactness of the parameter sets, it can be deduced that parameter-dependent stability implies exponential stability of the LPV system. Note that the stability condition (3) is given in terms of the scheduling parameters and the LPV system matrices. Generally, it consists of infinite number of constraints and is hard to verify. Although gridding method can be used to divide the whole parameter space $\Theta \times \mathcal{V} \times \mathit{\Delta}$, past experience reveals that the computational cost resulting from gridding parameter space is quite significant. Therefore there is great incentive to reduce the number of gridded parameters whenever it is possible.

The full block S-procedure [13, 19] would effectively translate the stability and performance tests for uncertain systems into their equivalent formulations. This approach is applicable to a general LPV problem as well. For this purpose, we introduce the class of full block multipliers

$$\mathcal{P} = \left\{ P \in \mathcal{S}^{2n_p \times 2n_p} : \ \begin{bmatrix} \Delta^T & I \end{bmatrix} P \begin{bmatrix} \Delta \\ I \end{bmatrix} > 0, \ \forall \Delta \in \mathit{\Delta} \right\}$$

Note that such a multiplier is specified over the parameter set $\mathit{\Delta}$. It consists of infinitely many LMI constraints in general. However, the multiplier set can be simplified under various assumptions on the parametric uncertainty Δ [13]. This will be further discussed later in this section.

Using a full block scaling matrix $P \in \mathcal{P}$, the original stability condition (3) can be reformulated as

$$\begin{bmatrix} \star \\ \hline \star \end{bmatrix}^T \left[\begin{array}{cc|c} \sum_{i=1}^s v_i \frac{\partial X}{\partial \theta_i} & X(\theta) & 0 \\ X(\theta) & 0 & \\ \hline 0 & & P \end{array} \right] \times \left[\begin{array}{cc} I & 0 \\ A(\theta, v) & B_0(\theta, v) \\ \hline 0 & I \\ C_0(\theta, v) & D_{00}(\theta, v) \end{array} \right] < 0 \tag{4}$$

for all $(\theta, v, \Delta) \in \Theta \times \mathcal{V} \times \Delta$. Note that the above matrix inequality (4) is specified using nominal LPV system and its inter-connection data. It is a convex constraint in the form of LMI, and can be solved by LMI optimization tools efficiently [5, 12]. Moreover, the resulting LMI condition does not explicitly rely on scheduling parameter Δ.

In light of stability condition, we have the following result to determine the stability and performance for the class of LPV systems using a parameter-dependent control Lyapunov function, which has restricted parameter dependency on θ only. However, the analysis test derived from full block S-procedure is not necessarily lossless for the LPV systems (1)–(2) due to its general parameter dependency.

Theorem 2.1 *Given compact sets Θ, $\mathcal{V}$ and Δ, If there exist continuously differentiable function $X : \mathbf{R}^s \rightarrow \mathcal{S}_+^{n\times n}$ and scaling matrix $P \in \mathcal{P}$, such that*

$$\left[\begin{array}{c} \star \\ \hline \star \\ \hline \star \end{array}\right]^T \left[\begin{array}{cc|c|cc} \sum_{i=1}^s v_i \frac{\partial X}{\partial \theta_i} & X(\theta) & 0 & 0 & 0 \\ X(\theta) & 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & P & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 0 & -\gamma I & 0 \\ \multicolumn{2}{c|}{0} & 0 & 0 & \frac{1}{\gamma} I \end{array}\right]$$

$$\times \left[\begin{array}{ccc} I & 0 & 0 \\ A(\theta,v) & B_0(\theta,v) & B_1(\theta,v) \\ \hline 0 & I & 0 \\ C_0(\theta,v) & D_{00}(\theta,v) & D_{01}(\theta,v) \\ \hline 0 & 0 & I \\ C_1(\theta,v) & D_{10}(\theta,v) & D_{11}(\theta,v) \end{array}\right] < 0 \qquad (5)$$

for all $(\theta, v, \Delta) \in \Theta \times \mathcal{V} \times \Delta$, then the LPV system (1)–(2) is parameter-dependent stable. In addition, if $d \in \mathcal{L}_2$ and $x(0) = 0$, then $\|e\|_2 < \gamma \|d\|_2$.

The multiplier set $P \in \mathcal{P}$ itself presents some interesting questions in terms of computational complexity. First, $\mathcal{P}$ is clearly a convex set. But it is generally semi-infinite dimensional in the sense that every possible elements in the parameter set Δ needs to be checked. Seemingly computational demanding, however, the multiplier set can be simplified considerably by imposing additional assumptions on the set Δ. Given the block diagonal assumption on the parameter Δ, it is possible to confine the search of multipliers to a smaller set defined as

$$\mathcal{P}^c = \left\{ P \in \mathcal{P} : P = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix},\ Q < 0 \right\}$$

Obviously, $\mathcal{P}^c \subset \mathcal{P}$. In additional, it can easily be shown [16, 13, 19] that the multiplier set $\mathcal{P}^c$ is fully characterized by finitely many (2^f) LMIs specified at the vertices of Δ, i.e., $P \in \mathcal{P}^c$ if and only if

$$Q < 0 \quad \text{and} \quad \begin{bmatrix} \Delta_v^T & I \end{bmatrix} P \begin{bmatrix} \Delta_v \\ I \end{bmatrix} > 0,\ \forall \Delta_v \in \{\underline{\Delta}, \bar{\Delta}\}$$

The restriction of search of multipliers to $\mathcal{P}^c$ generally would cause more conservatism in the analysis condition (5).

3 LPV Output-Feedback Control

Now, consider a generalized open-loop LPV system, which may have all the weighting functions included

$$\begin{bmatrix} \dot{x} \\ q \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\theta) & B_0(\theta) & B_1(\theta) & B_2(\theta) \\ C_0(\theta) & 0 & 0 & D_{02}(\theta) \\ C_1(\theta) & 0 & 0 & D_{12}(\theta) \\ C_2(\theta) & D_{20}(\theta) & D_{21}(\theta) & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ d \\ u \end{bmatrix} \qquad (6)$$

$$p(t) = \Delta(t) q(t) \qquad (7)$$

where $\theta \in \mathcal{F}_\Theta^\nu$, $\Delta \in \mathcal{F}_\Delta$, $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$. All matrix valued state-space data are continuous and have appropriate dimensions. For simplicity, we assume that

(A1) $(A(\theta), B_2(\theta), C_2(\theta))$ is parameter-dependent stabilizable and detectable for all θ,

(A2) $\begin{bmatrix} D_{02}(\theta) \\ D_{12}(\theta) \end{bmatrix}$ and $\begin{bmatrix} D_{20}(\theta) & D_{21}(\theta) \end{bmatrix}$ have full column and row rank respectively for all θ,

The first assumption is required for the existence of a stabilizing output-feedback controller. Assumptions ($A2$) and ($A3$) can be relaxed at the expense of more complicated controller formulae (see [26, 9] for details).

The class of LPV controllers is in the form of

$$\left[\begin{array}{c} \dot{x}_k \\ \hline u \\ q_k \end{array}\right] = \begin{bmatrix} A_k(\theta,\dot{\theta}) & B_k(\theta,\dot{\theta}) \\ C_k(\theta,\dot{\theta}) & D_k(\theta,\dot{\theta}) \end{bmatrix} \left[\begin{array}{c} x_k \\ \hline y \\ p_k \end{array}\right] \qquad (8)$$

$$p_k(t) = \Delta_k(\Delta(t)) q_k(t) \qquad (9)$$

where $x_k \in \mathbf{R}^{n_k}$, $p_k \in \mathbf{R}^{n_{p_k}}$ and $q_k \in \mathbf{R}^{n_{q_k}}$. The dimension of controller state n_k and scheduling function $\Delta_k : \mathbf{R}^f \rightarrow \mathbf{R}^{n_{p_k} \times n_{q_k}}$ is yet to be determined. The controller is scheduled by parameters $\theta, \dot{\theta}$ and Δ. In particular, the scheduling function $\Delta_k(\cdot)$ reflects the controller's dependency on Δ and enters the controller state-space data in linear fractional fashion. Note that the previous LPV control technique in LFT approach [15, 22] assumed that $\Delta_k(\Delta) = \Delta$. Consequently, block diagonal scaling matrices is adequate for this type of gain-scheduling control problem. For full block scaling, It was shown in [19] that one has to employ a quadratic function which is constructed from the corresponding multipliers.

Our goal is to design a LPV controller in the form of (8)–(9) to stabilizes the LPV system (6)–(7) with induced $\mathcal{L}_2$ norm of closed-loop system minimized.

In the LPV control synthesis phase, the full block extended multiplier is restricted to the following form

$$P_{cl} = \begin{bmatrix} Q_{cl} & S_{cl} \\ S_{cl}^T & R_{cl} \end{bmatrix} = \left[\begin{array}{cc|cc} Q & Q_{12} & S & S_{12} \\ Q_{12}^T & Q_{22} & S_{12}^T & S_{22} \\ \hline \multicolumn{2}{c|}{\star} & R & R_{12} \\ & & R_{12}^T & R_{22} \end{array}\right]$$

with $Q_{cl} < 0, R_{cl} > 0$ and P_{cl} satisfies

$$\begin{bmatrix} \Delta_{cl}^T & I \end{bmatrix} P_{cl} \begin{bmatrix} \Delta_{cl} \\ I \end{bmatrix} > 0.$$

This multiplier set will be denoted as $\mathcal{P}_e^c$. Admitted to possible conservatism for this refinement, the purpose for employing this type of full block multiplier in synthesis stage is two folds: When $Q_{cl} < 0$, the constraint associated with such a multiplier set is much simplified. In fact, it becomes finite number of LMIs specified at the vertices of the parameter set $\mathit{\Delta} \times \mathit{\Delta}_k$. Secondly, it renders the implicit scheduling function to an explicit quadratic form [20] and alleviates the controller implementation requirements.

We are now ready to state the main theorem for the LPV output-feedback control synthesis problem.

Theorem 3.1 *Given compact sets $\mathit{\Theta}$ and $\mathit{\Delta}$, a performance level $\gamma > 0$, and the open-loop LPV system in (6)–(7), there exist continuously differentiable matrix functions $X, Y : \mathbf{R}^s \to \mathcal{S}_+^{n\times n}$ and a pair of scaling matrices $P \in \mathcal{P}^c$, $\tilde{P} \in \tilde{\mathcal{P}}^c$, such that for all $\theta \in \mathit{\Theta}$,*

$$\mathcal{N}_X^T \left[\begin{array}{c} \star \\ \hline \star \\ \hline \star \end{array}\right]^T \left[\begin{array}{cc|c|cc} 0 & X(\theta) & 0 & 0 & \\ X(\theta) & -\{\underline{\nu}, \bar{\nu}\}\frac{dX}{d\theta} & & & \\ \hline \multicolumn{2}{c|}{0} & \tilde{P} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 0 & -\frac{1}{\gamma}I & 0 \\ & & & 0 & \gamma I \end{array}\right] \times \left[\begin{array}{ccc} -A^T(\theta) & -C_0^T(\theta) & -C_1^T(\theta) \\ I & 0 & 0 \\ \hline -B_0^T(\theta) & 0 & 0 \\ 0 & I & 0 \\ \hline -B_1^T(\theta) & 0 & 0 \\ 0 & 0 & I \end{array}\right] \mathcal{N}_X > 0 \tag{10}$$

$$\mathcal{N}_Y^T \left[\begin{array}{c} \star \\ \hline \star \\ \hline \star \end{array}\right]^T \left[\begin{array}{cc|c|cc} \{\underline{\nu}, \bar{\nu}\}\frac{dY}{d\theta} & Y(\theta) & 0 & 0 & \\ Y(\theta) & 0 & & & \\ \hline \multicolumn{2}{c|}{0} & P & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 0 & -\gamma I & 0 \\ & & & 0 & \frac{1}{\gamma}I \end{array}\right] \times \left[\begin{array}{ccc} I & 0 & 0 \\ A(\theta) & B_0(\theta) & B_1(\theta) \\ \hline 0 & I & 0 \\ C_0(\theta) & 0 & 0 \\ \hline 0 & 0 & I \\ C_1(\theta) & 0 & 0 \end{array}\right] \mathcal{N}_Y < 0 \tag{11}$$

$$\begin{bmatrix} Y(\theta) & I \\ I & X(\theta) \end{bmatrix} \geq 0 \tag{12}$$

where

$$\mathcal{N}_X(\theta) = \mathrm{Ker}\begin{bmatrix} B_2^T(\theta) & D_{02}^T(\theta) & D_{12}^T(\theta) \end{bmatrix},$$
$$\mathcal{N}_Y(\theta) = \mathrm{Ker}\begin{bmatrix} C_2(\theta) & D_{20}(\theta) & D_{21}(\theta) \end{bmatrix}$$

Let

$$(P - \tilde{P}^{-1})^{-1} = U\mathrm{diag}\{\Lambda_-, \Lambda_+\}\, U^T$$

where U is an orthonormal matrix, $\Lambda_- < 0$ and $\Lambda_+ > 0$. Then the scaling matrix P_{cl} can be chosen as

$$P_{cl} = \begin{bmatrix} Q_{cl} & S_{cl} \\ S_{cl}^T & R_{cl} \end{bmatrix} = \left[\begin{array}{c} \star \\ \hline \star \end{array}\right]^T \begin{bmatrix} P & U \\ U^T & \mathrm{diag}\{\Lambda_-, \Lambda_+\} \end{bmatrix} \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{array}\right]$$

Now solve for a pair of matrix equations

$$\left[\begin{array}{cc|cccc} \multicolumn{2}{c|}{0} & D_{20} & 0 & D_{21} & 0 \\ & & 0 & I & 0 & 0 \\ \hline D_{20}^T & 0 & \tilde{Q}_{cl}^{-1} & 0 & S_{cl}R_{cl}^{-1} & 0 \\ 0 & I & 0 & -\gamma I & 0 & 0 \\ D_{21}^T & 0 & 0 & 0 & -R_{cl}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma I \end{array}\right] \left[\begin{array}{c} \hat{B}_k^T(\theta) \\ \hline \star \end{array}\right] = -\left[\begin{array}{c} C_2 \\ 0 \\ \hline B_0^T Y \\ 0 \\ B_1^T Y \\ C_1 \end{array}\right] \tag{13}$$

$$\left[\begin{array}{cc|cccc} \multicolumn{2}{c|}{0} & 0 & D_{02}^T & 0 & D_{12}^T \\ & & 0 & 0 & I & 0 \\ \hline 0 & 0 & \tilde{Q}_{cl}^{-1} & 0 & S_{cl}R_{cl}^{-1} & 0 \\ D_{02} & 0 & 0 & -\gamma I & 0 & 0 \\ 0 & I & R_{cl}^{-1}S_{cl}^T & 0 & -R_{cl}^{-1} & 0 \\ D_{12} & 0 & 0 & 0 & 0 & -\gamma I \end{array}\right] \left[\begin{array}{c} \hat{C}_k(\theta) \\ \hline \star \end{array}\right] = -\left[\begin{array}{c} B_2^T \\ 0 \\ \hline B_1^T \\ C_0 X \\ 0 \\ C_1 X \end{array}\right] \tag{14}$$

and set $\hat{A}_k$ as

$$\begin{aligned} \hat{A}_k(\theta, \dot{\theta}) &= \dot{\theta}Y\frac{dX}{d\theta} + \dot{\theta}N\frac{dM^T}{d\theta} - A^T \\ &+ \left[Y\begin{bmatrix} B_0 & 0 & B_1 \end{bmatrix} + \hat{B}_k\begin{bmatrix} D_{20} & 0 & D_{21} \\ 0 & I & 0 \end{bmatrix} \quad \begin{bmatrix} C_0^T & 0 & C_1^T \end{bmatrix}\right] \\ &\times \left[\begin{array}{cc|cc} \tilde{Q}_{cl}^{-1} & 0 & S_{cl}R_{cl}^{-1} & 0 \\ 0 & -\gamma I & 0 & 0 \\ \hline R_{cl}^{-1}S_{cl}^T & 0 & -R_{cl}^{-1} & 0 \\ 0 & 0 & 0 & -\gamma I \end{array}\right]^{-1} \\ &\times \left[\begin{bmatrix} B_0 & 0 & B_1 \end{bmatrix} \quad X\begin{bmatrix} C_0^T & 0 & C_1^T \end{bmatrix} + \hat{C}_k^T\begin{bmatrix} D_{02}^T & 0 & D_{12}^T \\ 0 & I & 0 \end{bmatrix}\right]^T \end{aligned}$$

Then the state-space data of one n^{th} order, proper output-feedback controller is given by

$$A_k(\theta,\dot{\theta}) = N^{-1}\left\{\hat{A}_k - Y\begin{bmatrix}B_2 & 0\end{bmatrix}\hat{C}_k - \hat{B}_k\begin{bmatrix}C_2\\0\end{bmatrix}X\right.$$
$$\left. -YAX\right\}M^{-T} \tag{15}$$
$$B_k(\theta) = N^{-1}\hat{B}_k \tag{16}$$
$$C_k(\theta) = \hat{C}_k M^{-T} \tag{17}$$
$$D_k(\theta) = 0 \tag{18}$$

where $M(\theta)N^T(\theta) := I - X(\theta)Y(\theta)$. *The scheduling function* $\Delta_k(\dot{})$ *of size* $\dim(\Lambda_-)\times\dim(\Lambda_+)$. *is given by*

$$\Delta_k(\Delta) = \Lambda_-^{-1}V_-^T\left\{\begin{bmatrix}\Delta^T & I\end{bmatrix}P\begin{bmatrix}\Delta\\I\end{bmatrix} - V_-\Lambda_-^{-1}V_-^T\right\}^{-1}V_+$$

with $\begin{bmatrix}V_-(\Delta) & V_+(\Delta)\end{bmatrix} = \begin{bmatrix}\Delta^T & I\end{bmatrix}U$ *having a column partition compatible with* Λ_-, Λ_+.

Remark 3.1 *With a slight abuse of notation,* $\{\underline{\nu},\bar{\nu}\}\frac{d}{d\theta}$ *in (10)–(11) represents the combination of derivative terms in the form of* $\nu_i\frac{\partial}{\partial\theta_i}$ *when* ν_i *is taken as either* $\underline{\nu}_i$ *or* $\bar{\nu}_i$. *Therefore each inequality means* 2^s *different LMIs which must be checked.*

The general LPV synthesis conditions (10)–(12) do not include the controller data. Instead the controller state-space matrices are constructed from the solutions $X(\theta), Y(\theta)$ and $P, \tilde{P}$ explicitly. This approach has several advantages:

1. For a general LPV system we are interested in, the controller would have arbitrary dependency on the parameter θ. Consequently, a priori parameterization of controller data on the parameter will result in performance sacrifice.
2. It reduces decision variables involved in the LMI synthesis conditions. Moreover, the explicit control formula is essential for real-time implementation.
3. It gives further insight into the structure of the LPV controllers.

The solvability conditions provided in (10)–(12) are clearly infinite-dimensional, as is the solution space. To approximate, we restrict the search of parameter-dependent Lyapunov function to a span of finite number of basis functions. That is, let

$$X(\theta) = \sum_{i=1}^{N_x} h_i(\theta)X_i, \quad Y(\theta) = \sum_{j=1}^{N_y} g_j(\theta)Y_j$$

where $\{h_i(\theta)\}_{i=1}^{N_x}$ and $\{g_j(\theta)\}_{i=1}^{N_y}$ are user-specified scalar basis functions. X_i, Y_j are new optimization variables to be determined. After such a parameterization, the LPV synthesis conditions can be solved using an ad-hoc gridding method over parameter space.

The unified LPV synthesis framework is computationally attractive compared with the current parameter-dependent Lyapunov function approach. Given the parameter-dependent Lyapunov function having functional dependency on s parameters over total $(s+f)$ number of scheduling parameter. If the parameter sets Θ and Δ were gridded by N_s and N_f points, respectively, then the current approach requires $(2^{s+1}+1)N_sN_f$ LMI constraints. On the other hand, the unified LPV synthesis conditions involve $(2^{s+1}+1)N_s + 2^{f+1}$ LMIs. The comparison of decision variables for both approaches is $\frac{n(n+1)(N_x+N_y)}{2}$ vs. $\frac{n(n+1)(N_x+N_y)}{2} + 4n_p(n_p+1)$. Typically, N_s, N_f are large numbers which cause the LPV synthesis problem computationally expensive. With moderate increase of decision variables, the proposed approach would significantly reduce the number of LMI constraints required to solve the LPV synthesis problems. Moreover, it is always possible to scale full block multiplier back to block diagonal scaling matrices for large-scale gain scheduling problems. Then the number of decision variables in LMI optimization will further be decreased.

4 Concluding Remarks

A new framework for the study of a general class of LPV systems was developed. The LPV state-space data is assumed to depend on parameters θ and Δ, with Δ enters the plant dynamics in linear fractional fashion. Combining the power of parameter-dependent Lyapunov function and full block S-procedure, it was shown that the solution to the general LPV analysis and synthesis problems can be formulated into convex conditions in terms of LMI optimization. This research work extended the applicability of full block S-procedure to general LPV systems, and provided a useful framework that unifies two prevailing methods in LPV control theory, namely, scaled small-gain [15, 2, 19] and control Lyapunov function approach [4, 27, 28, 1]. From past experience, the computational cost associated with parameter-dependent LMIs becomes prohibitive when the number of scheduling parameters is more than 3. This framework provides a flexible approach for control engineers to trade between performance improvement, controller complexity and design effort required.

The synthesized output-feedback controller has the dependency on parameter derivative θ, which may be an undesirable feature from implementation viewpoint. But this can be easily remedied by forcing either X or Y as parameter independent [3, 1]. In many cases, such a restriction would not cause significant performance degradation.

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