On Stable $\mathcal{H}_\infty$ Controller Parameterization using Doubly Coprime Fractional Representation

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Abstract

Suboptimal $\mathcal{H}_\infty$ controller is said to internally stabilize the closed loop transfer matrix. However, the stability of controller is not assured in conventional $\mathcal{H}_\infty$ control theory. Unstable controller can damage the whole system when the sensor fails or actuator saturates. The sufficient condition for the existence of stable $\mathcal{H}_\infty$ controller is suggested. This paper presents the stable n-dimensional $\mathcal{H}_\infty$ controller and its parameterization based on the suggested sufficient condition for the linear time invariant systems using doubly coprime factorization. The stability of $\mathcal{H}_\infty$ controller and closed-loop stability are guaranteed if the positive semi-definite solutions for the suggested three Riccati equations exist.

1 Introduction

The stability of controller has been neglected in the design procedure for a controller. However, if sensor failure or actuator saturation happens, then stable controllers can relatively protect the entire control system comparing to unstable controllers. Also, the unstable controller brings the undesired right half plane zeros in the closed loop and it degrades the tracking performances and affects the sensitivity to disturbances. The problem dealing with the stable controller which stabilizes the closed-loop is said to be strong stabilizability. The necessary and sufficient condition on the existence of stable controller for strong stabilizability is the parity interlacing property, in other words, the plant is strongly stabilizable iff the number of poles of the plant between any pair of real right half plane blocking zeros of plant is even[4]. Hence, there exists a stable controller iff the plant has the parity interlacing property. Design method for stable LQG controller is suggested in [3], but this method brings the constraints for the covariance and weighting matrices. Zeren et al[6] studied the stable 2n-dimensional $\mathcal{H}_\infty$ controller by designing only Youla $\boldsymbol{Q}$ parameter, where n is the order of plant. Their method is based on the outer factorization for $\mathcal{H}_\infty$ central controller in the case that the controller obtained is unstable.

The primary goal of this paper is to parameterize a stable $\mathcal{H}_\infty$ controller using a doubly coprime factorization technique. However, we suggest only the parameterization based on the sufficient condition on the existence of stable $\mathcal{H}_\infty$ controller. The n-dimensional stable stabilizing $\mathcal{H}_\infty$ controllers are parameterized using free stable another parameter, not Youla $\boldsymbol{Q}$ parameter. For future notations, the Hardy space of stable and proper function is expressed by $\mathcal{RH}_\infty$, which denotes an analytic function in right half region of complex plane. The solution of Riccati equation satisfies:

1. $\boldsymbol{X} = \boldsymbol{X}^* \geq \boldsymbol{0}$,
2. $\boldsymbol{A}^*\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A} + \boldsymbol{X}\boldsymbol{R}\boldsymbol{X} + \boldsymbol{Q} = \boldsymbol{0}$ and
3. $\boldsymbol{A} + \boldsymbol{R}\boldsymbol{X}$ is stable,

where $\boldsymbol{A}^*$ expresses the Hermitian of $\boldsymbol{A}$. The lower linear fractional transformation of $\boldsymbol{G}$ on $\boldsymbol{K}$ is represented by

$$\mathcal{F}_l(\boldsymbol{G},\boldsymbol{K}) = \boldsymbol{G}_{11} + \boldsymbol{G}_{12}\boldsymbol{K}(\boldsymbol{I} - \boldsymbol{G}_{22}\boldsymbol{K})^{-1}\boldsymbol{G}_{21}$$

where $\boldsymbol{I}$ is an identity matrix of suitable dimension. Also, the proper controller $\boldsymbol{K}$ is said to be admissible if it internally stabilizes the plant $\boldsymbol{G}$.

2 Preliminaries

Let us assume that the generalized plant $\boldsymbol{G}(s)$ is minimal and expressed by

$$\boldsymbol{G}: \begin{bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \hline \boldsymbol{C}_1 & \boldsymbol{0} & \boldsymbol{D}_{12} \\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{bmatrix} \quad (1)$$

where $\boldsymbol{x}$ is the state vector, $\boldsymbol{u}$ is the control input vector, $\boldsymbol{z}$ is the controlled output vector, $\boldsymbol{y}$ is the output vector and $\boldsymbol{w}$ is the disturbance caused by unmodeled dynamics, modeling error, sensor noise and etc. Here, we assume that the following properties are satisfied for a given system of (1):

1. $(\boldsymbol{A},\boldsymbol{B}_2)$ is stabilizable and $(\boldsymbol{A},\boldsymbol{C}_2)$ is detectable,
2. $\boldsymbol{D}_{12}^*[\boldsymbol{C}_1 \;\; \boldsymbol{D}_{12}] = [\boldsymbol{0} \;\; \boldsymbol{I}]$,
3. $\begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{D}_{21} \end{bmatrix} \boldsymbol{D}_{21}^* = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$,

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4. $\begin{bmatrix} \boldsymbol{A} - j\varpi \boldsymbol{I} & \boldsymbol{B}_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$ has full column rank and

5. $\begin{bmatrix} \boldsymbol{A} - j\varpi \boldsymbol{I} & \boldsymbol{B}_1 \\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} \end{bmatrix}$ has full row rank for all ϖ,

where $\boldsymbol{0}$ is the zero matrix of suitable dimension and ϖ expresses the frequency. Assumption 4 and 5 mean that the feedback connection between the generalized plant and controller is detectable and stabilizable, respectively. If the assumption 2 and 3 are not satisfied for the generalized plant, then the loop transformation and scaling may be required to satisfy assumption 2 and 3[2, 7]. Suppose $\boldsymbol{X} \geq \boldsymbol{0}$ satisfying the following Riccati equation exists:

$$\boldsymbol{A}^*\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A} + \gamma^{-2}\boldsymbol{X}\boldsymbol{B}_1\boldsymbol{B}_1^*\boldsymbol{X} - \boldsymbol{X}\boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X} + \boldsymbol{C}_1^*\boldsymbol{C}_1 = \boldsymbol{0}. \tag{2}$$

In $\mathcal{H}_\infty$ control theory, there are the characteristics which the contraction mapping and internal stability are preserved under the inner linear fractional transformation. The following Lemma (Lemma 13.29 and Corollary 13.30 in [7]) can guide the inner linear fractional transformation for the generalized plant (1).

Lemma **1** *Suppose* $\boldsymbol{N} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right]$ *is stable and minimal, and* $\boldsymbol{X}$ *is the Observability Gramian:*

$$\boldsymbol{A}^*\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A} + \boldsymbol{C}^*\boldsymbol{C} \;=\; \boldsymbol{0}.$$

Then $\boldsymbol{N}$ *is an inner iff*

1. $\boldsymbol{D}^*\boldsymbol{C} + \boldsymbol{B}^*\boldsymbol{X} = \boldsymbol{0}$
2. $\boldsymbol{D}^*\boldsymbol{D} = \boldsymbol{I}$.

This inner characteristic was utilized to develop $\mathcal{H}_\infty$ state-space controller[1]. The following equation has the characteristic of inner function of Lemma 1:

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{z} \\ \gamma\boldsymbol{r} \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_F & \gamma^{-1}\boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \hline \boldsymbol{C}_{1F} & \boldsymbol{0} & \boldsymbol{D}_{12} \\ -\gamma^{-1}\boldsymbol{B}_1^*\boldsymbol{X} & \boldsymbol{I} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \gamma\boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}. \tag{3}$$

Also, the generalized plant of (1) can be modified using above inner linear fractional transformation as follows:

$$\boldsymbol{G}_{tmp1} : \begin{bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{v} \\ \boldsymbol{y} \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp1} & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \hline \boldsymbol{B}_2^*\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{r} \\ \boldsymbol{u} \end{bmatrix} \tag{4}$$

with new variables

$$\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{B}_2^*\boldsymbol{X}\boldsymbol{x} \qquad \boldsymbol{r} := \boldsymbol{w} - \gamma^{-2}\boldsymbol{B}_1^*\boldsymbol{X}\boldsymbol{x}$$

where

$$\begin{aligned} \boldsymbol{A}_F &= \boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X} \\ \boldsymbol{C}_{1F} &= \boldsymbol{C}_1 - \boldsymbol{D}_{12}\boldsymbol{B}_2^*\boldsymbol{X} \\ \boldsymbol{A}_{tmp1} &= \boldsymbol{A} + \gamma^{-2}\boldsymbol{B}_1\boldsymbol{B}_1^*\boldsymbol{X}. \end{aligned}$$

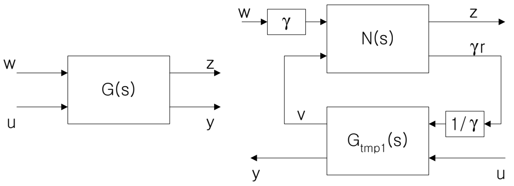

Figure 1: Inner LFT

This transformation is shown pictorially in the diagram of Figure 1: where $\boldsymbol{N}(s)$ is an inner function satisfying conditions of Lemma 1. In this setup, (3) is the inner linear fractional transformation and the following Lemma (Lemma 16.8 in [7]) is utilized in the sequel.

Lemma **2** *Assume that* $\boldsymbol{X} \geq \boldsymbol{0}$ *satisfying Riccati equation (2) exists. Then the controller* $\boldsymbol{K}$ *is admissible for* $\boldsymbol{G}$ *and* $\|\mathcal{F}_l(\boldsymbol{G}, \boldsymbol{K})\|_\infty < \gamma$ *iff* $\boldsymbol{K}$ *is admissible for* $\boldsymbol{G}_{tmp1}$ *and* $\|\mathcal{F}_l(\boldsymbol{G}_{tmp1}, \boldsymbol{K})\|_\infty < \gamma$.

To begin with, we assume that $(\boldsymbol{A}_{tmp1}, \boldsymbol{B}_2)$ and $(\boldsymbol{A}_{tmp1}, \boldsymbol{C}_2)$ for new generalized plant (4) are stabilizable and detectable. These assumptions will be removed later by showing the existence of feedback and observer gains. We repeat same procedures such as those given in previous paragraph using the dual concept. Also, we should perceive that $\mathcal{H}_\infty$ norm of the original system is equal to that of the dual system and $\boldsymbol{K}$ internally stabilizes $\boldsymbol{G}_{tmp1}$ iff $\boldsymbol{K}^*$ internally stabilizes the dual system $\boldsymbol{G}_{tmp1}^*$. Consider the dual system of (4) as the following form

$$\boldsymbol{G}_{tmp1}^* : \begin{bmatrix} \dot{\boldsymbol{x}}^* \\ \boldsymbol{v}^* \\ \boldsymbol{y}^* \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp1}^* & \boldsymbol{X}\boldsymbol{B}_2 & \boldsymbol{C}_2^* \\ \hline \boldsymbol{B}_1^* & \boldsymbol{0} & \boldsymbol{D}_{21}^* \\ \boldsymbol{B}_2^* & \boldsymbol{I} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{r}^* \\ \boldsymbol{u}^* \end{bmatrix} \tag{5}$$

where, $\boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{y}^*, \boldsymbol{r}^*$ and $\boldsymbol{u}^*$ mean the corresponding vectors for the dual system. Suppose $\boldsymbol{Y} \geq \boldsymbol{0}$ satisfying the following Riccati equation exists:

$$\begin{array}{c} \boldsymbol{A}_{tmp1}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{A}_{tmp1}^* - \boldsymbol{Y}\boldsymbol{C}_2^*\boldsymbol{C}_2\boldsymbol{Y} + \\ \gamma^{-2}\boldsymbol{Y}\boldsymbol{X}\boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{B}_1\boldsymbol{B}_1^* = \boldsymbol{0}. \end{array} \tag{6}$$

If the inner linear fractional transformation using above Riccati equation of (6) is applied to (5)

$$\begin{bmatrix} \dot{\boldsymbol{x}}^* \\ \boldsymbol{v}^* \\ \gamma\eta^* \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_L^* & \gamma^{-1}\boldsymbol{X}\boldsymbol{B}_2 & \boldsymbol{C}_2^* \\ \hline \boldsymbol{B}_{1L}^* & \boldsymbol{0} & \boldsymbol{D}_{21}^* \\ -\gamma^{-1}\boldsymbol{B}_2^*\boldsymbol{X}\boldsymbol{Y} & \boldsymbol{I} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x}^* \\ \gamma\boldsymbol{r}^* \\ \zeta^* \end{bmatrix}, \tag{7}$$

then we can obtain new modified generalized plant as following form:

$$\boldsymbol{G}_{tmp2}^* : \begin{bmatrix} \dot{\boldsymbol{x}}^* \\ \zeta^* \\ \boldsymbol{y}^* \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp2}^* & \boldsymbol{X}\boldsymbol{B}_2 & \boldsymbol{C}_2^* \\ \hline \boldsymbol{C}_2\boldsymbol{Y} & \boldsymbol{0} & \boldsymbol{I} \\ \widetilde{\boldsymbol{B}}_2^* & \boldsymbol{I} & \boldsymbol{0} \end{array}\right] \begin{bmatrix} \boldsymbol{x}^* \\ \eta^* \\ \boldsymbol{u}^* \end{bmatrix} \tag{8}$$

with new variables

$$\zeta^* := \boldsymbol{u}^* + \boldsymbol{C}_2\boldsymbol{Y}\boldsymbol{x}^* \qquad \eta^* := \boldsymbol{r}^* - \gamma^{-2}\boldsymbol{B}_2^*\boldsymbol{X}\boldsymbol{Y}\boldsymbol{x}^*$$

where

$$\begin{aligned}\boldsymbol{A}_L^* &= \boldsymbol{A}_{tmp1}^* - \boldsymbol{C}_2^*\boldsymbol{C}_2\boldsymbol{Y}\\ \boldsymbol{B}_{1L}^* &= \boldsymbol{B}_1^* - \boldsymbol{D}_{21}^*\boldsymbol{C}_2\boldsymbol{Y}\\ \widetilde{\boldsymbol{B}}_2^* &= \boldsymbol{B}_2^* + \gamma^{-2}\boldsymbol{B}_2^*\boldsymbol{X}\boldsymbol{Y}\\ \boldsymbol{A}_{tmp2}^* &= \boldsymbol{A}_{tmp1}^* + \gamma^{-2}\boldsymbol{X}\boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X}\boldsymbol{Y}.\end{aligned}$$

Also, (7) is the inner linear fractional transformation, therefore, the following Lemma for the dual system of (8) can be applied in the sequel and its proof can be easily achieved by using Lemma 2.

Lemma **3** *Assume that $\boldsymbol{X} \geq \boldsymbol{0}$ and $\boldsymbol{Y} \geq \boldsymbol{0}$ satisfying Riccati equations (2) and (6) exist. Then the controller $\boldsymbol{K}$ is admissible for $\boldsymbol{G}$ and $\|\mathcal{F}_l(\boldsymbol{G}, \boldsymbol{K})\|_\infty < \gamma$ iff $\boldsymbol{K}$ is admissible for $\boldsymbol{G}_{tmp2}$ and $\|\mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K})\|_\infty < \gamma$.*

3 Stable $\mathcal{H}_\infty$ Controller Parameterization

In this section, we will parameterize stable $\mathcal{H}_\infty$ controllers for the dual system of (8). Lemma 3 make it possible to parameterize $\mathcal{H}_\infty$ controller for the modified generalized plant (8) in place of the generalized plant (1). To begin with, we suggests the $\mathcal{H}_\infty$ controller parameterization by using left/right coprime factorization. And then, the stable n-dimensional $\mathcal{H}_\infty$ controller will be obtained using left/right coprime factors and the sufficient condition on the existence of stable $\mathcal{H}_\infty$ controller.

3.1 $\mathcal{H}_\infty$ Controller Parameterization

From now on, we are to parameterize $\mathcal{H}_\infty$ controller for the modified generalized plant:

$$\boldsymbol{G}_{tmp2} : \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp2} & \boldsymbol{Y}\boldsymbol{C}_2^* & \widetilde{\boldsymbol{B}}_2 \\ \hline \boldsymbol{B}_2^*\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_2 & \boldsymbol{I} & \boldsymbol{0} \end{array}\right] \tag{9}$$

where

$$\begin{aligned}\boldsymbol{A}_{tmp2} &= \boldsymbol{A}_{tmp1} + \gamma^{-2}\boldsymbol{Y}\boldsymbol{X}\boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X},\\ \widetilde{\boldsymbol{B}}_2 &= \boldsymbol{B}_2 + \gamma^{-2}\boldsymbol{Y}\boldsymbol{X}\boldsymbol{B}_2\end{aligned}$$

and $\boldsymbol{X}, \boldsymbol{Y}$ should be solutions for two Riccati equations of (2) and (6) by Lemma 3. For the new generalized plant $\boldsymbol{G}_{tmp2}$, if we assume that $(\boldsymbol{A}_{tmp2}, \widetilde{\boldsymbol{B}}_2)$ is stabilizable and $(\boldsymbol{A}_{tmp2}, \boldsymbol{C}_2)$ is detectable, then it is assured that $\mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K})$ is internally stable iff $\boldsymbol{K}$ internally stabilizes $(\boldsymbol{G}_{tmp2})_{22}$. These assumptions will be removed in the next section. Following paragraph presents the modified version for several Theorems on internal stability theory and Youla Parameterization in [2, 4, 7].

Consider $(\boldsymbol{G}_{tmp2})_{22} = \boldsymbol{C}_2(s\boldsymbol{I} - \boldsymbol{A}_{tmp2})^{-1}\widetilde{\boldsymbol{B}}_2$ and choose matrices $\boldsymbol{F}$ and $\boldsymbol{L}$ such that $\boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F}$ and $\boldsymbol{A}_{tmp2} + \boldsymbol{L}\boldsymbol{C}_2$ are stable. Then we can obtain the left and right coprime factorizations $(\boldsymbol{G}_{tmp2})_{22} = \boldsymbol{M}_l^{-1}\boldsymbol{N}_l = \boldsymbol{N}_r\boldsymbol{M}_r^{-1}$. The set of all proper controllers achieving internal stability is parameterized either by

$$\begin{aligned}\boldsymbol{K} &= -(\boldsymbol{X}_l - \boldsymbol{M}_r\boldsymbol{Q})(\boldsymbol{Y}_l + \boldsymbol{N}_r\boldsymbol{Q})^{-1}\\ &= -(\boldsymbol{Y}_r + \boldsymbol{Q}\boldsymbol{N}_l)^{-1}(\boldsymbol{X}_r - \boldsymbol{Q}\boldsymbol{M}_l)\end{aligned} \tag{10}$$

with $\boldsymbol{Q} \in \mathcal{RH}_\infty$. Moreover, $\boldsymbol{X}_l, \boldsymbol{Y}_l, \boldsymbol{X}_r, \boldsymbol{Y}_r$ are chosen such that

$$\begin{bmatrix} \boldsymbol{Y}_r & \boldsymbol{X}_r \\ -\boldsymbol{N}_l & \boldsymbol{M}_l \end{bmatrix}\begin{bmatrix} \boldsymbol{M}_r & -\boldsymbol{X}_l \\ \boldsymbol{N}_r & \boldsymbol{Y}_l \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}. \tag{11}$$

Then, the doubly coprime factorization for $(\boldsymbol{G}_{tmp2})_{22}$ can be chosen as

$$\begin{bmatrix} \boldsymbol{Y}_r & \boldsymbol{X}_r \\ -\boldsymbol{N}_l & \boldsymbol{M}_l \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp2} + \boldsymbol{L}\boldsymbol{C}_2 & \widetilde{\boldsymbol{B}}_2 & -\boldsymbol{L} \\ \hline -\boldsymbol{F} & \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{C}_2 & \boldsymbol{0} & \boldsymbol{I} \end{array}\right] \tag{12}$$

and

$$\begin{bmatrix} \boldsymbol{M}_r & -\boldsymbol{X}_l \\ \boldsymbol{N}_r & \boldsymbol{Y}_l \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F} & \widetilde{\boldsymbol{B}}_2 & -\boldsymbol{L} \\ \hline \boldsymbol{F} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C}_2 & \boldsymbol{0} & \boldsymbol{I} \end{array}\right]. \tag{13}$$

All stabilizing controllers (10) which internally stabilize $\boldsymbol{G}_{tmp2}$ are obtained using the free parameter $\boldsymbol{Q}$. However, this controllers are not $\mathcal{H}_\infty$ controller because they do not assure of satisfying $\mathcal{H}_\infty$ norm constraint for the closed loop. We should choose the feedback gain $\boldsymbol{F}$ and the observer gain $\boldsymbol{L}$ to be $\mathcal{H}_\infty$ controller. Now, we are to suggest the main Theorem which can guide the $\mathcal{H}_\infty$ controller parameterization.

Theorem **1** *Assume that $\boldsymbol{X} \geq \boldsymbol{0}$ of Riccati equation (2) and $\boldsymbol{Y} \geq \boldsymbol{0}$ of (6) exist and the set of all controllers (10) is applied to $\boldsymbol{G}_{tmp2}$, then the closed loop transfer matrix satisfies next relation:*

$$\mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K}) = \boldsymbol{Q},$$

if $\boldsymbol{F} = -\boldsymbol{B}_2^*\boldsymbol{X}$ and $\boldsymbol{L} = -\boldsymbol{Y}\boldsymbol{C}_2^*$.

Proof. First, consider the closed loop transfer matrix as follows:

$$\begin{aligned}\mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K}) &= (\boldsymbol{G}_{tmp2})_{11}\\ &+ (\boldsymbol{G}_{tmp2})_{12}\boldsymbol{K}(\boldsymbol{I} - (\boldsymbol{G}_{tmp2})_{22}\boldsymbol{K})^{-1}(\boldsymbol{G}_{tmp2})_{21}.\end{aligned} \tag{14}$$

And we utilize only the first form of (10) as $\boldsymbol{K}$ because the second form can be proved using same procedures. Second, we can derive next relations using coprime factors (12) and (13) after simple algebraic manipulations

$$\begin{aligned}(\boldsymbol{G}_{tmp2})_{12} &= \boldsymbol{N}_{r12}\boldsymbol{M}_r^{-1}\\ (\boldsymbol{G}_{tmp2})_{21} &= \boldsymbol{M}_l^{-1}\boldsymbol{N}_{l21}\\ (\boldsymbol{G}_{tmp2})_{22} &= \boldsymbol{M}_l^{-1}\boldsymbol{N}_l\end{aligned} \tag{15}$$

where

$$\boldsymbol{N}_{r12} = \left[\begin{array}{c|c} \boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F} & \widetilde{\boldsymbol{B}}_2 \\ \hline \boldsymbol{B}_2^*\boldsymbol{X} + \boldsymbol{F} & \boldsymbol{I} \end{array}\right] \tag{16}$$

and

$$\boldsymbol{N}_{l21} = \left[\begin{array}{c|c} \boldsymbol{A}_{tmp2} + \boldsymbol{L}\boldsymbol{C}_2 & \boldsymbol{Y}\boldsymbol{C}_2^* + \boldsymbol{L} \\ \hline \boldsymbol{C}_2 & \boldsymbol{I} \end{array}\right]. \tag{17}$$

Then the closed loop transfer matrix of (14) is reduced to

$$\begin{aligned} \mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K}) \;\; = \;\; & (\boldsymbol{G}_{tmp2})_{11} - \boldsymbol{N}_{r12}\boldsymbol{M}_r^{-1}\boldsymbol{X}_l\boldsymbol{N}_{l21} \\ & + \boldsymbol{N}_{r12}\boldsymbol{Q}\boldsymbol{N}_{l21} \end{aligned} \tag{18}$$

using coprime factors of (15) and Bezout identity of (11). Also, we know that $\boldsymbol{N}_{r12} = \boldsymbol{I}$ and $\boldsymbol{N}_{l21} = \boldsymbol{I}$ in (18) if we choose $\boldsymbol{F} = -\boldsymbol{B}_2^*\boldsymbol{X}$ in (16) and $\boldsymbol{L} = -\boldsymbol{Y}\boldsymbol{C}_2^*$ in (17). Therefore, we conclude that the closed loop transfer matrix of (18) is reduced to

$$\mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K}) \;\; = \;\; \boldsymbol{Q},$$

because $(\boldsymbol{G}_{tmp2})_{11} = \boldsymbol{M}_r^{-1}\boldsymbol{X}_l$. Finally, we show that both the system matrices of coprime factors:

$$\begin{aligned} \boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F} &= \boldsymbol{A} + \gamma^{-2}\boldsymbol{B}_1\boldsymbol{B}_1^*\boldsymbol{X} - \boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X} \\ \boldsymbol{A}_{tmp2} + \boldsymbol{L}\boldsymbol{C}_2 &= \boldsymbol{A}_{tmp1} + \gamma^{-2}\boldsymbol{Y}\boldsymbol{X}\boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{X} - \boldsymbol{Y}\boldsymbol{C}_2^*\boldsymbol{C}_2 \end{aligned}$$

are stable iff the stabilizing solutions $\boldsymbol{X} \geq \boldsymbol{0}$ and $\boldsymbol{Y} \geq \boldsymbol{0}$ exist for Riccati equations (2) and (6). ☐

Since Theorem 1 means that $\|\mathcal{F}_l(\boldsymbol{G}_{tmp2}, \boldsymbol{K})\|_\infty$ is equal to $\|\boldsymbol{Q}\|_\infty$, the controller of (10) are $\mathcal{H}_\infty$ controller iff $\|\boldsymbol{Q}\|_\infty < \gamma$. Additionally, we can see that the assumptions on the stabilizability and detectability for the systems $\boldsymbol{G}_{tmp2}$ and $\boldsymbol{G}_{tmp1}$ are removed because the matrices $\boldsymbol{F}$ and $\boldsymbol{L}$ which $\boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F}$, $\boldsymbol{A}_{tmp2} + \boldsymbol{L}\boldsymbol{C}_2$, $\boldsymbol{A}_{tmp1} + \boldsymbol{B}_2\boldsymbol{F}$ and $\boldsymbol{A}_{tmp1} + \boldsymbol{L}\boldsymbol{C}_2$ are stabilizable simultaneously are found in Theorem 1. Therefore, all stabilizing $\mathcal{H}_\infty$ controllers are parameterized using a free stable parameter $\boldsymbol{Q}$ as follows:

$$\boldsymbol{K} \;\; = \;\; \mathcal{F}_l(\boldsymbol{J}, \boldsymbol{Q})$$

where

$$\boldsymbol{J} \;\; = \;\; \left[\begin{array}{cc} -\boldsymbol{Y}_r^{-1}\boldsymbol{X}_r & \boldsymbol{Y}_r^{-1} \\ \boldsymbol{Y}_l^{-1} & -\boldsymbol{Y}_l^{-1}\boldsymbol{N}_r \end{array}\right],$$

and its state space representation is

$$\boldsymbol{J} \;\; = \;\; \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F} + \boldsymbol{L}\boldsymbol{C}_2 & -\boldsymbol{L} & \widetilde{\boldsymbol{B}}_2 \\ \hline \boldsymbol{F} & \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{C}_2 & \boldsymbol{I} & \boldsymbol{0} \end{array}\right].$$

Also, all stabilizing controllers with $\boldsymbol{F}$ and $\boldsymbol{L}$ in Theorem 1 are the suboptimal $\mathcal{H}_\infty$ controllers for the generalized plant (1) by Lemma 3 iff $\|\boldsymbol{Q}\|_\infty < \gamma$ with $\boldsymbol{Q} \in \mathcal{RH}_\infty$. However, we can not say that $\mathcal{H}_\infty$ controller obtained through above procedures is stable. Though we can easily show that both $\boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F}$ and $\boldsymbol{A}_{tmp2} + \boldsymbol{L}\boldsymbol{C}_2$ are stable by Theorem 1, the system matrix of $\mathcal{H}_\infty$ controller may be unstable, i.e., $\boldsymbol{A}_{tmp2} + \widetilde{\boldsymbol{B}}_2\boldsymbol{F} + \boldsymbol{L}\boldsymbol{C}_2$ may not be a stability matrix.

3.2 Stable $\mathcal{H}_\infty$ Controller Parameterization

From now on, we deal with the stability of $\mathcal{H}_\infty$ controller (10) using gains in Theorem 1. To be stable $\mathcal{H}_\infty$ controller, the denominator parts of controller should be unimodular. If either $\boldsymbol{Y}_l$ or $\boldsymbol{Y}_r$ is unimodular, then $\boldsymbol{Q}$ design problem is not required because the central controller is stable. Though it is not unimodular, if the parity interlacing property is satisfied, then we can make $(\boldsymbol{Y}_l + \boldsymbol{N}_r\boldsymbol{Q})$ or $(\boldsymbol{Y}_r + \boldsymbol{Q}\boldsymbol{N}_l)$ to be unimodular by designing stable $\boldsymbol{Q}$ parameter under the constraint $\|\boldsymbol{Q}\|_\infty < \gamma$. Hence, the $\boldsymbol{Q}$ design problem is formulated as follows:

$$\begin{array}{c} \text{Find } \;\boldsymbol{Q} \in \mathcal{RH}_\infty \quad \text{subject to} \\ \|\boldsymbol{Q}\|_\infty < \gamma \text{ and } (\boldsymbol{Y}_r + \boldsymbol{Q}\boldsymbol{N}_l)^{-1} \in \mathcal{RH}_\infty. \end{array} \tag{19}$$

Also, the another formulation for $\boldsymbol{Q}$ design problem is possible using $(\boldsymbol{Y}_l + \boldsymbol{N}_r\boldsymbol{Q})^{-1} \in \mathcal{RH}_\infty$. Now, let us define the new coprime factors as following forms:

$$\begin{aligned} \boldsymbol{X}_{rs} &= \boldsymbol{X}_r - \boldsymbol{Q}\boldsymbol{M}_l \\ \boldsymbol{Y}_{rs} &= \boldsymbol{Y}_r + \boldsymbol{Q}\boldsymbol{N}_l, \end{aligned} \tag{20}$$

then we can easily know that $\boldsymbol{N}_{rs} = \boldsymbol{N}_r$ and $\boldsymbol{M}_{rs} = \boldsymbol{M}_r$ from $\boldsymbol{Y}_{rs}\boldsymbol{M}_{rs} + \boldsymbol{X}_{rs}\boldsymbol{N}_{rs} = \boldsymbol{I}$ and the definition of (20). The sufficient condition for the existence of stable n-dimensional $\mathcal{H}_\infty$ controller is derived in the following Lemma.

Lemma 4 The stable n-dimensional $\mathcal{H}_\infty$ controller exists if there exists the coprime factor $\boldsymbol{X}_{rs} \in \mathcal{RH}_\infty$ satisfying:

$$\begin{aligned} \| & \left[\boldsymbol{X}_r\boldsymbol{M}_l^{-1}, \gamma(\boldsymbol{M}_r^{-1} - \boldsymbol{I})\right] \\ & - \boldsymbol{X}_{rs}\left[\boldsymbol{M}_l^{-1}, \gamma\boldsymbol{N}_r\boldsymbol{M}_r^{-1}\right] \|_\infty < \gamma. \end{aligned} \tag{21}$$

Proof. First, we convert $\boldsymbol{Q}$ design problem (19) into $\boldsymbol{X}_{rs}$ design problem. We know from the definition of (20) that

$$\|\boldsymbol{Q}\|_\infty \;\; = \;\; \|(\boldsymbol{X}_r - \boldsymbol{X}_{rs})\boldsymbol{M}_l^{-1}\|_\infty < \gamma.$$

Also, the unimodularity of $\boldsymbol{Y}_{rs}$ is assured if the following condition is satisfied

$$\|\boldsymbol{Y}_{rs} - \boldsymbol{I}\|_\infty \;\; = \;\; \|(\boldsymbol{M}_r^{-1} - \boldsymbol{I}) - \boldsymbol{X}_{rs}\boldsymbol{N}_r\boldsymbol{M}_r^{-1}\|_\infty < 1,$$

where $\boldsymbol{Y}_{rs}$ is derived from $\boldsymbol{Y}_{rs}\boldsymbol{M}_r + \boldsymbol{X}_{rs}\boldsymbol{N}_r = \boldsymbol{I}$. Therefore, we obtain (21) from above two norm constraints. Second, if the coprime factor $\boldsymbol{X}_{rs}$ satisfying (21) exists, then $\boldsymbol{Y}_{rs}$ can be easily obtained and the controller $\boldsymbol{K} = -\boldsymbol{Y}_{rs}^{-1}\boldsymbol{X}_{rs}$ is n-dimensional. Therefore, we conclude that the stable n-dimensional $\mathcal{H}_\infty$ controller exists if $\boldsymbol{X}_{rs}$ satisfying (21) exists. ☐

To begin with, we rewrite (21) into the LFT form of

$$\|\mathcal{F}_l(\boldsymbol{G}_{tmp3}, -\boldsymbol{X}_{rs})\|_\infty < \gamma \tag{22}$$

where

$$\begin{aligned}\boldsymbol{G}_{tmp3} &= \left[\begin{array}{cc}\left[\begin{array}{cc}\boldsymbol{X}_r\boldsymbol{M}_l^{-1}, & \gamma(\boldsymbol{M}_r^{-1}-\boldsymbol{I})\end{array}\right] & \boldsymbol{I} \\ \left[\begin{array}{cc}\boldsymbol{M}_l^{-1}, & \gamma\boldsymbol{N}_r\boldsymbol{M}_r^{-1}\end{array}\right] & \boldsymbol{0}\end{array}\right] \\ &= \left[\begin{array}{c|cc}\boldsymbol{A}_{tmp2} & \left[\boldsymbol{L}, -\gamma\widetilde{\boldsymbol{B}}_2\right] & \boldsymbol{0} \\ \hline \boldsymbol{F} & \left[\begin{array}{cc}\boldsymbol{0} & \boldsymbol{0}\end{array}\right] & \boldsymbol{I} \\ -\boldsymbol{C}_2 & \left[\begin{array}{cc}\boldsymbol{I} & \boldsymbol{0}\end{array}\right] & \boldsymbol{0}\end{array}\right]. \end{aligned} \tag{23}$$

This is the closed-loop estimation problem discussed in chapter 8 of [2]. The design problem of (22) is solvable iff the following Riccati equation has the solution $\boldsymbol{Z}$:

$$\begin{aligned}&(\boldsymbol{A}_{tmp2}+\boldsymbol{L}\boldsymbol{C}_2)\boldsymbol{Z}+\boldsymbol{Z}(\boldsymbol{A}_{tmp2}+\boldsymbol{L}\boldsymbol{C}_2)^* \\ &-\boldsymbol{Z}\boldsymbol{C}_2^*\boldsymbol{C}_2\boldsymbol{Z}+\gamma^{-2}\boldsymbol{Z}\boldsymbol{F}^*\boldsymbol{F}\boldsymbol{Z}+\gamma^2\widetilde{\boldsymbol{B}}_2\widetilde{\boldsymbol{B}}_2^*=\boldsymbol{0}.\end{aligned} \tag{24}$$

Then the solution $\boldsymbol{X}_{rs}$ satisfying (22) is obtained as following form:

$$\boldsymbol{X}_{rs}=\left[\begin{array}{c|c}\boldsymbol{A}_{tmp2}+\boldsymbol{H}\boldsymbol{C}_2 & -\boldsymbol{H} \\ \hline -\boldsymbol{F} & \boldsymbol{0}\end{array}\right] \tag{25}$$

where $\boldsymbol{H}=\boldsymbol{L}-\boldsymbol{Z}\boldsymbol{C}_2^*=-(\boldsymbol{Y}+\boldsymbol{Z})\boldsymbol{C}_2^*$. The obtained $\boldsymbol{X}_{rs}$ is different from $\boldsymbol{X}_r$ in that the additive observer gain term $(-\boldsymbol{Z}\boldsymbol{C}_2)$ is added to the conventional observer gain $\boldsymbol{L}$. Also, the Youla $\boldsymbol{Q}$ parameter can be found from $\boldsymbol{X}_{rs}$ as following form:

$$\begin{aligned}\boldsymbol{Q} &= (\boldsymbol{X}_r-\boldsymbol{X}_{rs})\boldsymbol{M}_l^{-1} \\ &= \left[\begin{array}{c|c}\boldsymbol{A}_{tmp2}+\boldsymbol{H}\boldsymbol{C}_2 & \boldsymbol{L}-\boldsymbol{H} \\ \hline \boldsymbol{F} & \boldsymbol{0}\end{array}\right],\end{aligned} \tag{26}$$

iff the solution $\boldsymbol{Z}$ of Riccati equation (24) exists. This $\boldsymbol{Q}$ parameter changes the observer gain matrix $\boldsymbol{L}$ into the new gain $\boldsymbol{H}$. As a matter of fact, the $\boldsymbol{Q}$ parameter of (26) has the observer gain replacement property and it offers the clue to derive the stable n-dimensional $\mathcal{H}_\infty$ controller from unstable $\mathcal{H}_\infty$ controller. We can confirm once more that the above $\boldsymbol{Q}$ parameter has the observer gain replacement property by calculating the coprime factor $\boldsymbol{Y}_{rs}$ from the definition of (20):

$$\boldsymbol{Y}_{rs}=\left[\begin{array}{c|c}\boldsymbol{A}_{tmp2}+\boldsymbol{H}\boldsymbol{C}_2 & \widetilde{\boldsymbol{B}}_2 \\ \hline -\boldsymbol{F} & \boldsymbol{I}\end{array}\right].$$

So, the doubly coprime factorization satisfying new Bezout identity:

$$\left[\begin{array}{cc}\boldsymbol{Y}_{rs} & \boldsymbol{X}_{rs} \\ -\boldsymbol{N}_{ls} & \boldsymbol{M}_{ls}\end{array}\right]\left[\begin{array}{cc}\boldsymbol{M}_{rs} & -\boldsymbol{X}_{ls} \\ \boldsymbol{N}_{rs} & \boldsymbol{Y}_{ls}\end{array}\right]=\left[\begin{array}{cc}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}\end{array}\right]$$

are obtained by replacing the observer gain $\boldsymbol{L}$ with $\boldsymbol{H}$ in coprime factors (12) and (13). In spite of replacing the observer gain with new one in coprime factors (12) and (13), $\boldsymbol{M}_{rs}$ and $\boldsymbol{N}_{rs}$ remain invariable form, in other words, $\boldsymbol{M}_{rs}=\boldsymbol{M}_r$ and $\boldsymbol{N}_{rs}=\boldsymbol{N}_r$. Here, the controller $\boldsymbol{K}=-\boldsymbol{Y}_{rs}^{-1}\boldsymbol{X}_{rs}=-\boldsymbol{X}_{ls}\boldsymbol{Y}_{ls}^{-1}$ and the plant $(\boldsymbol{G}_{tmp2})_{22}=\boldsymbol{M}_{ls}^{-1}\boldsymbol{N}_{ls}=\boldsymbol{N}_{rs}\boldsymbol{M}_{rs}^{-1}$. Then, the proper n-dimensional stable $\mathcal{H}_\infty$ controller is obtained as following form:

$$\boldsymbol{K} = -\boldsymbol{Y}_{rs}^{-1}\boldsymbol{X}_{rs}=\left[\begin{array}{c|c}\boldsymbol{A}_{tmp2}+\boldsymbol{H}\boldsymbol{C}_2+\widetilde{\boldsymbol{B}}_2\boldsymbol{F} & -\boldsymbol{H} \\ \hline \boldsymbol{F} & \boldsymbol{0}\end{array}\right].$$

Also, the set of proper stable controllers achieving internal stability are parameterized by either

$$\begin{aligned}\boldsymbol{K}_s &= -(\boldsymbol{Y}_{rs}+\boldsymbol{R}\boldsymbol{N}_{ls})^{-1}(\boldsymbol{X}_{rs}-\boldsymbol{R}\boldsymbol{M}_{ls}) \\ &= -(\boldsymbol{X}_{ls}-\boldsymbol{M}_{rs}\boldsymbol{R})(\boldsymbol{Y}_{ls}+\boldsymbol{N}_{rs}\boldsymbol{R})^{-1}\end{aligned} \tag{27}$$

where the free parameter $\boldsymbol{R}\in\mathcal{RH}_\infty$ and

$$\boldsymbol{K}_s=\mathcal{F}_l(\boldsymbol{J}_s,\boldsymbol{R})$$

and the state space representation of $\boldsymbol{J}_s$ is

$$\boldsymbol{J}_s=\left[\begin{array}{c|cc}\boldsymbol{A}_{tmp2}+\widetilde{\boldsymbol{B}}_2\boldsymbol{F}+\boldsymbol{H}\boldsymbol{C}_2 & -\boldsymbol{H} & \widetilde{\boldsymbol{B}}_2 \\ \hline \boldsymbol{F} & \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{C}_2 & \boldsymbol{I} & \boldsymbol{0}\end{array}\right]. \tag{28}$$

If the stable controller (27) is applied to $\boldsymbol{G}_{tmp2}$, then the closed-loop transfer matrix is not reduced to the simple form such as Theorem 1, but it has the relation in following Theorem.

Theorem **2** *Assume that the Riccati solutions* $\boldsymbol{X}\geq\boldsymbol{0},\boldsymbol{Y}\geq\boldsymbol{0}$ *and* $\boldsymbol{Z}\geq\boldsymbol{0}$ *for (2),(6) and (24) exist respectively. If the set of stable controllers (27) is applied to* $\boldsymbol{G}_{tmp2}$*, then the closed loop transfer matrix satisfies next relation:*

$$\mathcal{F}_l(\boldsymbol{G}_{tmp2},\boldsymbol{K}_s)=\boldsymbol{Q}+\boldsymbol{R}\boldsymbol{W}$$

where $\boldsymbol{Q}=(\boldsymbol{X}_r-\boldsymbol{X}_{rs})\boldsymbol{M}_l^{-1}$ *and* $\boldsymbol{W}=\boldsymbol{M}_{ls}\boldsymbol{M}_l^{-1}$.

Proof. First, consider the closed loop transfer matrix as follows:

$$\begin{aligned}&\mathcal{F}_l(\boldsymbol{G}_{tmp2},\boldsymbol{K}_s)=(\boldsymbol{G}_{tmp2})_{11} \\ &+(\boldsymbol{G}_{tmp2})_{12}(\boldsymbol{I}-\boldsymbol{K}_s(\boldsymbol{G}_{tmp2})_{22})^{-1}\boldsymbol{K}_s(\boldsymbol{G}_{tmp2})_{21}\end{aligned} \tag{29}$$

If the first form of controller (27) is applied to above equation, then the closed loop transfer matrix of (29) is reduced to

$$\mathcal{F}_l(\boldsymbol{G}_{tmp2},\boldsymbol{K}_s)=\boldsymbol{X}_r\boldsymbol{M}_l^{-1}-(\boldsymbol{X}_{rs}-\boldsymbol{R}\boldsymbol{M}_{ls})\boldsymbol{M}_l^{-1} \tag{30}$$

by using

$$\begin{aligned}(\boldsymbol{G}_{tmp2})_{11} &= \boldsymbol{X}_r\boldsymbol{M}_l^{-1} \\ (\boldsymbol{G}_{tmp2})_{12} &= \boldsymbol{M}_r^{-1} \\ (\boldsymbol{G}_{tmp2})_{21} &= \boldsymbol{M}_l^{-1}\end{aligned}$$

If we let $\boldsymbol{W}=\boldsymbol{M}_{ls}\boldsymbol{M}_l^{-1}$, then we conclude that the closed loop transfer matrix of (30) is reduced to

$$\mathcal{F}_l(\boldsymbol{G}_{tmp2},\boldsymbol{K}_s) \;=\; \boldsymbol{Q}+\boldsymbol{R}\boldsymbol{W}$$

because $\boldsymbol{Q}=(\boldsymbol{X}_r-\boldsymbol{X}_{rs})\boldsymbol{M}_l^{-1}$. Also, the second form of controller (27) brings the same conclusion. □

Since Theorem 2 means that $\|\mathcal{F}_l(\boldsymbol{G}_{tmp2},\boldsymbol{K}_s)\|_\infty$ is equal to $\|\boldsymbol{Q}+\boldsymbol{R}\boldsymbol{W}\|_\infty$, we can know that the controllers (27) are the stable $\mathcal{H}_\infty$ controllers for the generalized plant (1) by Lemma 3 iff $\|\boldsymbol{Q}+\boldsymbol{RW}\|_\infty < \gamma$. If we are to design $\boldsymbol{R}$ to achieve another objective, then the norm constraint of $\|\boldsymbol{Q}+\boldsymbol{RW}\|_\infty < \gamma$ can make the $\boldsymbol{R}$ design problem difficult. The following Theorem reduces the constraint of $\|\boldsymbol{Q}+\boldsymbol{RW}\|_\infty < \gamma$ to the simplified form.

Theorem **3** *Assume that the positive semi-definite solution matrices $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ for Riccati equations (2),(6) and (24) exist respectively. If the following condition is satisfied*

$$\left(\sqrt{\frac{\gamma^4-\gamma^2+1}{\gamma^2-1}}\boldsymbol{Z}-\boldsymbol{Y}\right)\boldsymbol{X} \le \gamma^2\boldsymbol{I} \qquad (31)$$

for $\gamma > 1$, then $\|\boldsymbol{R}\|_\infty < \gamma^{-1}$ implies that $\|\boldsymbol{Q}+\boldsymbol{RW}\|_\infty < \gamma$ by the Bounded Real Lemma.

Proof. First, we use the LFT for $\boldsymbol{Q}+\boldsymbol{RW}$

$$\boldsymbol{Q}+\boldsymbol{RW} = \mathcal{F}_l\left(\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{W} & \boldsymbol{0} \end{bmatrix}, \boldsymbol{R}\right),$$

where

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{W} & \boldsymbol{0} \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{A}_{tmp2}+\boldsymbol{HC}_2 & \boldsymbol{L}-\boldsymbol{H} & \boldsymbol{0} \\ \hline \boldsymbol{F} & \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{C}_2 & \boldsymbol{I} & \boldsymbol{0} \end{array}\right], \qquad (32)$$

because the explicit expression on $\boldsymbol{W}$ is

$$\boldsymbol{W} = \left[\begin{array}{c|c} \boldsymbol{A}_{tmp2}+\boldsymbol{HC}_2 & \boldsymbol{L}-\boldsymbol{H} \\ \hline -\boldsymbol{C}_2 & \boldsymbol{I} \end{array}\right].$$

Second, if $\left\|\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{W} & \boldsymbol{0} \end{bmatrix}\right\|_\infty < \gamma$, then $\|\boldsymbol{R}\|_\infty < \gamma^{-1}$ implies that $\|\boldsymbol{Q}+\boldsymbol{RW}\|_\infty < \gamma$ by the contraction mapping. Now, we apply the Bounded Real Lemma[2, 7] to (32) for $\gamma > 1$:

$$\begin{gathered} \boldsymbol{V}(\boldsymbol{A}_{tmp2}+\boldsymbol{HC}_2-\tfrac{1}{\gamma^2-1}\boldsymbol{ZC}_2^*\boldsymbol{C}_2)^*+ \\ (\boldsymbol{A}_{tmp2}+\boldsymbol{HC}_2-\tfrac{1}{\gamma^2-1}\boldsymbol{ZC}_2^*\boldsymbol{C}_2)\boldsymbol{V}+\tfrac{1}{\gamma^2-1}\boldsymbol{ZC}_2^*\boldsymbol{C}_2\boldsymbol{Z} \\ +\tfrac{\gamma^2}{\gamma^2-1}\boldsymbol{VF}^*\boldsymbol{FV}+\tfrac{\gamma^2}{\gamma^2-1}\boldsymbol{VC}_2^*\boldsymbol{C}_2\boldsymbol{V} \le \boldsymbol{0}, \end{gathered} \qquad (33)$$

if we let $\boldsymbol{V} = \boldsymbol{Z}$, then (33) is simplified using Riccati equation (24) as following form:

$$\begin{gathered} \left(\tfrac{\gamma^2}{\gamma^2-1}-\tfrac{1}{\gamma^2}\right)\boldsymbol{ZF}^*\boldsymbol{FZ}-\gamma^2\widetilde{\boldsymbol{B}}_2\widetilde{\boldsymbol{B}}_2^* \le \boldsymbol{0}. \\ \to (\gamma^4-\gamma^2+1)\boldsymbol{ZXB}_2\boldsymbol{B}_2^*\boldsymbol{XZ} \le \gamma^4(\gamma^2-1)\widetilde{\boldsymbol{B}}_2\widetilde{\boldsymbol{B}}_2^*. \end{gathered} \qquad (34)$$

The condition satisfying above inequality (34) can be reduced to (31). Hence, the matrix $\boldsymbol{Z}$ satisfies the Bounded Real Lemma assuring that $\left\|\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{W} & \boldsymbol{0} \end{bmatrix}\right\|_\infty < \gamma$, if the condition (31) is satisfied. ☐

If the condition (31) is satisfied, then the design constraint for free parameter $\boldsymbol{R}$ can be reduced to $\|\boldsymbol{R}\|_\infty < \gamma^{-1}$ by Theorem 3. And the stable n-dimensional $\mathcal{H}_\infty$ controller parameterization (27) can be achieved with $\|\boldsymbol{R}\|_\infty < \gamma^{-1}$ in place of the somewhat ambiguous condition $\|\boldsymbol{Q}+\boldsymbol{RW}\|_\infty < \gamma$. However, it is not assured that the suggested condition (31) is satisfied for the general cases. The following Theorem examines the stability of the proper stable n-dimensional $\mathcal{H}_\infty$ controllers.

Theorem **4** *The system matrix of controller (27) is stable if the solution matrix $\boldsymbol{Z} \ge \boldsymbol{0}$ of (24) exists.*

Proof. We can rearrange the Riccati equation (24) by letting $\boldsymbol{B}_q = (\gamma\widetilde{\boldsymbol{B}}_2 - \gamma^{-1}\boldsymbol{ZF}^*)$ as following form:

$$\begin{gathered} (\boldsymbol{A}_{tmp2}+\boldsymbol{LC}_2+\widetilde{\boldsymbol{B}}_2\boldsymbol{F})\boldsymbol{Z}+\boldsymbol{Z}(\boldsymbol{A}_{tmp2}+\boldsymbol{LC}_2+\widetilde{\boldsymbol{B}}_2\boldsymbol{F})^* \\ -\boldsymbol{ZC}_2^*\boldsymbol{C}_2\boldsymbol{Z}+\boldsymbol{B}_q\boldsymbol{B}_q^* = \boldsymbol{0}. \end{gathered}$$

From above equation, we conclude that the system matrix $\boldsymbol{A}_{tmp2}+\boldsymbol{HC}_2+\widetilde{\boldsymbol{B}}_2\boldsymbol{F}$ is stable. ☐

4 Concluding Remarks

We suggested the parameterization for a stable n-dimensional $\mathcal{H}_\infty$ controller by using the doubly coprime factorization technique. To parameterize the stable $\mathcal{H}_\infty$ controller, we need to solve three coupled Riccati equations. As a matter of fact, the solutions for two Riccati equations are required to obtain the $\mathcal{H}_\infty$ controller and the third Riccati equation should be solved to guarantee the stability of $\mathcal{H}_\infty$ controller. However, the suggested controllers are only a part of all stable $\mathcal{H}_\infty$ controllers because it is derived from the sufficient condition on the existence of the stable controller.

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