# LMI-Based Filter Design for Fault Detection and Isolation

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### Abstract

A linear matrix inequality (LMI) based filter design approach for fixed-order robust fault detection and isolation (FDI) is examined in this paper. The proposed filter design provides necessary and sufficient conditions for the existence of a solution to the detection and isolation of faults using an $H_\infty$ formulation. These conditions are expressed in terms of LMIs with matrix rank constraints, and a parameterization of all admissible filters is provided, which correspond to a feasible solution. A convex LMI problem is obtained for the full-order FDI filter design. Finally, the proposed methods are demonstrated using a structural system simulation example, which include faulty actuators, sensors and external disturbances.

Keywords: Fault detection and isolation; Linear Matrix Inequalities; $H_\infty$ filtering; Linear systems

### 1. Introduction

The complexity of today's control systems requires fault tolerance schemes to provide early warning of faulty sensors, actuators or system components. Such schemes need to detect and isolate faults before they lead to catastrophes, so that appropriate actions and control reconfiguration can be accomplished. Consequently, for the past two decades, the FDI problem has received considerable attention in the control systems literature [Bea71, Wil76, Fra90, Pat94].

The fault detection and isolation tasks may be classified into three levels: detection, isolation, and estimation [Wil76]. Level one is concerned with giving an indication (alarm) of the existence of a fault. Level two is the ability of pointing out the source of the fault (location of the fault in the plant, actuators, or sensors). Level three is the estimation of the extent of the fault.

FDI techniques based on observer schemes, parameter estimation methods, or statistical approaches have been developed. Detailed surveys of different FDI methods can be found in [Pat94, Fra97]. Recently, optimization-based FDI schemes have been proposed where an appropriately selected performance index or error function is optimized to provide robust FDI in the presence of disturbances. Lately, $H_\infty$ optimization methods for FDI have received increased attention for providing disturbance rejection and robustness properties to the FDI schemes [Din90, Qiu93, Nie96, Che99].

In this work, an LMI-based $H_\infty$ filtering formulation is presented for continuous and discrete-time FDI problems. LMIs provide a flexible algebraic formulation and efficient computational tools for the solution of the $H_\infty$ filter design [Gri97, Wat98]. When compared with Riccati equation-based methods, LMI methods are more general, less restrictive in the assumptions, and lead to less conservative solutions for robust synthesis problems. Besides, fixed-order controller synthesis may be included directly in the formulation of the problem. These properties are present also in the proposed FDI filter method.

The proposed formulation allows the development of necessary and sufficient solvability conditions for the fixed-order FDI filter design and a parameterization of all admissible filters. The full-order $H_\infty$ FDI filter design is characterized in terms of convex LMIs, which can be solved using recently developed interior point optimization algorithms [Van96]. The fixed-order $H_\infty$ FDI filter design is characterized by LMIs with additional coupling non-convex matrix rank constraints. A simulation example of a structural system is presented to demonstrate the proposed methods.

The notation used in this work is presented next. The transpose of a real matrix $A$ is denoted by $A^T$, and the standard notation $>, \geq (<, \leq)$ is used to denote positive (negative) definite and semidefinite ordering of matrices. The $H_\infty$ norm of a rational transfer function *F(s)* is defined as $\|F(s)\|_\infty = \max_w \bar{s}(F(jw))$, where $\bar{s}(\cdot)$ denotes the maximum singular value of a matrix. The $L_2$ norm of a vector valued function *f(t)* is defined as $\|f(t)\|_{L_2} = \{\int_0^\infty f^T(t) f(t) dt\}^{1/2}$. The induced matrix norm is $\|A\| = \bar{s}(A) = \{ l_{\max}(AA^T)\}^{1/2}$. Given a real $n \times m$ matrix $A$ with rank $r$, an orthogonal complement $A^\perp$ is defined as the possibly non-unique matrix that satisfies $A^\perp A = 0$ and $A^\perp A^{\perp T} > 0$. Hence, the orthogonal complement may be computed from the singular-value decomposition of a matrix

$$A = [U_1 \quad U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

as $A^\perp = TU_2^T$ where T is an arbitrary nonsingular matrix.

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## 2. Problem Formulation

In order to formulate the FDI filter problem in a $H_\infty$ filtering framework, it will be represented in a linear fraction transformation (LFT) form. Consider the following plant *P* of order *n* with state-space representation

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p d + F_p f \\ y_p &= C_p x_p + D_p d + H_p f \end{aligned}, \tag{1}$$

where $x_p$ is the state vector, *d* is the disturbance vector, *f* is the fault vector, and $A_p$, $B_p$, $C_p$, $D_p$, $F_p$ and $H_p$ are real matrices of appropriate dimensions. For simplicity it is chosen not to include the control input, but this does not imply loss of generality. The block diagram in Figure 1 represents the proposed configuration.

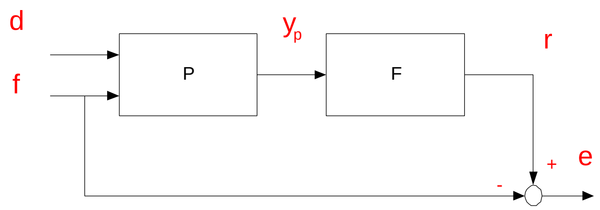


Figure 1 – Block diagram of the proposed FDI filter scheme

In Figure 1, *P* represents the plant to be monitored and *F* the unknown filter. The estimation error is defined as $e = r - f$ where *r* is the residual generated vector of the FDI filter *F*. The objective is to design *F* such that *r* provides an estimate of the fault vector *f*. Hence, detection and isolation of faults can be accomplished by examining the values and pattern of vector *r*.

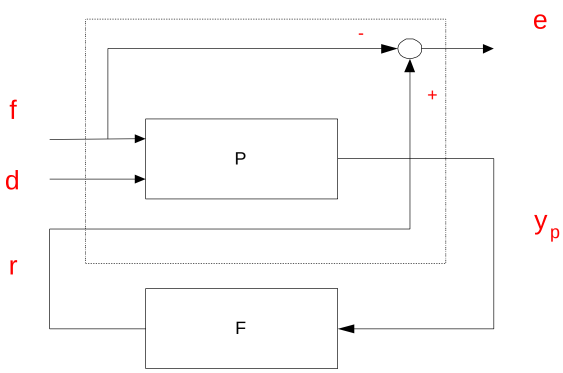


Figure 2 – Rearranging the original configuration

Based on the above formulation, the proposed $H_\infty$ optimal filtering problem is to find an FDI dynamic filter *F* to minimize the worst-case estimation error energy $\|e\|_{L_2}$ over all bounded energy generalized disturbance $w = \begin{bmatrix} d^T & f^T \end{bmatrix}^T$, that is

$$\min_F \sup_{w \in L_2 - \{0\}} \frac{\|e\|_{L_2}}{\|w\|_{L_2}} \tag{2}$$

This is equivalent to minimizing the $H_\infty$ norm of the transfer function $T_{we}$ between the generalized disturbance input and the error of the fault estimation. The $\gamma$ suboptimal $H_\infty$ FDI filtering problem is to find (if exists) a filter such that $\|T_{we}\|_\infty < \gamma$ where $\gamma$ is a given positive scalar.

The block diagram of Figure 1 can be rearranged as presented in Figure 2, where *w* represents the combined vectors of disturbances and faults, and then represented as the LFT scheme of Figure 3. To provide good FDI capability different weighting functions can be used to penalize the fault, the error output or the disturbance vectors. These weighting functions can be easily included in the representation of the plant *P*, increasing its order, but without significant difference to the treatment here presented.

Using Figure 3, the corresponding state-space formulation is given by

$$\begin{aligned} \dot{x} &= A_s x + B_1 w + B_2 r \\ e &= C_1 x + D_{11} w + D_{12} r \\ y &= C_2 x + D_{21} w + D_{22} r \end{aligned} \tag{3}$$

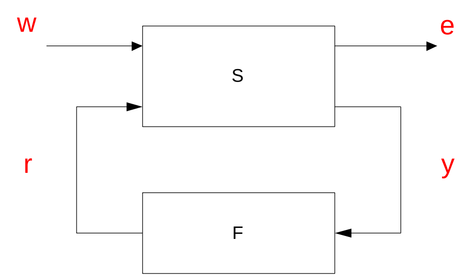


Figure 3 – LFT scheme for the filter

In packed notation, and substituting for the original matrices, the system may be represented as

$$S(s) = \begin{bmatrix} A_p & \begin{bmatrix} B_p & F_p \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & -I \end{bmatrix} & I \\ C_p & \begin{bmatrix} D_p & H_p \end{bmatrix} & 0 \end{bmatrix} \tag{4}$$

The objective now, according to Figure 3 and equation (4), is to design the unknown filter *F* as if it were a feedback controller attaining the performance index (2). This approach permits to use any appropriate controller design method to design the FDI filter.

## 3. Continuous-Time FDI Filter Design

To solve this problem using an LMI-based approach in the continuous-time domain, it will be necessary for completeness to present two lemmas and two theorems. Consider a stable *n*th-order linear time-invariant system with state-space representation

$$\begin{aligned} \dot{x}_s &= A_s x_s + B_1 w \\ e &= D_{11} w + r \\ y &= C_s x + D_{21} w \end{aligned} \tag{5}$$

where $x_s$ is the state vector, *y* the output vector, *w* the generalized disturbance vector and $A_s$, $B_1$, $C_2$, $D_{11}$ and $D_{21}$ are real matrices of appropriate dimensions according to (3) and (4). For this system the objective is to design a stable linear dynamic filter, with state-space representation

$$\begin{aligned} \dot{x}_f &= K_f x_f + L_f y \\ r &= M_f x_f + N_f y \end{aligned} \tag{6}$$

whose output *r* is to estimate the fault vector. The vector $x_f$ is the filter state vector, and $K_f$, $L_f$, $M_f$ and $N_f$ are real matrices of appropriate dimensions to be computed. The order of the filter $n_f$ is restricted to be less or equal to the order of the system $n$.

The method here used is based on the Bounded Real Lemma [Ske98], presented next.

**LEMMA 1** *Consider a stable linear time-invariant system*

$$\begin{aligned} \dot{x} &= A_c x + B_c w \\ y &= C_c x + D_c w \end{aligned} \qquad (7)$$

*with transfer function* $T_c(s) = C_c(sI - A_c) + B_c + D_c$ *and let* $\gamma$ *be a given positive scalar. Then* $\|T_c\|_\infty < \gamma$ *if and only if there exists a matrix* $P > 0$ *that satisfies*

$$\begin{bmatrix} PA_c + A_c^T P & PB_c & C_c^T \\ B_c^T P & -\gamma^2 I & D_c^T \\ C_c & D_c^T & -I \end{bmatrix} < 0 \qquad (8)$$

To find a parameterization of all solutions of an LMI such as the above, the following lemma may be applied [Ske98].

**LEMMA 2** *Let G, L and* $Q = Q^T$ *be given matrices. There exists a matrix F to solve the matrix inequality*

$$\Gamma F \Lambda + \Lambda^T F^T \Gamma^T + \Theta < 0 \qquad (9)$$

*if and only if the following conditions are satisfied*

$$\Gamma^\perp \Theta \Gamma^{\perp T} < 0 \qquad (10)$$

$$\Lambda^{T\perp} \Theta \Lambda^{T\perp T} < 0 \ . \qquad (11)$$

These two lemmas may be specialized for the case of FDI filtering to provide the necessary and sufficient conditions for the existence of such filters and the parameterization of all solutions. The following theorem gives the solution to the $\gamma$-suboptimal $H_\infty$ FDI filtering problem.

**THEOREM 1** *There exists an* $n_f$*th-order filter F to solve the* $\gamma$ *suboptimal* $H_\infty$ *FDI filtering problem if and only if there exist matrices X and Y with* $Y \geq X > 0$ *such that the following conditions are satisfied*

$$\begin{bmatrix} XA_s + A_s^T X & XB_1 \\ B_1^T X & -\gamma^2 I \end{bmatrix} < 0 \qquad (12)$$

$$\begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^\perp \begin{bmatrix} YA_s + A_s^T Y & YB_1 \\ B_1^T Y & D_{11}^T D_{11} - \gamma^2 I \end{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^{\perp T} < 0 \qquad (13)$$

$$rank(X - Y) \leq n_f \qquad (14)$$

**PROOF** Defining $x^T = \begin{bmatrix} x_s^T & x_f^T \end{bmatrix}$, the closed-loop system *$F_l(S,F)$*, from the LFT scheme of Figure 3, is described by the following state-space equations:

$$\dot{x} = \begin{bmatrix} A_s & 0 \\ L_f C_2 & K_f \end{bmatrix} x + \begin{bmatrix} B_1 \\ L_f D_{21} \end{bmatrix} w$$

$$e = \begin{bmatrix} C_1 + N_f C_2 & M_f \end{bmatrix} x + \begin{bmatrix} D_{11} + N_f D_{21} \end{bmatrix} w \ .$$

Adopting the formulation

$$\begin{aligned} \dot{x} &= (A + BFM)x + (D + BFE)w \\ e &= (C + HFM)x + (G + HFE)w \end{aligned} \qquad (15)$$

it is found for the respective matrices

$$A = \begin{bmatrix} A_s & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad M = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}$$

$$D = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad E = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} I & 0 \end{bmatrix} \quad G = D_{11}$$

for the unknown filter defined as $F = \begin{bmatrix} N_f & M_f \\ L_f & K_f \end{bmatrix}$.

Applying Lemma 1, the inequality (8) becomes

$$\begin{bmatrix} P(A+BFM) + (A+BFM)^T P & P(D+BFE) & (HFM)^T \\ (D+BFE)^T P & -\gamma^2 I & (G+HFE)^T \\ HFM & G+HFE & -I \end{bmatrix} < 0 \qquad (17)$$

Using Lemma 2, it can be easily devised from (9) that

$$\Gamma = \begin{bmatrix} PB \\ 0 \\ H \end{bmatrix} \quad \Lambda^T = \begin{bmatrix} M^T \\ E^T \\ 0 \end{bmatrix} \text{ and } \Theta = \begin{bmatrix} PA + A^T P & PD & 0 \\ D^T P & -\gamma^2 I & G^T \\ 0 & G & -I \end{bmatrix}$$

For condition (10) of Lemma 2, considering that

$$\Gamma^\perp = \begin{bmatrix} \begin{bmatrix} B \\ H \end{bmatrix}^\perp & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

and defining

$$\alpha = \begin{bmatrix} B \\ H \end{bmatrix}^\perp = \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \ \psi = \begin{bmatrix} AP^{-1} + P^{-1}A^T & 0 \\ 0 & -I \end{bmatrix}, \text{ and}$$

$$\delta = \begin{bmatrix} D \\ G \end{bmatrix}$$

it leads to

$$\alpha \psi \alpha^T + \frac{1}{\gamma^2} \alpha \delta \delta^T \alpha^T < 0 \ .$$

Partitioning $P$ as $P^{-1} = \begin{bmatrix} Z & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}$, making $X = Z^{-1}$, and substituting for the LFT matrices in (16), it yields the following inequality

$$XA_s + A_s^T X + \frac{1}{\gamma^2} XB_1 B_1^T X < 0 \, ,$$

which corresponds to condition (12) of Theorem 1.

For condition (11) of Lemma 2, considering that

$$\Lambda^{T\perp} = \begin{bmatrix} \begin{bmatrix} M^T \\ E^T \end{bmatrix}^\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & I \end{bmatrix} .$$

and the following definitions,

$$f = \begin{bmatrix} M^T \\ E^T \end{bmatrix}^{\perp}, \; y = \begin{bmatrix} PA + A^T P & PD \\ D^T P & -g^2 I \end{bmatrix} \text{ and } d = [0 \quad G],$$

it can be easily shown that it becomes

$$fy\, f^T + f d^T\, df^T < 0 .$$

Substituting, it yields

$$\begin{bmatrix} M^T \\ E^T \end{bmatrix}^{\perp} \left( \begin{bmatrix} PA + A^T P & PD \\ D^T P & -g^2 I \end{bmatrix} + \begin{bmatrix} 0 \\ G^T \end{bmatrix} [0 \quad G] \right) \begin{bmatrix} M^T \\ E^T \end{bmatrix}^{\perp T} < 0$$

Partitioning $P$ as $P = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$ and considering that

$$f = \begin{bmatrix} M^T \\ E^T \end{bmatrix}^{\perp} = \begin{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix},$$

the inequality (13) results after some simple algebraic manipulations. ?

Notice that the conditions (10) and (11) of Theorem 1 are convex LMI constraints on the matrix parameters $X$ and $Y$. For the full-order FDI filtering problem, where $n_f = n$, the rank constraint is redundant and the problem results in a convex LMI. For the case of reduced order FDI filtering a non-convex problem is obtained. Methods to solve such LMI problems with rank constraints have been recently proposed [Ske98, Gri99].

The following result provides a parametrization of all feasible filters.

**THEOREM 2** *All the $g$-suboptimal $H_\infty$ FDI $n_f$th-order filters that correspond to a feasible matrix pair (X, Y) are given by*

$$F = \begin{bmatrix} K_f & L_f \\ M_f & N_f \end{bmatrix} = -R^{-1}\Gamma^T \Phi \Lambda^T \Psi + \Omega^{1/2} J \Psi^{1/2}$$

*where $F$. R, and J are free matrix parameters subject to*

$$\Phi = (\Gamma R^{-1} \Gamma^T - \Theta)^{-1} > 0 \,,\; R > 0,\; \|J\| < 1$$

*and $W$ and $Y$ defined by*

$$\Omega = R^{-1} - R^{-1}\Gamma^T (\Phi - \Phi \Lambda^T \Psi \Lambda \Phi) \Gamma R^{-1}$$
$$\Psi = (\Lambda \Phi \Lambda)^{-1}$$

The proof follows a similar approach as in [Gri98].

### *4. Discrete-Time FDI Filter Design*

The $H_\infty$ FDI filter design problem in the discrete-time domain is addressed in this section following a similar algebraic approach. Consider a stable $n$th-order linear time-invariant discrete-time system with the state-space representation

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_p w(k) \\ y_p(k) &= C_p x_p(k) + D_p w(k) \end{aligned} \tag{19}$$

where the vectors and matrices are similar to the continuous-time model.

For this system the objective is to design a stable linear dynamic filter, with discrete-time state-space representation

$$\begin{aligned} x_f(k+1) &= K_f x_f(k) + L_f y_p(k) \\ r(k) &= M_f x_f(k) + N_f y_p(k) \end{aligned} \tag{20}$$

where the order of the filter $n_f$ is less or equal to the order of the system. The result for the discrete-time $g$-suboptimal $H_\infty$ FDI filter is outlined using the same approach as before, beginning with the discrete Bounded Real Lemma [Ske98] .

**LEMMA 3** *Consider a stable linear time-invariant discrete-time system*

$$\begin{aligned} x(k+1) &= A_c x(k) + B_c w(k) \\ y(k) &= C_c x(k) + D_c w(k) \end{aligned} \tag{21}$$

*with transfer function $T_c(z) = C_c(zI - A_c) + B_c + D_c$ and let $g$ be a given positive scalar. Then $\|T_c\|_\infty < g$ if and only if there exists a matrix $P > 0$ that satisfies*

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & g^2 I \end{bmatrix} \tag{22}$$

Solvability conditions for a general quadratic matrix inequality, like the above one, is presented next [Ske98] .

**LEMMA 4** *Let G, L, $Q > 0$ and $R > 0$ be given matrices. There exists a matrix F to solve the matrix inequality*

$$(\Theta + \Gamma F \Lambda)^T R (\Theta + \Gamma F \Lambda) < Q \tag{23}$$

*if and only if the following conditions are satisfied*

$$\Gamma^{\perp} (R^{-1} - \Theta Q^{-1} \Theta^T) \Gamma^{\perp T} > 0 \tag{24}$$

$$\Lambda^{T\perp} (Q - \Theta^T R \Theta) \Lambda^{T\perp T} > 0 \tag{25}$$

These two lemmas may be specialized for the case of FDI filtering. The following theorem provides the necessary and sufficient conditions for the solution of the discrete-time $g$-suboptimal $H_\infty$ FDI filtering problem.

**THEOREM 3** *There exists an $n_f$th-order filter F to solve the discrete-time $g$-suboptimal $H_\infty$ FDI filtering problem if and only if there exists matrices X and Y with $Y \geq X > 0$ such that the following conditions are satisfied*

$$\begin{bmatrix} X - A_s^T X A_s & -A_s^T X B_1 \\ -B_1^T X A_s & g^2 I - B_1^T X B_1 \end{bmatrix} > 0 \tag{26}$$

$$\begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^{\perp} \begin{bmatrix} Y - A_s^T Y A_s & -A_s^T Y B_1 \\ -B_1^T Y A_s & g^2 I - B_1^T Y B_1 - D_{11}^T D_{11} \end{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^{\perp T} > 0 \tag{27}$$

$$rank(X - Y) \leq n_f \tag{28}$$

**PROOF** For the closed-loop system formulated in discrete-time, the state-space equations are

$$\begin{aligned} x(k+1) &= (A + BFM) x(k) + (D + BFE) w(k) \\ e(k) &= (HFM) x(k) + (G + HFE) w(k) \end{aligned}$$

where the matrices are defined as already presented. Applying Lemma 3 it can be found that the Bounded Real Lemma condition can be written as in (23), where

$$\Gamma = \begin{bmatrix} B \\ H \end{bmatrix}, \ \Lambda = \begin{bmatrix} M & E \end{bmatrix},$$

$$\Theta = \begin{bmatrix} A & D \\ 0 & G \end{bmatrix}, Q = \begin{bmatrix} P & 0 \\ 0 & g^2 I \end{bmatrix}, R = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix},$$

and the unknown filter matrices are in $F = \begin{bmatrix} N_f & M_f \\ L_f & K_f \end{bmatrix}$.

Using the solvability condition (24) of Lemma 4, and considering that $\Gamma^{\perp} = \begin{bmatrix} B \\ H \end{bmatrix}^{\perp} = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} Z & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}$, it provides

$$Z - A_s Z A_s^T - \frac{1}{g^2} B_1 B_1^T > 0$$

which can be written in the form of the inequality (26) for $X = Z^{-1}$, and applying the Schur complement formulas two times.

For the condition (25) of Lemma 4, consider that

$$\Lambda^{T\perp} = \begin{bmatrix} M^T \\ E^T \end{bmatrix}^{\perp} = \begin{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix},$$

then it can be easily shown that the solvability condition leads to inequality (27) for $P = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$. The rank constraint follows from the constraint between *Y* and *Z*. ?

The following result parameterizes all discrete-time $g$ suboptimal $H_\infty$ FDI filters

**THEOREM 4** *All discrete-time $g$ suboptimal $H_\infty$ FDI filter F that correspond to a feasible pair (X, Y) are given by*

$$\begin{bmatrix} K_f & L_f \\ M_f & N_f \end{bmatrix} = F_1 + F_2 U F_3$$

*where U is any matrix such that $\|U\| < 1$, and*

$$F_1 = -\Omega \Gamma^T R \Theta^T \Phi \Lambda (\Lambda \Phi \Lambda^T)^{-1}$$

$$F_2 = \{\Omega - \Omega \Gamma^T R \Theta [\Phi - \Phi \Lambda^T (\Lambda \Phi \Lambda^T)^{-1} \Lambda \Phi] \Theta^T R \Gamma \Omega\}^{1/2}$$

$$F_3 = (\Lambda \Phi \Lambda^T)^{-1/2}$$

and where

$$\Phi = (Q - \Theta^T R \Theta + \Theta^T R \Gamma \Omega \Gamma^T R \Theta)^{-1}$$

$$\Omega = (\Gamma^T R \Gamma)^{-1}$$

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & I \\ I & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} A_s & 0 & B_1 \\ 0 & 0 & 0 \\ 0 & 0 & D_{11} \end{bmatrix}, \ Q = \begin{bmatrix} P & 0 \\ 0 & g^2 I \end{bmatrix} \text{ and } R = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$$

The proof follows a similar approach as in [Gri98].

As in the continuous-time case, a convex LMI optimization solves the full-order FDI filtering, but non-convex problem results for the reduced-order FDI filtering.

### *5. Numerical Examples*

To demonstrate the applicability of the method a simple example is presented next, for both the continuous-time and the discrete-time cases, and solved for the convex full-order filter design. A single degree-of-freedom mechanical system that consists of a mass, spring and damper is considered, with one plant output and one fault source. External disturbances and measurement noise are included as well. The plant matrices, according to the previous presented formulation, are

$$\dot{x}_p = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_p + \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} d + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f$$

$$y_p = \begin{bmatrix} 3 & 3 \end{bmatrix} x_p + 0.05 d + f$$

For the continuous-time case, the second order filter matrices that resulted from the proposed $H_\infty$ FDI filter method are

$$\dot{x}_f = \begin{bmatrix} -4.0846 & 0.8631 \\ -4.0121 & -0.1450 \end{bmatrix} x_f + \begin{bmatrix} 0.8255 \\ 0.5981 \end{bmatrix} y_p .$$

$$r = \begin{bmatrix} -4.2108 & 0.4256 \end{bmatrix} x_f + 0.9975 y_p$$

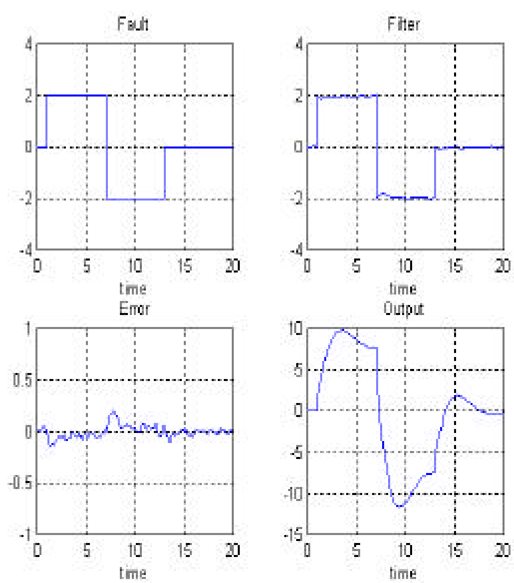


Figure 4 – Simulated results for the continuous-time example

The optimal value for the $H_\infty$ performance index $g$ was found to be 0.239. Simulation results may be seen in Figure 4.

The simulated fault was considered to be a combination of one positive and one negative pulse, as shown in Figure 4. A random disturbance input signal of zero mean and unitary variance is applied to the system disturbance input. The system output and the optimal FDI filter results are also shown in Figure 4.

It can be seen that the filter output is close to the fault input, which is considered unknown. The error, also shown in Figure 4, is under 5% of the output of the filter, and is mainly due to the random disturbance input.

For the discrete-time case, a sample interval of 0.01 seconds was used to discretize the state-space equations. The respective discrete-time $H_\infty$ FDI filter design results are shown in Figure 5.

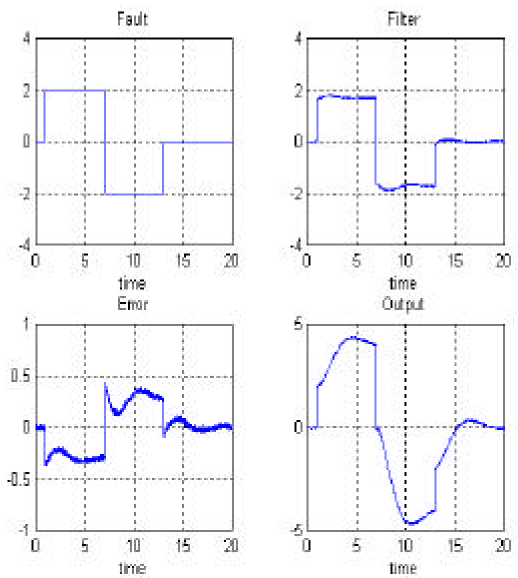


Figure 5 – Simulated results for the discrete-time example

The value of the $H_\infty$ performance index $g$ was found to be 0.243, which is a little worst compared to the continuous-time case. But, in spite of a higher error, the fault input estimation is practically as good as in the continuous case.

### 6. Conclusions

An LMI based approach was examined for the design of continuous-time and discrete-time $H_\infty$ FDI filters. The proposed approach provides necessary and sufficient conditions for the existence of admissible full and reduced-order FDI filters and a parameterization of all solutions. The full-order FDI filtering problem corresponds to the solution of convex LMIs, but the reduced-order design requires the solution of a non-convex problem with LMIs and a coupling rank constraint. The results were demonstrated using a simple simulation example of a spring-mass-damper model, for both the continuous and discrete-time cases.

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