

Guaranteed Cost Inequalities for Robust Stability and Performance Analysis

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1. Introduction

For unstructured time-varying or complex uncertainty, the small gain theorem provides a nonconservative test for robust stability [1]-[5]. For structured, and possibly real uncertainty, however, the small gain theorem is known to be conservative, and structured singular value bounds, which involve multipliers and complex scalings, can be used [6],[7]. Linear matrix inequalities (LMI's) are also used to guarantee robust stability [8]-[14].

Within the context of robust H_2 performance, the small gain theorem is equivalent to the bounded real bound [1]-[3], which plays the role of a guaranteed cost bound [5]. Various guaranteed cost bounds have been developed including quadratic and nonquadratic bounds. Quadratic bounds include the bounded real [1], [2], [3], positive real [20], and Popov bounds [5], [24], while nonquadratic bounds include the absolute value and linear bounds [15]-[17].

In the present paper we review the bounded real, Popov, inverse, shifted bounded real [21], and shifted Popov [23] bounds. In addition, we present the shifted linear, and shifted inverse bounds. Finally, we reformulate all of these bounds as guaranteed cost inequalities and then recast them as linear matrix inequalities.

The guaranteed cost bounds that we consider are either parameter independent or parameter dependent. Parameter-independent bounds, such as the bounded real bound, use a common Lyapunov function, whereas parameter-dependent bounds, such as the Popov bound, use a family of Lyapunov functions. For polytopic uncertainty we show that the least conservative common Lyapunov function can be determined by solving an optimization problem involving a set of linear matrix inequalities.

Notation

\mathbf{R}^d	$d \times 1$ real column vectors
$\mathbf{R}^{m \times n}$	$m \times n$ real matrices
$I_n, 0_n$	$n \times n$ identity matrix, $n \times n$ zero matrix
\mathbf{S}^n	$n \times n$ symmetric matrices
\mathbf{N}^n	$n \times n$ nonnegative-definite matrices
\mathbf{P}^n	$n \times n$ positive-definite matrices
$A \leq B$	$B - A$ is nonnegative definite
$\mathcal{E}(\cdot), \text{tr}, H $	expectation, trace, $(HH^T)^{1/2}$, where $H \in \mathbf{R}^{k_1 \times k_2}$
vec, \oplus, \otimes	Col. stacking Op., Kronecker sum, Kronecker product

2. Robust Performance and Guaranteed Cost Bounds

Let $\mathcal{U} \subset \mathbf{R}^{n \times n}$ denote an uncertainty set and consider the system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \quad (1)$$

where $x \in \mathbf{R}^n$ and $w \in \mathbf{R}^d$ are the state and disturbance, respectively, $\Delta A \in \mathcal{U}$, and the disturbance w is a standard zero-mean white noise process. In this paper, it is assumed that the nominal dynamics matrix A is asymptotically stable. Next, consider the performance variable

$$z(t) = Ex(t), \quad (2)$$

and define the worst-case H_2 performance measure

$$J(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \limsup_{t \rightarrow \infty} \mathcal{E} \left\{ \frac{1}{t} \int_0^t [x^T(s)Rx(s)] ds \right\},$$

where $R \triangleq E^T E$. If $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then $J(\mathcal{U})$ is given by

$$J(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \text{tr } P_{\Delta A} V, \quad (3)$$

where $V \triangleq DD^T$ and $P_{\Delta A}$ is the unique, nonnegative-definite solution to the Lyapunov equation

$$0 = (A + \Delta A)^T P_{\Delta A} + P_{\Delta A} (A + \Delta A) + R. \quad (4)$$

The following result, which is an extension of Theorem 3.1 of [24], provides a bound for the worst-case performance $J(\mathcal{U})$.

Theorem 1 Let $\Omega : \mathcal{N} \subseteq \mathbf{S}^n \rightarrow \mathbf{S}^n$, assume there exists $P \in \mathcal{N}$ satisfying

$$A^T P + PA + \Omega(P) + R \leq 0, \quad (5)$$

and let $P_0 : \mathcal{U} \rightarrow \mathbf{S}^n$ be such that

$$0 \leq P + P_0(\Delta A), \quad \Delta A \in \mathcal{U}, \quad (6)$$

and

$$0 \leq \Omega(P) - \Delta A^T P - P \Delta A \quad (7)$$

$$- [(A + \Delta A)^T P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)], \quad \Delta A \in \mathcal{U}.$$

Then $(A + \Delta A, E)$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case,

$$P_{\Delta A} \leq P + P_0(\Delta A), \quad \Delta A \in \mathcal{U}, \quad (8)$$

where $P_{\Delta A} \in \mathbf{N}^n$ is given by (4), and $J(\mathcal{U}) \leq \bar{J}(\mathcal{U})$, where

$$J(\mathcal{U}) = \text{tr } PV + \sup_{\Delta A \in \mathcal{U}} \text{tr } P_0(\Delta A)V. \quad (9)$$

If, in addition, there exists $\bar{P}_0 \in \mathbf{S}^n$ such that

$$P_0(\Delta A) \leq \bar{P}_0, \quad \Delta A \in \mathcal{U}, \quad (10)$$

then $J(\mathcal{U}) \leq \bar{J}(\mathcal{U})$, where

$$\bar{J}(\mathcal{U}) = \text{tr}[(P + \bar{P}_0)V]. \quad (11)$$

The pair (Ω, P_0) is a *bound*. A bound is *parameter independent* if P_0 is constant. In this case, $P_0(\Delta A)$ is replaced with P_0 and $\bar{P}_0 = P_0$ so that $\bar{J}(\mathcal{U}) = J(\mathcal{U})$. A bound is *parameter dependent* if P_0 depends on ΔA .

For a given bound (Ω, P_0) , the following result yields a bound $(\hat{\Omega}, \hat{P}_0)$. This result will be used in the proofs of Corollary 1 and Corollary 3.

Proposition 1 Let $\Omega : \mathcal{N} \subseteq \mathbf{S}^n \rightarrow \mathbf{S}^n$, $P \in \mathcal{N}$, and $P_0 : \mathcal{U} \rightarrow \mathbf{S}^n$ satisfy (5)-(7), and let $\bar{P}_0 \in \mathbf{S}^n$ satisfy (10). Let $\tilde{P}_0 \in \mathbf{S}^n$, and define $\hat{\mathcal{N}} \subseteq \mathbf{S}^n$, $\hat{\Omega} : \hat{\mathcal{N}} \rightarrow \mathbf{S}^n$ and $\hat{P}_0 : \mathcal{U} \rightarrow \mathbf{S}^n$ by

$$\hat{\mathcal{N}} \triangleq \mathcal{N} + \bar{P}_0 - \tilde{P}_0, \quad (12)$$

$$\hat{\Omega}(\hat{P}) \triangleq \Omega(\hat{P} - \bar{P}_0 + \tilde{P}_0) - A^T(\bar{P}_0 - \tilde{P}_0) - (\bar{P}_0 - \tilde{P}_0)A, \quad \hat{P} \in \hat{\mathcal{N}}, \quad (13)$$

and

$$\hat{P}_0(\Delta A) \triangleq P_0(\Delta A) - \bar{P}_0 + \tilde{P}_0. \quad (14)$$

Then (5)-(7) and (10) are satisfied with $\hat{\mathcal{N}}$, $\hat{\Omega}$, P , P_0 , and \bar{P}_0 replaced by $\hat{\mathcal{N}}$, $\hat{\Omega}$, $P + \bar{P}_0 - \tilde{P}_0$, \hat{P}_0 , and \tilde{P}_0 .

Remark 1 Let (Ω, P_0) be a bound. Choosing $\tilde{P}_0 \in \mathbf{S}^n$, Proposition 1 yields the bound $(\hat{\Omega}, \hat{P}_0)$. Now, using Proposition 1 and replacing (Ω, P_0) with $(\hat{\Omega}, \hat{P}_0)$ and letting $\tilde{P}_0 = \bar{P}_0$, yields (Ω, P_0) . This shows that (Ω, P_0) and $(\hat{\Omega}, \hat{P}_0)$ are equivalent.

Remark 2 Let (Ω, P_0) be a parameter-independent bound with $\bar{P}_0 = P_0$ and let $\tilde{P}_0 = 0$. Then Proposition 1 yields the equivalent parameter independent bound $(\hat{\Omega}, 0)$. Thus, without loss of generality, we can consider parameter-independent bounds of the form $(\Omega, 0)$.

Remark 3 If there exists $\Delta A \in \mathcal{U}$ such that $P_0(\Delta A) = 0$, then (8) implies $P \geq 0$, and without loss of generality we can assume $\mathcal{N} \subseteq \mathbf{N}^n$.

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In the following sections, \mathcal{U} is given by either the *parametric uncertainty set*

$$\mathcal{U} = \left\{ \Delta A : \Delta A = \sum_{i=1}^r \delta_i A_i, \text{ where } (\delta_1, \dots, \delta_r) \in \mathcal{R} \right\}, \quad (15)$$

where $A_i \in \mathbf{R}^{n \times n}$, $i = 1, \dots, r$, and $\mathcal{R} \subseteq \mathbf{R}^r$, or the *factored uncertainty set*

$$\mathcal{U} = \{ \Delta A : \Delta A = B_0 F C_0, \text{ where } F \in \mathcal{F} \}, \quad (16)$$

where $\mathcal{F} \subseteq \mathbf{R}^{l_1 \times l_2}$, $B_0 \in \mathbf{R}^{n \times l_1}$, and $C_0 \in \mathbf{R}^{l_2 \times n}$. To show that (15) can be written in the form (16), let $B_i \in \mathbf{R}^{n \times k_i}$ and $C_i \in \mathbf{R}^{k_i \times n}$ satisfy $A_i = B_i C_i$, $i = 1, \dots, r$, and let

$$B_0 = [B_1 \ \dots \ B_r] \in \mathbf{R}^{n \times k}, \quad C_0 = \begin{bmatrix} C_1 \\ \vdots \\ C_r \end{bmatrix} \in \mathbf{R}^{k \times n}, \quad (17)$$

where $k = \sum_{i=1}^r k_i$. Then $\Delta A = \sum_{i=1}^r \delta_i A_i = B_0 F C_0$, where $F \in \mathcal{F}_d$, and $\mathcal{F}_d \subset \mathbf{S}^k$ is defined by

$$\mathcal{F}_d \triangleq \{ F \in \mathbf{S}^k : F = \text{block-diag}(\delta_1 I_{k_1}, \dots, \delta_r I_{k_r}), (\delta_1, \dots, \delta_r) \in \mathcal{R} \}. \quad (18)$$

3. Vertex LMI's for Robust Performance

In this section linear matrix inequalities are used to determine a bound for $J(\mathcal{U})$ that is less conservative than parameter independent bounds $(\Omega, 0)$. First, LMI's are used to bound $J(\mathcal{U})$. Let (15) be given by the *polytopic uncertainty set*

$$\mathcal{U} = \left\{ \Delta A : \Delta A = \sum_{i=1}^r \delta_i A_i, \text{ where } |\delta_i| \leq \gamma, i = 1, \dots, r \right\}, \quad (19)$$

which is a special case of (15) with

$$\mathcal{R} = \{ (\delta_1, \dots, \delta_r) : |\delta_i| \leq \gamma, i = 1, \dots, r \}.$$

Proposition 2 Let \mathcal{U} be given by (19). Then $P \in \mathbf{N}^n$ satisfies

$$A^T P + P A \pm \gamma(A_1^T P + P A_1) \pm \dots \pm \gamma(A_r^T P + P A_r) + R \leq 0, \quad (20)$$

if and only if P satisfies

$$(A + \Delta A)^T P + P(A + \Delta A) + R \leq 0, \quad \Delta A \in \mathcal{U}. \quad (21)$$

Define

$$\mathcal{P} \triangleq \{ P \in \mathbf{N}^n : P \text{ satisfies (20)} \}.$$

Proposition 3 Let \mathcal{U} be given by (19), and let $(\Omega, 0)$ be a bound. If $P \in \mathbf{N}^n$ satisfies (5), then $P \in \mathcal{P}$.

The next result uses Proposition 2 to obtain the best parameter-independent bound.

Proposition 4 Let \mathcal{U} be given by (19), and define $\Omega_{\text{LMI}} : \mathcal{P} \rightarrow \mathbf{N}^n$ by

$$\Omega_{\text{LMI}}(P) \triangleq -R - A^T P - P A. \quad (22)$$

Then $(\Omega_{\text{LMI}}, 0)$ is a bound. Furthermore, if $(\Omega, 0)$ is a bound with $J(\mathcal{U})$ given by (9), then

$$\min_{P \in \mathcal{P}} \text{tr} PV \leq J(\mathcal{U}). \quad (23)$$

Note that if $(\Omega, 0)$ is a bound, then (5) and (7) imply (21). The next result provides an alternative vertex LMI.

Proposition 5 Let \mathcal{U} be given by (19), and let $\hat{P} \in \mathbf{N}^n$ and $Y_1, \dots, Y_r \in \mathbf{S}^n$ satisfy

$$A^T \hat{P} + \hat{P} A + \sum_{i=1}^r Y_i + R \leq 0 \quad (24)$$

and

$$-Y_i \leq \gamma(A_i^T \hat{P} + \hat{P} A_i) \leq Y_i, \quad i = 1, \dots, r. \quad (25)$$

Then (5)-(7) are satisfied with $P = \hat{P}$, $P_0 = 0$, $\mathcal{N} = \mathbf{N}^n$, and

$$\Omega(P) = \sum_{i=1}^r Y_i, \quad P \in \mathbf{N}^n. \quad (26)$$

4. Shifted Bounded Real Bound

Define $\mathcal{F}_{\text{BR}} \subset \mathbf{R}^{l_1 \times l_2}$ by

$$\mathcal{F}_{\text{BR}} \triangleq \{ F \in \mathbf{R}^{l_1 \times l_2} : F^T F \leq M \}, \quad (27)$$

where $M \in \mathbf{N}^{l_2}$. The following result concerns the classical *bounded real bound* [3], [4].

Proposition 6 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_{\text{BR}}$, and let $\mathcal{N} = \mathbf{N}^n$,

$$\Omega(P) = P B_0 B_0^T P + C_0^T M C_0, \quad (28)$$

and $P_0 = 0$. Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

For \mathcal{U} given by (19), the factored uncertainty set is given by (16), where $\mathcal{F} = \mathcal{F}_{\text{BRd}}$, and $\mathcal{F}_{\text{BRd}} \subset \mathbf{S}^k$ is defined by

$$\mathcal{F}_{\text{BRd}} \triangleq \{ F \in \mathcal{F}_d : |F| \leq \gamma I \}. \quad (29)$$

Next, define

$$\mathcal{F}_{\text{BRs}} \triangleq \{ F \in \mathbf{R}^{l_1 \times l_2} : N^T N + N^T F C_0 + C_0^T F^T N + C_0^T F^T F C_0 \leq M_s \}, \quad (30)$$

where $N \in \mathbf{R}^{l_1 \times n}$ and $M_s \in \mathbf{N}^n$. Note that if $N = 0$ and $M_s \geq C_0^T M C_0$, then $\mathcal{F}_{\text{BR}} \subseteq \mathcal{F}_{\text{BRs}}$. The following result concerns the *shifted bounded real bound* [21].

Proposition 7 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_{\text{BRs}}$, and let $\mathcal{N} = \mathbf{N}^n$,

$$\Omega(P) = P B_0 B_0^T P - (B_0 N)^T P - P B_0 N + M_s, \quad (31)$$

and $P_0 = 0$. Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

The *shifted bounded real inequality* has the form

$$(A - B_0 N)^T P + P(A - B_0 N) + P B_0 B_0^T P + M_s + R \leq 0. \quad (32)$$

For \mathcal{U} given by (19) with C_0 given by (17), define $\mathcal{F}_{\text{BRsd}} \subset \mathbf{S}^k$ by

$$\mathcal{F}_{\text{BRsd}} \triangleq \{ F \in \mathcal{F}_d : N^T N + N^T F C_0 + C_0^T F^T N + C_0^T F^T F C_0 \leq M_s \}, \quad (33)$$

where $M_s \in \mathbf{N}^n$. Note that if

$$N^T N + \gamma^2 C_0^T C_0 + \sum_{i=1}^r \delta_i (N_i^T C_i + C_i^T N_i) \leq M_s, \quad |\delta_i| \leq \gamma, \quad i = 1, \dots, r, \quad (34)$$

where $N_i \in \mathbf{R}^{k_i \times n}$, $i = 1, \dots, r$, and $N \triangleq [N_1^T \ \dots \ N_r^T]^T$, then $\mathcal{F}_{\text{BRd}} \subseteq \mathcal{F}_{\text{BRsd}}$. To satisfy (34), M_s can be chosen as

$$M_s = N^T N + \gamma^2 C_0^T C_0 + Y, \quad (35)$$

where $Y \in \mathbf{N}^n$ satisfies

$$\sum_{i=1}^r \delta_i (N_i^T C_i + C_i^T N_i) \leq Y, \quad |\delta_i| \leq \gamma, \quad i = 1, \dots, r. \quad (36)$$

With this choice of M_s , (32) becomes

$$(A - B_0 N)^T P + P(A - B_0 N) + P B_0 B_0^T P + N^T N + \gamma^2 C_0^T C_0 + Y + R \leq 0. \quad (37)$$

The next proposition gives three choices of Y that satisfy (36).

Proposition 8 Consider the following statements:

i) $Y = \sum_{i=1}^r Y_i$, where $Y_1, \dots, Y_r \in \mathbf{N}^n$ satisfy

$$Y_i = \gamma |N_i^T C_i + C_i^T N_i|, \quad i = 1, \dots, r. \quad (38)$$

ii) $Y = \sum_{i=1}^r Y_i$, where $Y_1, \dots, Y_r \in \mathbf{N}^n$ satisfy

$$-Y_i \leq \gamma (N_i^T C_i + C_i^T N_i) \leq Y_i, \quad i = 1, \dots, r. \quad (39)$$

iii) $Y \in \mathbf{N}^n$ satisfies

$$\pm \gamma (N_1^T C_1 + C_1^T N_1) \pm \dots \pm \gamma (N_r^T C_r + C_r^T N_r) \leq Y. \quad (40)$$

Then i) \implies ii) \implies iii) \iff (36).

With $Y = \sum_{i=1}^r Y_i$, where Y_1, \dots, Y_r are given by (38), (32) becomes

$$A^T P + P A + (P B_0 - N^T)(B_0^T P - N) \quad (41)$$

$$+ \gamma \sum_{i=1}^r |N_i^T C_i + C_i^T N_i| + \gamma^2 C_0^T C_0 + R \leq 0.$$

Letting $\beta \in \mathbf{R}$ and $N = \beta B_0^T$, (41) becomes

$$A^T P + P A + (P - \beta I) B_0 B_0^T (P - \beta I) \quad (42)$$

$$+ \gamma |\beta| \sum_{i=1}^r |A_i + A_i^T| + \gamma^2 C_0^T C_0 + R \leq 0.$$

The following result provides a sufficient condition under which (42) has a positive-definite solution.

Proposition 9 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_{\text{BRsd}}$. Suppose there exist $\nu_1, \dots, \nu_r > 0$ such that

$$\nu_i(A + A^T) + \gamma\beta|A_i + A_i^T| + R < 0, \quad i = 1, \dots, r, \quad (43)$$

where $\beta = \sum_{i=1}^r \nu_i$. Furthermore, let $C_i = \gamma^{-1}[-\nu_i(A + A^T) - \gamma\beta|A_i + A_i^T| - R]^{1/2}$ and $B_i = A_i C_i^{-1}$, $i = 1, \dots, r$. Then $P = \beta I$ satisfies (42).

Suppose A is dissipative, that is, $A + A^T$ is negative definite. Then there exist $\nu_1, \dots, \nu_r > 0$ and $\gamma > 0$ such that (43) holds. If, in addition, A_1, \dots, A_r are skew symmetric, then there exists $\nu > 0$ such that (43) holds with $\nu_1 = \dots = \nu_r = \nu$ for all $\gamma > 0$. Furthermore, $P = r\nu I$ satisfying (42) is independent of γ .

Applying Proposition 1 to Proposition 7 with $N = \beta B_0^T$ and $\hat{P}_0 = \beta I$ yields the following result.

Corollary 1 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_{\text{BRsd}}$, let $\beta \in \mathbf{R}$, and suppose there exists $P \in \mathbf{N}^n - \beta I$ satisfying

$$A^T P + P A + P B_0 B_0^T P + \gamma|\beta| \sum_{i=1}^r |A_i + A_i^T| + \beta(A + A^T) + \gamma^2 C_0^T C_0 + R \leq 0. \quad (44)$$

Then (6) and (7) are satisfied with $P_0 = \beta I$.

Next, an LMI is formulated to obtain a feasible solution $P \in \mathbf{N}^n$ to the shifted bounded real inequality (32) along with M_s and N_1, \dots, N_r . Let $N = \begin{bmatrix} N_1^T & \dots & N_r^T \end{bmatrix}^T$.

Proposition 10 Let \mathcal{U} be given by (19) with $\mathcal{F} = \mathcal{F}_{\text{BRsd}}$, and let $P \in \mathbf{N}^n$, $N_i \in \mathbf{R}^{k_i \times n}$, $i = 1, \dots, r$, and $Y \in \mathbf{S}^n$. Then P , N_1, \dots, N_r , Y satisfy (40) and

$$\begin{bmatrix} A^T P + P A + \gamma^2 C_0^T C_0 + Y + R & P B_0 - N^T \\ B_0^T P - N & -I \end{bmatrix} \leq 0 \quad (45)$$

if and only if P , N_1, \dots, N_r , Y satisfy (36) and (37).

5. Shifted Popov Bound

For convenience, define $\mathcal{F}_P \in \mathbf{S}^n$, $\mathcal{H}_P \in \mathbf{P}^k$, and $\mathcal{N}_P \in \mathbf{R}^{k \times k}$ by

$$\mathcal{F}_P \triangleq \{F \in \mathbf{S}^k : M_L \leq F \leq M_U\}, \quad (46)$$

$$\mathcal{H}_P \triangleq \{H \in \mathbf{P}^k : H F = F^T H, \quad F \in \mathcal{F}_P\}, \quad (47)$$

$$\mathcal{N}_P \triangleq \{N \in \mathbf{R}^{k \times k} : N^T F = F N, \quad F \in \mathcal{F}_P\}, \quad (48)$$

where $M_L, M_U \in \mathbf{S}^k$ are such that $M \triangleq M_U - M_L$ is positive definite. The following result concerns the *Popov bound* [24].

Proposition 11 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_P$, $\mathcal{N} = \mathbf{N}^n$, $N \in \mathcal{N}_P$ and $H \in \mathcal{H}_P$. Let

$$\Omega(P) = (H C_0 + N C_0 \tilde{A}_P + B_0^T P)^T R_0^{-1} (H C_0 + N C_0 \tilde{A}_P + B_0^T P) + (B_0 M_L C_0)^T P + P B_0 M_L C_0, \quad (49)$$

where

$$R_0 \triangleq (H M^{-1} - N C_0 B_0) + (H M^{-1} - N C_0 B_0)^T > 0, \quad (50)$$

$\tilde{A}_P \triangleq A + B_0 M_L C_0$, and

$$P_0(\Delta A) = C_0^T (F - M_L) N C_0. \quad (51)$$

Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

Remark 4 If $\mu \in \mathbf{S}^k$ satisfies $\mu \geq (F - M_L)N$ for all $F \in \mathcal{F}_P$, then $\tilde{P}_0 = C_0^T \mu C_0$ satisfies (10).

The following result concerns the *shifted Popov bound* [23].

Proposition 12 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_P$, $\mathcal{N} = \mathbf{N}^n$, $H \in \mathcal{H}_P$, $N \in \mathcal{N}_P$, $X \in \mathbf{R}^{k \times k}$, and $Y \in \mathbf{S}^n$. Let

$$\begin{aligned} \Omega(P) &= (H C_0 + N C_0 \tilde{A}_P + B_0^T P - X B_0^T)^T \\ &\quad \times R_0^{-1} (H C_0 + N C_0 \tilde{A}_P + B_0^T P - X B_0^T) \\ &\quad + (B_0 M_L C_0)^T P + P B_0 M_L C_0 + Y, \end{aligned} \quad (52)$$

and let $P_0(\Delta A)$ be given by (51), where Y satisfies

$$B_0 X^T (F - M_L) C_0 + C_0^T (F - M_L) X B_0^T \leq Y, \quad F \in \mathcal{F}_P, \quad (53)$$

and R_0 is given by (50). Then, for all $P \in \mathcal{N}$, (7) and (6) are satisfied.

Remark 5 Setting $X = 0$ and $Y = 0$ in Proposition 12 yields Proposition 11.

For \mathcal{U} given by (19), let B_0 and C_0 be given by (17). Define $\mathcal{F}_{\text{Pd}} \subset \mathbf{S}^k$, $\mathcal{H}_{\text{Pd}} \subset \mathbf{P}^k$, $\mathcal{N}_{\text{Pd}} \subset \mathbf{S}^k$, and $\hat{I}_i \in \mathbf{S}^k$ by

$$\mathcal{F}_{\text{Pd}} \triangleq \{F \in \mathcal{F}_d : -\gamma I \leq F \leq \gamma I\}, \quad (54)$$

$$\mathcal{H}_{\text{Pd}} \triangleq \{H \in \mathbf{P}^k : H = \text{block-diag}(H_1, \dots, H_r)\}, \quad (55)$$

$$\mathcal{N}_{\text{Pd}} \triangleq \{N \in \mathbf{S}^k : N = \text{block-diag}(N_1, \dots, N_r)\}, \quad (56)$$

and $\hat{I}_i \triangleq \text{block-diag}(0_{k_1}, \dots, 0_{k_{i-1}}, I_{k_i}, 0_{k_{i+1}}, \dots, 0_{k_r})$, where $k = \sum_{i=1}^r k_i$, $H_i \in \mathbf{P}^{k_i}$, and $N_i \in \mathbf{S}^{k_i}$, $i = 1, \dots, r$.

Proposition 13 Let \mathcal{U} be given by (16) with $\mathcal{F} = \mathcal{F}_{\text{Pd}}$, and suppose there exist $P \in \mathbf{N}^n$, $\tilde{P}_0 \in \mathbf{N}^n$, $\tilde{X} \in \mathbf{R}^{k \times k}$, $Y \in \mathbf{S}^n$, $\tilde{N} \in \mathcal{N}_{\text{Pd}}$, and $\tilde{H} \in \mathcal{H}_{\text{Pd}}$ such that

$$\begin{bmatrix} A_P^T P + P A_P + Y + R & C_0^T \tilde{H} + \tilde{A}_P^T C_0^T \tilde{N} + P B_0 - B_0 \tilde{X}^T \\ \tilde{H} C_0 + \tilde{N} C_0 \tilde{A}_P + B_0^T P - \tilde{X} B_0^T & \tilde{N} C_0 B_0 + B_0^T C_0^T \tilde{N} - \frac{1}{\gamma} \tilde{H} \end{bmatrix} < 0, \quad (57)$$

$$\begin{aligned} \pm \gamma (B_0 \tilde{X}^T C_0 + C_0^T \tilde{X} B_0^T) \pm \gamma (B_0 \tilde{X}^T \hat{I}_1 C_0 + C_0^T \hat{I}_1 \tilde{X} B_0^T) \\ \pm \dots \pm \gamma (B_0 \tilde{X}^T \hat{I}_r C_0 + C_0^T \hat{I}_r \tilde{X} B_0^T) \leq Y, \end{aligned} \quad (58)$$

and

$$\gamma C_0^T (\pm I \pm \hat{I}_1 \pm \hat{I}_2 \pm \dots \pm \hat{I}_r) \tilde{N} C_0 \leq \tilde{P}_0. \quad (59)$$

Then Y satisfies (53), and (5)-(7) and (10) are satisfied with $\Omega(P)$ given by (52) and $P_0(\Delta A)$ given by (51).

Corollary 2 Let $Y_1, \dots, Y_{r+1} \in \mathbf{S}^n$ satisfy

$$-Y_i \leq \gamma B_0 \tilde{X}^T \hat{I}_i C_0 + \gamma C_0^T \tilde{X} \hat{I}_i B_0^T \leq Y_i, \quad i = 1, \dots, r, \quad (60)$$

$$-Y_{r+1} \leq \gamma B_0 \tilde{X}^T C_0 + \gamma C_0^T \tilde{X} B_0^T \leq Y_{r+1}, \quad (61)$$

and let $P_1, \dots, P_{r+1} \in \mathbf{S}^n$ satisfy

$$-P_i \leq \gamma C_0^T \hat{I}_i \tilde{N} C_0 \leq P_i, \quad i = 1, \dots, r, \quad (62)$$

$$-P_{r+1} \leq \gamma C_0^T \tilde{N} C_0 \leq P_{r+1}. \quad (63)$$

Then $Y = \sum_{i=1}^{r+1} Y_i$ satisfies (53). Finally, let $P_0(\Delta A)$ be given by (51). Then $\tilde{P}_0 = \sum_{i=1}^{r+1} P_i$ satisfies (10).

6. Shifted Linear Bound

In this section, let the uncertainty set \mathcal{U} be given by (19), which is an alternative to the elliptical set, see [18]. The following result concerns the *linear bound* [16], [26].

Proposition 14 Let $\mathcal{N} = \mathbf{N}^n$, $\alpha > 0$,

$$\Omega(P) = \alpha r P + \frac{\gamma^2}{\alpha} \sum_{i=1}^r A_i^T P A_i, \quad (64)$$

and $P_0 = 0$. Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

Next, the *shifted linear bound* is obtained.

Proposition 15 Let $\mathcal{N} = \mathbf{N}^n \cap (\bigcap_{i=1}^r [\mathbf{N}^n + N_i])$, where N_1, \dots, N_r , $Y \in \mathbf{S}^n$, and let $\alpha > 0$. Let

$$\Omega(P) = \sum_{i=1}^r [\alpha(P - N_i) + \frac{\gamma^2}{\alpha} A_i^T (P - N_i) A_i] + Y, \quad (65)$$

where

$$\sum_{i=1}^r \delta_i (A_i^T N_i + N_i^T A_i) \leq Y, \quad |\delta_i| \leq \gamma, \quad i = 1, \dots, r, \quad (66)$$

and let $P_0 = 0$. Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

Remark 6 Setting $N_1, \dots, N_r = 0$ and $Y = 0$ in Proposition 15 yields Proposition 14.

The next result provides a method for computing P satisfying

$$A^T P + P A + \sum_{i=1}^r [\alpha(P - N_i) + \frac{\gamma^2}{\alpha} A_i^T (P - N_i) A_i] + Y + R = 0. \quad (67)$$

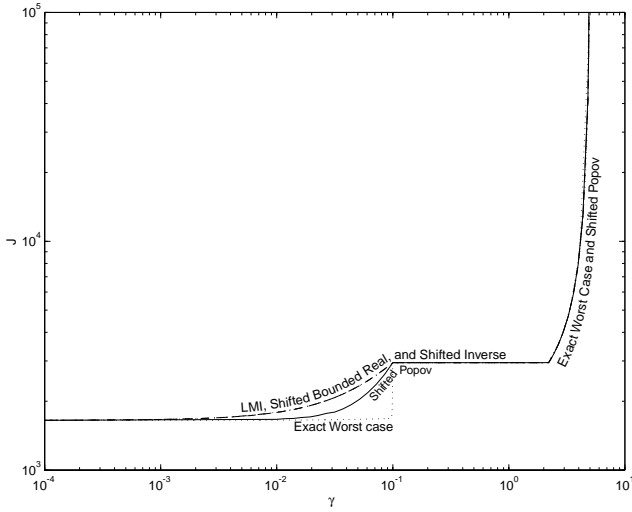


Figure 1: Cost bounds for Example 1, demonstrating a destabilizing uncertainty.

Proposition 16 Let $N_1, \dots, N_r \in \mathbf{S}^n$, $Y = \gamma \sum_{i=1}^r [A_i^T N_i + N_i^T A_i]$, and $\alpha > 0$. Suppose

$$\mathcal{A} \triangleq (A + \frac{\alpha}{2}I) \oplus (A + \frac{\alpha}{2}I) + \frac{\gamma^2}{\alpha} \sum_{i=1}^r (A_i \otimes A_i) \quad (68)$$

is invertible. Then $P = -\text{vec}^{-1}(\mathcal{A}^{-T} \text{vec } R_0)$ is the unique solution to (67), where $R_0 \triangleq \sum_{i=1}^r [\gamma |A_i^T N_i + N_i A_i| - \alpha N_i - \frac{\gamma^2}{\alpha} A_i^T N_i A_i] + R$. If, in addition, \mathcal{A} is asymptotically stable and R_0 is nonnegative definite, then $P \geq 0$.

Remark 7 The last statement of Proposition 16 follows from techniques used in [27].

Proposition 17 Let \mathcal{U} be given by (19). Suppose there exist $\alpha > 0$, $P \in \mathbf{N}^n$ and $N_1, \dots, N_r, Y \in \mathbf{S}^n$ satisfying

$$A^T P + P A + \sum_{i=1}^r [\alpha(P - N_i) + \frac{\gamma^2}{\alpha} A_i^T (P - N_i) A_i] + Y + R \leq 0 \quad (69)$$

and

$$\pm \gamma (A_1^T N_1 + N_1 A_1) \pm \dots \pm \gamma (A_r^T N_r + N_r A_r) \leq Y. \quad (70)$$

Then Y satisfies (66), and (5)-(7) are satisfied with $\Omega(P)$ given by (65).

Applying Proposition 1 to Proposition 15 with $N_1 = \dots = N_r = N$ and setting $P_0 = N$ yields the following specialization of Proposition 15.

Corollary 3 Let $\alpha > 0$, $P_0 \in \mathbf{S}^n$, and suppose there exists $P \in \mathbf{N}^n \cap (\mathbf{N}^n - P_0)$ satisfying

$$0 = A^T (P + P_0) + (P + P_0) A + \alpha r P + \sum_{i=1}^r \left[\frac{\gamma^2}{\alpha} A_i^T P A_i + \gamma |A_i^T P_0 + P_0 A_i| \right] + R. \quad (71)$$

Then (6) and (7) are satisfied.

7. Shifted Inverse Bound

In this section, let the uncertainty set \mathcal{U} be given by (19). The following result concerns the *inverse bound* [18].

Proposition 18 Let $\mathcal{N} = \mathbf{P}^n$, $\alpha > 0$,

$$\Omega(P) = \alpha \gamma r P + \frac{\gamma}{4\alpha} \sum_{i=1}^r (A_i^T P + P A_i) P^{-1} (A_i^T P + P A_i), \quad (72)$$

and $P_0 = 0$. Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

Note that the *inverse bound inequality*

$$A^T P + P A + \alpha \gamma r P + \frac{\gamma}{4\alpha} \sum_{i=1}^r (A_i^T P + P A_i) P^{-1} (A_i^T P + P A_i) + R \leq 0 \quad (73)$$

can be written as

$$\tilde{A}_{\text{inv}}^T P + P \tilde{A}_{\text{inv}} + \frac{\gamma}{4\alpha} \sum_{i=1}^r (A_i^T P A_i + P A_i P^{-1} A_i^T P) + R \leq 0, \quad (74)$$

where $\tilde{A}_{\text{inv}} \triangleq A + \alpha r \frac{\gamma}{2} I + \frac{\gamma}{4\alpha} \sum_{i=1}^r A_i^2$.

Next, the *shifted inverse bound* is obtained.

Proposition 19 Let $\mathcal{N} = \mathbf{N}^n \cap (\cap_{i=1}^r [\mathbf{P}^n + N_i])$, where $N_1, \dots, N_r, Y \in \mathbf{S}^n$, let $\alpha > 0$, $M_{1i}, M_{2i} \in \mathbf{R}^{n \times n}$,

$$\Omega(P) = \sum_{i=1}^r \left(\frac{\gamma}{4\alpha} [A_i^T (P - M_{1i}) + (P - M_{2i}) A_i] (P - N_i)^{-1} \right. \\ \left. \times [A_i^T (P - M_{1i}) + (P - M_{2i}) A_i]^T + \alpha \gamma (P - N_i) \right) + Y, \quad (75)$$

where

$$\sum_{i=1}^r \frac{\delta_i}{2} [A_i^T (M_{1i} + M_{2i}^T) + (M_{2i} + M_{1i}^T) A_i] \leq Y, \quad |\delta_i| \leq \gamma, \quad i = 1, \dots, r, \quad (76)$$

and $P_0 = 0$. Then, for all $P \in \mathcal{N}$, (6) and (7) are satisfied.

Substituting $\Omega(P)$ into (5) yields the *shifted inverse bound inequality*

$$A^T P + P A + \sum_{i=1}^r \left(\frac{\gamma}{4\alpha} [A_i^T (P - M_{1i}) + (P - M_{2i}) A_i] (P - N_i)^{-1} \right. \\ \left. \times [A_i^T (P - M_{1i}) + (P - M_{2i}) A_i]^T - \alpha \gamma N_i \right) + \alpha \gamma r P + Y + R \leq 0. \quad (77)$$

Let $M_{1i} = M_{2i} = M_i \in \mathbf{S}^n$, $i = 1, \dots, r$. The next result uses LMI's to find $P \in \mathbf{P}^n$, $M_1, \dots, M_r, N_1, \dots, N_r, Y \in \mathbf{S}^n$ and $\alpha > 0$ satisfying (5) with $\Omega(P)$ given by (75) and (76).

Proposition 20 Let \mathcal{U} be given by (19) and let $\alpha > 0$ be given. Suppose there exist $P \in \mathbf{N}^n$, and $N_1, \dots, N_r, M_1, \dots, M_r, Y \in \mathbf{S}^n$ satisfying

$$\begin{bmatrix} A^T P + P A + \alpha \gamma \sum_{i=1}^r (P - N_i) + Y + R & A_1^T \tilde{P}_1 + \tilde{P}_1 A_1 & \dots & A_r^T \tilde{P}_r + \tilde{P}_r A_r \\ A_1^T \tilde{P}_1 + \tilde{P}_1 A_1 & -\frac{4\alpha}{\gamma} (P - N_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ A_r^T \tilde{P}_r + \tilde{P}_r A_r & 0 & 0 & -\frac{4\alpha}{\gamma} (P - N_r) \end{bmatrix} < 0, \quad (78)$$

where $\tilde{P}_i = P - M_i$, $i = 1, \dots, r$, and

$$\pm \gamma (A_1^T M_1 + M_1 A_1) \pm \dots \pm \gamma (A_r^T M_r + M_r A_r) \leq Y. \quad (79)$$

Then Y satisfies (76), and (5)-(7) are satisfied with $\Omega(P)$ given by (75).

Remark 8 As in the result *ii*) \implies *iii*) from Proposition 8, (79) can be recast as $2rn^2$ constraints.

Corollary 4 Let $A + A^T < 0$, $A_i + A_i^T = 0$, $M_{1i} = M_{2i} = 0$, $N_i = \frac{1}{\alpha \gamma} [\beta(A + A^T) + R] + \beta I$, $i = 1, \dots, r$, and let β satisfy $\beta(A + A^T) + R < 0$. Then (6) and (7) are satisfied with $P = \beta I$ and $P_0 = 0$.

8. Examples

LMI methods can be used to calculate solutions along with optimal scalings for bounds such as the linear, bounded real, inverse, and Popov bounds, as well as their shifted counterparts. In the case of the inverse and linear bounds, the α scalings must be chosen separately.

Example 1 Let $A = \begin{bmatrix} -0.005 & 1 \\ -1 & -0.005 \end{bmatrix}$, $A_1 = \begin{bmatrix} .001 & .10 \\ -10 & .001 \end{bmatrix}$, $R = \begin{bmatrix} 0.25 & 0.12 \\ 0.12 & 2.5 \end{bmatrix}$, and $V = \begin{bmatrix} 0 & 0 \\ 0 & 12 \end{bmatrix}$. Although the uncertainty is nearly skew symmetric, it is destabilizing. Figure 1 shows the actual worst case cost. Now, let $B_1 = \varepsilon^{-1} 0.001 I_2$, and $C_1 = \varepsilon \begin{bmatrix} 1 & 10000 \\ -10000 & 1 \end{bmatrix}$, where $\varepsilon \neq 0$ is a free parameter used for optimization. With this choice, the bounded real and linear bounds guarantee stability for $|\delta| < 4 \times 10^{-4}$, while, as can be seen in Figure 1, the shifted linear bound yields an improved robust performance bound for $|\delta| < 4 \times 10^{-4}$. The LMI version of the shifted bounded real bound given by Proposition 10 coincides with the LMI bound given by Proposition 4. The shifted Popov bound guarantees stability for $|\delta| < 5$, and outperforms the LMI bound. The shifted inverse bound guarantees stability for $|\delta| < 5$ and coincides with the LMI bound.

Example 2 Let

$$A = \begin{bmatrix} -0.0002 & 0.2208 & 0 & 0 \\ -0.2208 & -0.0002 & 0 & 0 \\ 0 & 0 & -0.0103 & 1.4322 \\ 0 & 0 & -1.4322 & -0.0103 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0_{2 \times 2} & I_2 \\ -I_2 & 0_{2 \times 2} \end{bmatrix},$$

where the uncertainty represents modal coupling. Furthermore, let $R = I_4$ and $V = I_4$, and let B_1 and C_1 be given by $B_1 = \begin{bmatrix} \varepsilon I_2 & 0_{2 \times 2} \\ 0_{2 \times 2} & -\varepsilon I_2 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0_{2 \times 2} & \varepsilon^{-1} I_2 \\ \varepsilon^{-1} I_2 & 0_{2 \times 2} \end{bmatrix}$, where $\varepsilon \neq 0$ is a free parameter. When LMI methods are used to obtain a bound on the cost, ε is obtained as part of the optimization process. In the other cases, the optimal value ε is obtained by evaluating a mesh of points.

Each plot in Figure 3 shows the actual worst-case performance along with the LMI bound given by Proposition 4. As can be seen, the bounded real bound guarantees stability for $|\delta| < 0.0003$, whereas the shifted bounded real bound with $N = \beta B_1^T$ guarantees stability for $|\delta| < 0.01$, and from Proposition 9 stability is guaranteed for $|\delta| \geq 0.01$. Then the shifted bounded real bound guarantees stability for all $\delta \in \mathbf{R}$. Applying Proposition 10, the shifted bounded real bound coincides with the LMI bound given by Proposition 2. Next, the Popov bound guarantees stability for $|\delta| < 0.1$, while the shifted Popov bound outperforms the LMI bound given by Proposition 2. The linear bound guarantees stability for $|\delta| < 6 \times 10^{-4}$, while the shifted linear bound is less conservative for $|\delta| < 6 \times 10^{-4}$. Finally, the inverse bound guarantees stability for all $\delta \in \mathbf{R}$, while the shifted inverse bound, with $M_1 = 0$ coincides with the LMI bound given by Proposition 2. Figure 2 compares the shifted bounds.

Example 3 Consider a variation of Example 2 with two uncertain parameters. First, consider two nondestabilizing uncertain parameters. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (80)$$

and let B_0 and C_0 be given by (17), where

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T. \quad (81)$$

Figure 4 (a) shows the performance predicted by the shifted bounded real bound and the shifted Popov bound, which coincide with the LMI bound given by Proposition 4.

Next, let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (82)$$

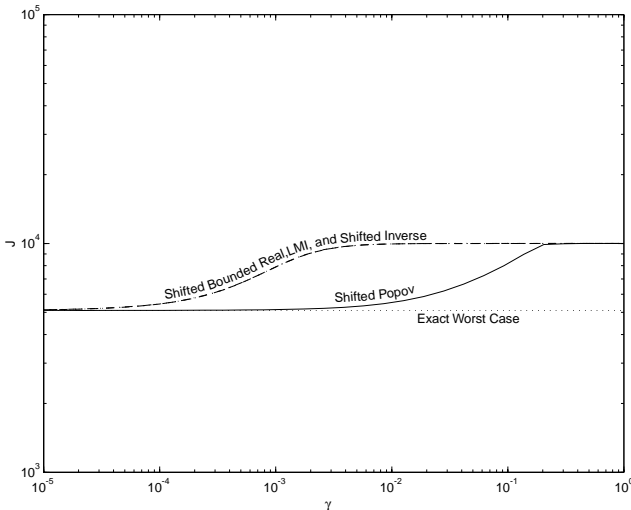


Figure 2: Cost bounds for Example 2.

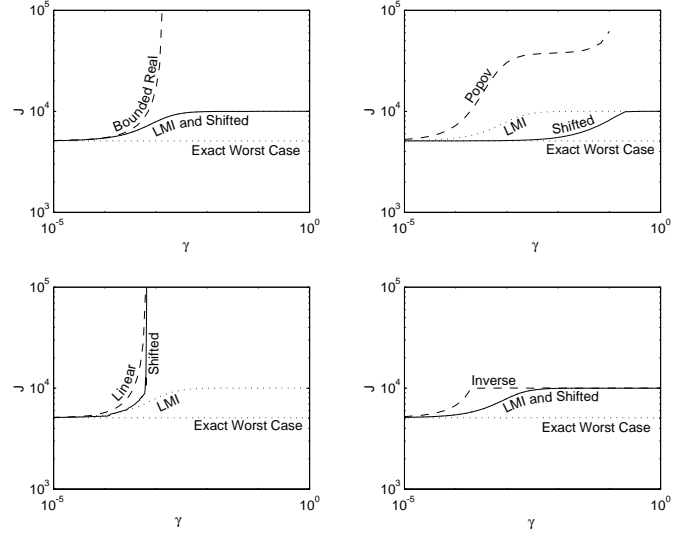


Figure 3: Comparison of shifted, unshifted, and LMI bounds for Example 2.

and let B_0 and C_0 be given by (17) where C_1 , B_2 , and C_2 are given in (81), and $B_1 = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}^T$. In this case, the first uncertain parameter is destabilizing. Figure 4 (b) shows the performance predicted by the shifted bounded real bound which coincides with the LMI bound, and the shifted Popov bound which does slightly better for higher levels of uncertainty.

Next, let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (83)$$

and let B_0 and C_0 be given by (17) where $B_1 = \begin{bmatrix} 0 & 0.1 & 0 & 0 \end{bmatrix}^T$, $C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, and B_2 and C_2 are given in (81). In this case, the first uncertain parameter is destabilizing. Figure 4 (c) shows the performance predicted by the shifted bounded real bound which coincides with the LMI bound, and the shifted Popov bound which, does slightly better for higher levels of uncertainty.

Finally, let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (84)$$

and let B_0 and C_0 be given by (17) where

$$B_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, both uncertain parameter are destabilizing. Figure 4 (d) shows the performance predicted by the shifted bounded real bound and the shifted Popov bound, which coincide with the LMI bound.

9. Conclusions

In this paper the shifted bounded real bound [21], the shifted linear bound, the shifted inverse bound, and the shifted Popov bound [23] have been considered. These bounds were compared with the bounded real bound, the linear bound, the inverse bound, the Popov bound and an LMI bound. It was shown that these shifted bounds can be recast as LMI's, and presented as guaranteed cost inequalities. For several examples, it was shown that the shifted bounded real bound and shifted inverse bound exhibit comparable performance to that of the best possible parameter-independent bound. It has also been shown for several numerical examples that the shifted Popov bound, which is a parameter-dependent bound, may be less conservative than parameter-independent bounds.

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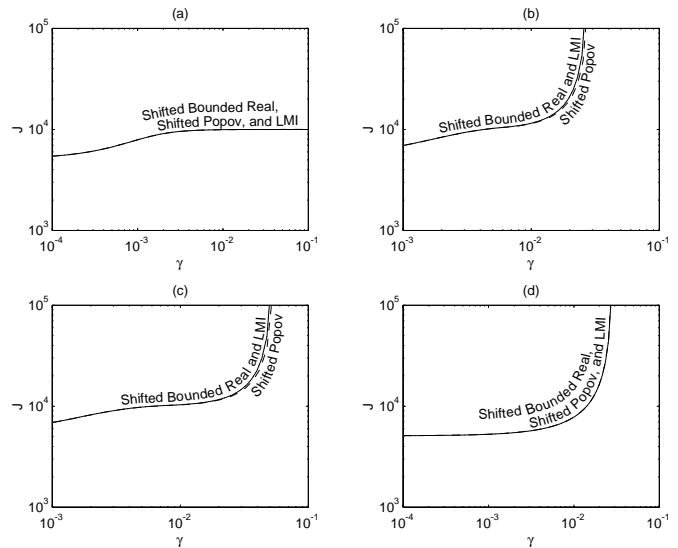


Figure 4: Cost bounds for Example 3, comparing the shifted bounded real, shifted Popov and LMI bounds with two uncertain parameters: (a) two skew-symmetric uncertainties, (b) symmetric and skew-symmetric uncertainty, (c) nilpotent and skew-symmetric uncertainty, (d) two nilpotent uncertainties.