Ellipsoidal Approximations of Reachable Sets for Linear Games [1]

Norihiko Shishido and Claire J. Tomlin
Hybrid Systems Laboratory
Department of Aeronautics and Astronautics
Stanford University, Stanford, CA 94305
shishido,tomlin@stanford.edu

Abstract

Verification of safety properties for continuous, discrete, and hybrid systems requires computation of the reachable sets of states for such systems. It is of great interest to develop efficient and scalable numerical algorithms for computation and representation of this reachable set. In this paper, we compute reachable sets for linear differential games, in which one player (the "control") tries to keep the state of the system outside of a given *unsafe* subset of the state space; and the second player (the "disturbance") tries to push the system into this subset. We model this unsafe set, the input set, and the disturbance set as ellipsoids, and we derive conditions under which the reachable set at each time t is an ellipsoid. We give an integral form equation whose solution represents this ellipsoid, and we present special cases in which this ellipsoid may be computed analytically. We conclude with a set of examples.

KEYWORDS: Verification, reachability, ellipsoids, differential games, hybrid systems.

1 Introduction

The computation of reachable sets of states for control systems is key to verifying their safety. Many safety critical systems depend on a correct and efficient computation. We have studied several examples in previous work, such as the design of vehicle maneuvers for ground and air transportation systems [1, 2], and the development of flight mode switching schemes for aircraft [3, 4], and we have developed a method for computing reachable sets (and control laws) for *hybrid control systems*, which incorporate both continuous and discrete state dynamics. For example, consider the continuous state, continuous time control system represented as the following differential equation evolving on state space X:

$$\dot{x}(t) = f(x(t), u(t), d(t)) \tag{1}$$

where $u \in U$, $U \subseteq \boldsymbol{R}^{n_u}$ is the set of control inputs, and $d \in D$, $D \subseteq \boldsymbol{R}^{n_d}$ is the set of disturbance inputs. The spaces of acceptable control and disturbance trajectories are the spaces of piecewise continuous functions $\mathcal{U} = \{u(\cdot) \in PC^0 \mid u(t) \in U,\ \forall t \in \boldsymbol{R}\}$, $\mathcal{D} = \{d(\cdot) \in PC^0 \mid d(t) \in D,\ \forall t \in \boldsymbol{R}\}$. The problem of computing the set of states which reaches a given *unsafe* set $G \subset X$ may be posed as follows: *Determine the set $V \subset X$ for which there exists a $t \le 0$ such that for all $x \in V$ and for all $u(\tau) \in \mathcal{U}$, there exists a $d(\tau) \in \mathcal{D}$ such that $x(t) = x$, $x(0) \in G$, and $x(\tau)$, for $t \le \tau \le 0$ is the solution to (1).*

This problem may be solved using the framework of differential game theory [5, 6] by constructing a value function $J(x, u, d, t)$ such that $J(x, u, d, 0)$ has zero level set on the boundary of G. In [7], the boundary of the *reachable set V* is shown to be the zero level set of the solution $J^*(x,t)$ to the following Hamilton-Jacobi equation:

$$-\frac{\partial J^*(x,t)}{\partial t} = \begin{cases} \frac{\partial J^*(x,t)}{\partial x} f(x, u^*, d^*) \\ \text{for } \{x \in X : J^*(x,t) > 0\} \\ \min\{0, \frac{\partial J^*(x,t)}{\partial x} f(x, u^*, d^*)\} \\ \text{for } \{x \in X : J^*(x,t) \le 0\} \end{cases} \tag{2}$$

with boundary condition $J^*(x,0)$, where (u^*, d^*) represents the $\max_u \min_d$ solution to the game. In [8], a computational method is designed to solve equation (2) for hybrid systems. The method is based on level set techniques for partial differential equations, and has been shown to work efficiently in state spaces of 2 or 3 dimensions, yet suffers from a complexity which grows exponentially with the dimension of the continuous state.

In response to the need for algorithms which work in high dimensions, approximation techniques have been proposed for solving the reachability problem for continuous state systems. One of the most promising classes of approximation techniques uses ellipsoids to over-approximate the reachable set at any given time [9, 10]. As a data structure, ellipsoids are efficient to store and easy to manipulate [11, 12]; in addition, efficient computational methods based on convex optimization have been developed to calculate both the

[1] Research supported by Kawasaki Heavy Industries, Ltd., and DARPA under the Software Enabled Control Program (administered by AFRL under contract F33615-99-C-3014).

minimum volume ellipsoid containing given points, and the maximum volume ellipsoid in a polyhedron [13, 14].

In this paper, we consider the following problem. Suppose the dynamics of (1) are simplified to the following linear time-invariant (LTI) system:

$$\dot{x}(t) = Ax(t) + Bu(t) + Cd(t) \tag{3}$$

with $x \in \boldsymbol{R}^n$, $u \in U$ and $d \in D$, where U and D are given by ellipsoids in $\boldsymbol{R}^{n_u}$ and $\boldsymbol{R}^{n_d}$ respectively. Assume the unsafe set G is also given as an ellipsoid. For this problem, we present the formula for the boundary of the set of states backwards reachable from G. We then derive conditions on U and D under which the reachable set, at each instant $t < 0$, is an ellipsoid, and we use these conditions to give the formula for an ellipsoidal over-approximation to the actual reachable set. We then present several examples in which this formula has an analytic solution. We conclude with a discussion of our current research in this area, in which we are investigating extensions to nonlinear and hybrid systems.

2 Linear Differential Games

Our aim is to provide an efficient way to over-approximate the reachable set using ellipsoids. The definition of the linear differential game and the notation of an ellipsoid used in this paper are given below.

Definition 1 (Linear Differential Game) *Consider the LTI system:*

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t) + Cd(t) \tag{4}$$

where $A \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times n_u}$, and $C \in \boldsymbol{R}^{n\times n_d}$. The spaces of acceptable control and disturbance trajectories are the spaces of piecewise continuous functions $\mathcal{U} = \{u(\cdot) \in PC^0 | u(t) \in U, \forall t \in \boldsymbol{R}\}$, $\mathcal{D} = \{d(\cdot) \in PC^0 | d(t) \in D, \forall t \in \boldsymbol{R}\}$, where U and D are closed bounded convex subsets of $\boldsymbol{R}^{n_u}$ and $\boldsymbol{R}^{n_d}$. Also, let G be a closed convex subset of $\boldsymbol{R}^n$. Equation (4) and the set G define a differential game: the controller wins if it can keep the system from entering the interior of the unsafe set G, denoted G^o; the disturbance wins if it can drive the system into the unsafe set. Denote by ∂G the boundary of G. The winning states for the controller are denoted by $W^* \subseteq \boldsymbol{R}^n$.

Assumption 1 (Shapes of the sets) *Assume that G, U, and D are ellipsoids:*

$$G = \mathcal{E}(q_g, Q_g) \quad U = \mathcal{E}(q_u, Q_u) \quad D = \mathcal{E}(q_d, Q_d)$$

where $\mathcal{E}(q, Q)$ denotes an ellipsoid which is defined as follows:

$$\mathcal{E}(q, Q) = \{x \in X | \, \|Qx - q\| \leq 1\} \tag{5}$$

where, for convenience, we choose Q such that $Q = Q^T \succ 0$ and X denotes $\boldsymbol{R}^n$, $\boldsymbol{R}^{n_u}$ and $\boldsymbol{R}^{n_d}$ respectively. The center of the ellipsoid is $Q^{-1}q$, the orientation of the principal axes are given by the eigenvectors of Q^{-1}, the lengths of the principal axes are the eigenvalues of Q^{-1}, and hence the volume of the ellipsoid is proportional to $\det Q^{-1}$. We will also use the following description: $\mathcal{E}(q, Q) = \{x \in X | \, l(x) \leq 0\}$ where $l(x)$ is a quadratic function:

$$l(x) = \frac{1}{2}(x^T Q^2 x - 2q^T Qx + q^T q - 1) \tag{6}$$

3 Linear Differential Game Solution

In this section, we apply optimal control theory to the linear differential game (4), and develop expressions for the optimal input and disturbance.

Consider the system (4) over the time interval $[t, 0]$, where $t < 0$. The value function of the game is defined in terms of the unsafe set $G = \mathcal{E}(q_g, Q_g)$:

$$\begin{aligned} & J(x, u(\cdot), d(\cdot), t) : X \times \mathcal{U} \times \mathcal{D} \times \boldsymbol{R}_- \to \boldsymbol{R} \\ & \text{such that} \\ & J(x, u(\cdot), d(\cdot), t) = l(x(0)) \end{aligned} \tag{7}$$

where $l(x)$ is given by (6) and has zero level set on the boundary of G. The value function may be interpreted as the cost of a trajectory $x(\cdot)$ which starts at x at initial time $t \leq 0$, evolves according to (4) with input $(u(\cdot), d(\cdot))$, and ends at the final state $x(0)$, with cost $l(x(0))$.

The optimal action of the controller is one which tries to maximize the minimum cost, to try to counteract the optimal disturbance action of pushing the system towards G. Therefore, define $J^*(x, t)$, the optimal cost, as

$$J^*(x, t) = \max_{u(\cdot)\in\mathcal{U}} \min_{d(\cdot)\in\mathcal{D}} J(x, u(\cdot), d(\cdot), t) \tag{8}$$

and corresponding optimal input and disturbance as

$$u^*(\cdot) \quad = \quad \arg \max_{u(\cdot)\in\mathcal{U}} \min_{d(\cdot)\in\mathcal{D}} J(x, u(\cdot), d(\cdot), t) \tag{9}$$

$$d^*(\cdot) \quad = \quad \arg \min_{d(\cdot)\in\mathcal{D}} J(x, u^*(\cdot), d(\cdot), t) \tag{10}$$

Applying the development in [5] (Chapter 4) and [7] (Chapter 4) to the linear differential game (4), we have

$$H \quad = \quad p^T(Ax + Bu + Cd) \tag{11}$$

$$\dot{p}^T \quad = \quad -\frac{\partial H}{\partial x} = -p^T A \tag{12}$$

$$p^T(0) \quad = \quad \nu^T = (Q_g x(0) - q_g)^T Q_g \tag{13}$$

$$u^*(\cdot) \quad = \quad \arg\max_{u\in U} \min_{d\in D} H(x, p, u, d) \tag{14}$$

$$d^*(\cdot) \quad = \quad \arg\min_{d\in D} H(x, p, u^*, d) \tag{15}$$

where $p \in \boldsymbol{R}^n$ is the costate, H is the Hamiltonian, and ν is the outward pointing normal to G. For each

$p(s)$, $t \le s \le 0$, the problem of computing $u^*(s)$ and $d^*(s)$ can be posed as maximizing and minimizing linear functions with quadratic constraints :

$$\begin{cases} \text{maximize} & p^T B u \\ \text{subject to} & \|Q_u u - q_u\| \le 1 \\ & (Q_u = Q_u^T \succ 0) \end{cases}$$
$$\begin{cases} \text{minimize} & p^T C d \\ \text{subject to} & \|Q_d d - q_d\| \le 1 \\ & (Q_d = Q_d^T \succ 0) \end{cases}$$

Lemma 1 *The optimal input and disturbance $u^*(s)$ and $d^*(s)$ are given by*

$$u^*(s) = Q_u^{-1}\left(\frac{r_u}{\|r_u\|} + q_u\right) \quad (16)$$
$$d^*(s) = Q_d^{-1}\left(-\frac{r_d}{\|r_d\|} + q_d\right) \quad (17)$$

where

$$r_u = Q_u^{-1} B^T p, \quad (18)$$
$$r_d = Q_d^{-1} C^T p. \quad (19)$$

The optimal Hamiltonian is given by

$$H^*(x,p) = p^T A x + \|r_u\| + r_u^T q_u - \|r_d\| + r_d^T q_d \quad (20)$$

Proof: *As in [15], by change of variables: $v = Q_u u - q_u$, $r_u = Q_u^{-1} B^T p$, we pose the above optimization problem as*

$$\textit{maximize} \quad r_u^T v + r_u^T q_u \quad (21)$$
$$\textit{subject to} \quad \|v\| \le 1 \quad (22)$$

The optimal solution is given by

$$v^* = \frac{r_u}{\|r_u\|} \quad (23)$$
$$\Rightarrow u^* = Q_u^{-1}\left(\frac{r_u}{\|r_u\|} + q_u\right)$$

The optimal value is given by

$$\max_{u \in U} p^T B u = r_u^T v^* + r_u^T q_u = \|r_u\| + r_u^T q_u$$

The computation of $d^*(s)$ is given similarly. Figure 1 shows the geometric interpretation of this optimization procedure.

∎

4 Ellipsoidal Approximation of Reachable Set

In this section, we integrate the system equation (4) with the optimal input and disturbance given in the previous section, and examine how the initial ellipsoid G is mapped to another geometrical figure $\mathcal{X}$ at time

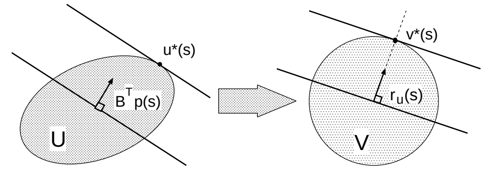

Figure 1: Maximizing a linear function with quadratic constraint

$t < 0$, where $\mathcal{X}$ is, in general, not an ellipsoid. Then, we present a sufficient condition on the shapes of the input and disturbance regions so that $\mathcal{X}(t)$ is an ellipsoid at any given time $t < 0$. Making use of this condition, we under-approximate the input region and over-approximate the disturbance region to derive an expression for an ellipsoidal over-approximation of the reach set $\mathcal{X}(t)$, which we denote $\mathcal{X}_e(t)$.

In the following, we will assume that the matrices B and C are square and invertible. This assumption may be easily generalized to over-actuated systems. The fundamental problem with under-actuated systems is that one loses the ability to under-approximate the set U in as many dimensions as B is rank-deficient (i.e., B cannot be "skinny").

For convenience, introduce n-vector w where $\|w\| = 1$, and express $p(0)$ and $x(0)$, the final values of x and p, in terms of w as follows:

$$\begin{aligned} x(0) &= Q_g^{-1}(w + q_g) \\ p(0) &= Q_g w \end{aligned} \quad (24)$$

Integrating equation (4) under the optimal control (16), disturbance (17) and the initial condition given above, and assuming no trajectory crossing, we have that the boundary of the reachable set at time s is given by:

$$\begin{aligned} \partial\mathcal{X}(s) &= \Phi(s) Q_g^{-1}(w + q_g) \\ &+ \int_0^s \Phi(s-\tau)\left[B Q_u^{-1}\left(\frac{Q_u^{-1} B^T \Phi^T(-\tau) Q_g w}{\|Q_u^{-1} B^T \Phi^T(-\tau) Q_g w\|} + q_u\right)\right. \\ &+ \left. C Q_d^{-1}\left(-\frac{Q_d^{-1} C^T \Phi^T(-\tau) Q_g w}{\|Q_d^{-1} C^T \Phi^T(-\tau) Q_g w\|} + q_d\right)\right] d\tau \end{aligned} \quad (25)$$

for w such that $\|w\| = 1$ and where $\Phi(s) = e^{As}$. The unsafe set $G(t)$ is the union of $\mathcal{X}(s)$:

$$G(t) = \cup_{t \le s \le 0} \mathcal{X}(s)$$

Note that $\mathcal{X}(s)$ $(s \ne 0)$ is not generally an ellipsoid, and equation (25) is not guaranteed to have an analytic expression. Hence, directly finding an ellipsoidal over-approximation of $\mathcal{X}(s)$ may not be successful as the dimension increases. In the following, we under/over-approximate U and D by time-varying ellipsoids $\mathcal{E}(q_u(s), Q_u(s))$ and $\mathcal{E}(q_d(s), Q_d(s))$ and find the ellipsoidal over-approximation $\mathcal{X}_e(s)$ of $\mathcal{X}(s)$.

Lemma 2 *Equation (25) expresses an ellipsoid if $Q_u(s)$ and $Q_d(s)$ are chosen so that*

$$Q_u(s) = Q_u(s)^T \succ 0 \quad (26)$$
$$Q_d(s) = Q_d(s)^T \succ 0 \quad (27)$$
$$Q_u(s)^2 = \frac{1}{\gamma_u(s)^2} B^T \Phi^T(-s) Q_g^2 \Phi(-s) B \quad (28)$$
$$Q_d(s)^2 = \frac{1}{\gamma_d(s)^2} C^T \Phi^T(-s) Q_g^2 \Phi(-s) C \quad (29)$$

where $\gamma_u(s)$ and $\gamma_d(s)$ are positive real "scaling factors" which we will optimize later in lemmas 4 and 5.

Proof: *Substituting equations (26) – (29) into (25) yields*

$$\partial\mathcal{X}_e(s) = K_1(s)\Phi(s)Q_g^{-1}w + \Phi(s)[Q_g^{-1}q_g + K_2(s)] \quad (30)$$

for $\|w\| = 1$, where

$$K_1(s) = 1 + \int_0^s [\gamma_u(\tau) - \gamma_d(\tau)]d\tau \quad (31)$$

$$K_2(s) = \int_0^s \Phi(-\tau)[BQ_u^{-1}(\tau)q_u(\tau) + CQ_d^{-1}(\tau)q_d(\tau)]d\tau \quad (32)$$

Thus, for a given s, where $t \le s \le 0$, equation (30) is simply an affine transformation of the unit circle centered at the origin $\{w \mid \|w\| = 1\}$, and since $\Phi(s)Q_g^{-1}$ is always non-singular, equation (30) is an ellipsoid. Also note that the affine transformation is a one-to-one mapping, thus no trajectory crossing occurs if $Q_u(s)$ and $Q_d(s)$ are chosen as in Lemma 2. ∎

In order to calculate expressions for $\gamma_u(s)$ and $\gamma_d(s)$ in equation (31), we choose $\gamma_u(s) > 0$ and $\gamma_d(s) > 0$ so that $\mathcal{E}(q_u(s), Q_u(s)) \subseteq \mathcal{E}(q_{uo}, Q_{uo})$ and $\mathcal{E}(q_d(s), Q_d(s)) \supseteq \mathcal{E}(q_{do}, Q_{do})$, where $\mathcal{E}(q_{uo}, Q_{uo}) = \mathcal{E}(q_u, Q_u) = U$ and $\mathcal{E}(q_{do}, Q_{do}) = \mathcal{E}(q_d, Q_d) = D$. The following lemma supports our choice of $\gamma_u(s)$ and $\gamma_d(s)$.

Lemma 3 (Inclusion) *Consider the linear differential game (4) with cost function given by (7), (6). Consider input sets U_0 and U_1, and disturbance sets D_0 and D_1, given by ellipsoids. Suppose $U_0 \supseteq U_1$ and $D_0 \subseteq D_1$ and*

$$J_0^*(x,t) = \max_{u(\cdot)\in\mathcal{U}_0} \min_{d(\cdot)\in\mathcal{D}_0} J(x, u(\cdot), d(\cdot), t)$$
$$J_1^*(x,t) = \max_{u(\cdot)\in\mathcal{U}_1} \min_{d(\cdot)\in\mathcal{D}_1} J(x, u(\cdot), d(\cdot), t)$$

Then $\mathcal{J}_0 = \{x \mid J_0^*(x,t) < 0\} \subseteq \mathcal{J}_1 = \{x \mid J_1^*(x,t) < 0\}$.

Proof:

$$\begin{aligned} J_1^*(x,t) &= \max_{u(\cdot)\in\mathcal{U}_1} \min_{d(\cdot)\in\mathcal{D}_1} J(x, u(\cdot), d(\cdot), t) \\ &\le \max_{u(\cdot)\in\mathcal{U}_0} \min_{d(\cdot)\in\mathcal{D}_1} J(x, u(\cdot), d(\cdot), t) \\ &\le \max_{u(\cdot)\in\mathcal{U}_0} \min_{d(\cdot)\in\mathcal{D}_0} J(x, u(\cdot), d(\cdot), t) \\ &= J_0^*(x,t) \end{aligned}$$

Thus, if $x \in \mathcal{J}_0$ then $x \in \mathcal{J}_1$. ∎

Next, we optimize $\gamma_u(s)$ and $\gamma_d(s)$. Here we use the term "optimal" in the sense that at each s such that $t \le s \le 0$, we would like $\mathcal{E}(q_u(s), Q_u(s))$ to be the largest volume ellipsoid inside $\mathcal{E}(q_{uo}, Q_{uo})$ and $\mathcal{E}(q_d(s), Q_d(s))$ to be the smallest volume ellipsoid including $\mathcal{E}(q_{do}, Q_{do})$ that guarantee $\partial\mathcal{X}_e$ is an ellipsoid (Figure 2). The optimal $\gamma_u(s)$ and $\gamma_d(s)$ are achieved by maximizing $\gamma_u(s)$ and minimizing $\gamma_d(s)$, since the volume of the ellipsoids is proportional to $\det Q_u^{-1}$, $\det Q_d^{-1}$ respectively. Since

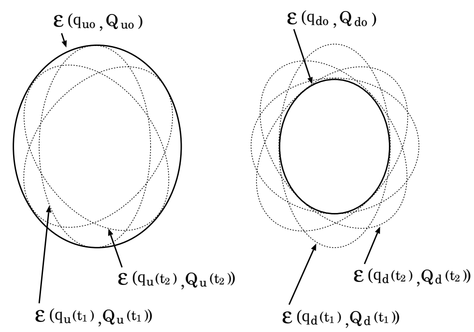

Figure 2: Showing the under-approximation of $\mathcal{E}(q_{uo}, Q_{uo})$ with $\mathcal{E}(q_u(s), Q_u(s))$ and the over-approximation of $\mathcal{E}(q_{do}, Q_{do})$ with $\mathcal{E}(q_d(s), Q_d(s))$ at $s = t_1, t_2 \ldots$, so that conditions (28) and (29) are met.

we want to maximize $\gamma_u(s)$ and minimize $\gamma_d(s)$, we restrict the centers of $\mathcal{E}(q_u(s), Q_u(s))$, $\mathcal{E}(q_d(s), Q_d(s))$ to coincide with the centers of $\mathcal{E}(q_{uo}, Q_{uo})$, $\mathcal{E}(q_{do}, Q_{do})$[1]:

$$Q_u^{-1}(s)q_u(s) = Q_{uo}^{-1}q_{uo}$$
$$Q_d^{-1}(s)q_d(s) = Q_{do}^{-1}q_{do}$$

Equation (32) further becomes

$$K_2(s) = \int_0^s \Phi(-\tau)d\tau[BQ_{uo}^{-1}q_{uo} + CQ_{do}^{-1}q_{do}] \quad (33)$$

Lemma 4 (Optimal γ_u) *The optimal $\gamma_u(s)$ is given by*

$$\gamma_u(s) = \underline{\sigma}(Q_g\Phi(-s)BQ_{uo}^{-1}) \quad (34)$$

where $\underline{\sigma}(M)$ is the minimum singular value of M.

[1] Since the problems of maximizing an ellipsoid and minimizing an ellipsoid are dual, we consider only the minimization here. If we include "q_d" as an optimization objective as well as "γ_d", then the problem becomes a Semi-Definite Program. Suppose we could minimize $\mathcal{E}_d(s)$ with its center not coincident with the center of $\mathcal{E}_{do}$, then, by symmetry, we would have another optimal ellipsoid. This contradicts the convexity of the problem.

Proof: *Without loss of generality , let $q_{uo} = 0$. Then, $u^*(s)$ is given by*

$$\begin{aligned} u^*(s) &= Q_u^{-1}(s)\left(\frac{Q_u^{-1}(s)B^T\Phi^T(-s)Q_g w}{\|Q_u^{-1}(s)B^T\Phi^T(-s)Q_g w\|}\right) \\ &= \gamma_u(s)B^{-1}\Phi(s)Q_g^{-1}w \end{aligned}$$

where $\|w\| = 1$. In order that $u^*(s)$ is inside $\mathcal{E}_{uo} = \{x \mid \|Q_{uo}x - q_{uo}\| \le 1 \;,\; q_{uo} = 0\}$, the following inequality should hold:

$$\begin{aligned} & u^*(s)^T Q_{uo}^2 u^*(s) \le 1 \\ \Leftrightarrow\; & \gamma_u(s)^2 w^T Q_g^{-1}\Phi^T(s)(B^{-1})^T Q_{uo}^2 B^{-1}\Phi(s)Q_g^{-1}w \le 1 \\ \Leftrightarrow\; & \gamma_u(s)^2 \lambda_{max}(Q_g^{-1}\Phi^T(s)(B^{-1})^T Q_{uo}^2 B^{-1}\Phi(s)Q_g^{-1}) \le 1 \end{aligned}$$

Since we want to maximize $\gamma_u(s)$, the optimal $\gamma_u(s)$ is given by

$$\begin{aligned} \gamma_u(s) &= \frac{1}{\sqrt{\lambda_{max}(Q_g^{-1}\Phi^T(s)(B^{-1})^T Q_{uo}^2 B^{-1}\Phi(s)Q_g^{-1})}} \\ &= \underline{\sigma}(Q_g\Phi(-s)BQ_{uo}^{-1}) \end{aligned}$$

■

Lemma 5 (Optimal γ_d) *The optimal $\gamma_d(s)$ is given by*

$$\gamma_d(s) = \overline{\sigma}(Q_g\Phi(-s)CQ_{do}^{-1}) \qquad (35)$$

where $\overline{\sigma}(M)$ is the maximum singular value of M.

Proof: *The proof is similar to the above.* ■

Although we use the term "optimal", this doesn't necessarily mean that the ellipsoid is the "tight" approximation [9] to the actual reach set (meaning that the ellipsoid and reach set have common supporting hyperplanes), rather that $\mathcal{E}(q_u(s), Q_u(s))$ is the largest volume ellipsoid inside $\mathcal{E}(q_{uo}, Q_{uo})$ and $\mathcal{E}(q_d(s), Q_d(s))$ is the smallest volume ellipsoid including $\mathcal{E}(q_{do}, Q_{do})$ that guarantee $\partial\mathcal{X}_e$ is an ellipsoid which over-approximates the actual reach set at each time s. In another technique which expresses the reach set by ellipsoids, Kurzhanski and Varaiya [9] introduced a method for computing the tight ellipsoidal approximation of the reach set. This technique enables one to exactly express the reach set by the intersection of the infinite number of ellipsoids, which is of great advantage in finding less conservative approximations.

Note from equations (30) and (31) that computation of $\partial\mathcal{X}_e$ requires integration of γ_u and γ_d: for some special cases, we can perform this integration analytically. Examples of this are presented in the next section. In general, our algorithm requires a relatively small amount of computation which has potential benefits for real time implementation.

5 Examples

In this section, we apply our technique to several examples in two and three dimensions, so that we can visualize the results. For simplicity, assume $q_{uo} = q_{do} = 0$ and hence $K_2(s) = 0$ from equation (33). In Case 1, we integrate $\gamma_u(\tau)$ and $\gamma_d(\tau)$ numerically. In Case 2, we treat a special case in which we can perform the integration analytically. In case 3, we try to approximate the integration. The examples were coded in MATLAB 5.3 on a Pentium MMX 233 MHz, with 80MB memory. For Case 1 (worst case, since numerical integration), for an integration time span of $[-10, 0]$ seconds, actual computation time was less than 0.1 second.

Case 1

In Case 1, the integration of $\gamma_u(s)$ and $\gamma_d(s)$ in equation (31) are performed numerically. An example of this case is shown in figure 3. The example system is a hyperbolic system: $\lambda_1 < 0 < \lambda_2$, where λ_i's are the eigenvalues of the system.

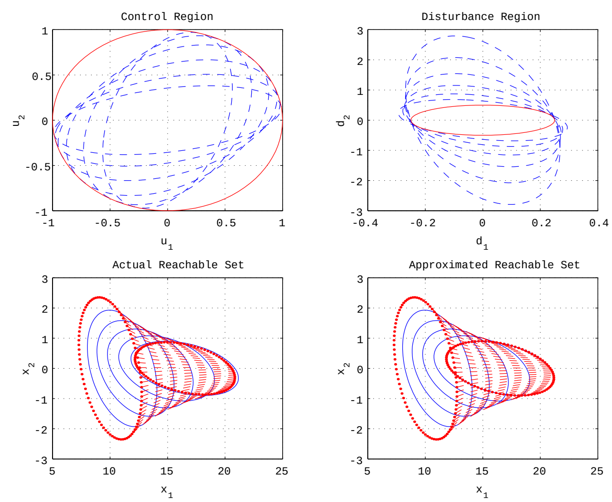

Figure 3: Example of Case 1

Case 2

Consider the case where $A = \lambda I$. In this case, equation (30) has an analytic expression. Figure 4 shows an example of this case for a three-dimensional system. The top figure shows the actual reach set and the bottom figure shows the ellipsoids representing the ellipsoidal approximation.

Case 3

Consider the case where A is skew-symmetric. In this case $\Phi(s) = e^{As}$ is orthogonal. $\gamma_u(s)$ and $\gamma_d(s)$ can be approximated as follows:

$$\begin{aligned} \gamma_u(s) &= \underline{\sigma}(Q_g\Phi(-s)BQ_{uo}^{-1}) \ge \underline{\sigma}(Q_g)\underline{\sigma}(BQ_{uo}^{-1}) \triangleq \tilde{\gamma}_u \\ \gamma_d(s) &= \overline{\sigma}(Q_g\Phi(-s)CQ_{do}^{-1}) \le \overline{\sigma}(Q_g)\overline{\sigma}(CQ_{do}^{-1}) \triangleq \tilde{\gamma}_d \end{aligned}$$

These constants can be used for $\gamma_u(s)$ and $\gamma_d(s)$, Figure 5 shows an example of this case.

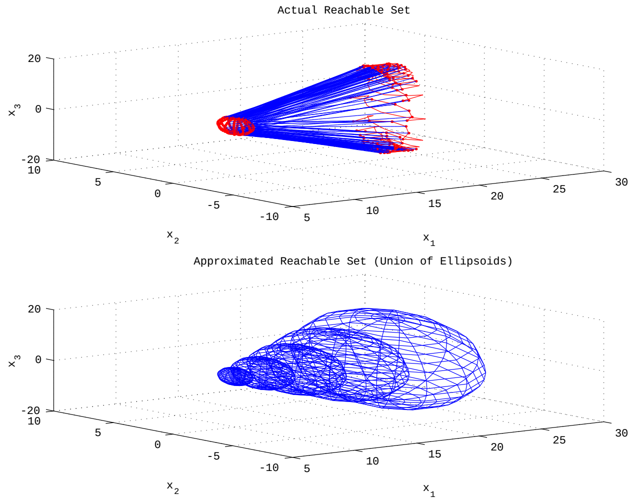

Figure 4: Example of Case 2

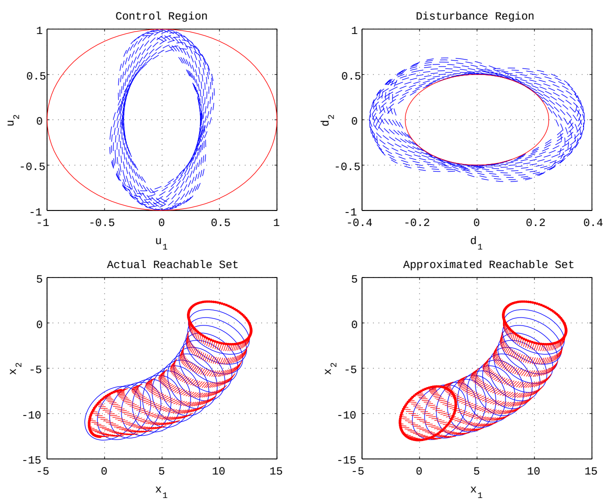

Figure 5: Example of Case 3

6 Conclusions and Current Research

Equation (30), with $K_1(s)$ and $K_2(s)$ given by (31) and (33) respectively, represents an ellipsoidal over-approximation of the reach set for the linear differential game defined in §2. Equations (34) and (35) give equations for the optimal $\gamma_u(s)$ and $\gamma_d(s)$, where we use the term "optimal" in the sense that $\mathcal{E}(q_u(s), Q_u(s))$ is the largest volume ellipsoid inside the input set U and $\mathcal{E}(q_d(s), Q_d(s))$ is the smallest volume ellipsoid including the disturbance set D that guarantee $\partial \mathcal{X}_e$ is an ellipsoid for each time s. Our technique requires a relatively small amount of calculation, and is hence appropriate for real time calculation in high dimensional spaces, such as multi-vehicle collision avoidance and aerodynamic envelope protection. We are currently extending these ellipsoidal methods to consider over-approximations of reachable sets of nonlinear dynamics. We have also begun work incorporating this technique into the computation of reachable sets for hybrid systems.

References

[1] J. Lygeros, D. N. Godbole, and S. Sastry. Verified hybrid controllers for automated vehicles. *IEEE Transactions on Automatic Control*, 43(4):522–539, April 1998.

[2] C. Tomlin, G. J. Pappas, and S. Sastry. Conflict resolution for air traffic management: A case study in multi-agent hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):509–521, April 1998.

[3] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3), 1999.

[4] M. Oishi and C. Tomlin. Switched nonlinear control of a VSTOL aircraft. In *Proceedings of the IEEE Conference on Decision and Control*, Phoenix, AZ, 1999.

[5] A. E. Bryson and Y-C. Ho. *Applied Optimal Control.* Blaisdell Publishing Company, Waltham, 1969.

[6] T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory.* Academic Press, second edition, 1995.

[7] Claire J. Tomlin. *Hybrid Control of Air Traffic Management Systems.* PhD thesis, Department of Electrical Engineering, University of California, Berkeley, 1998.

[8] I. Mitchell and C. Tomlin. Level set methods for computation in hybrid systems. In B. Krogh and N. Lynch, editors, *Hybrid Systems: Computation and Control*, LNCS. Springer Verlag, 2000.

[9] A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability analysis. In B. Krogh and N. Lynch, editors, *Hybrid Systems: Computation and Control.* Springer Verlag, 2000.

[10] O. Botchkarev and S. Tripakis. Verification of hybrid systems with linear differential inclusions using ellipsoidal approximations. In B. Krogh and N. Lynch, editors, *Hybrid Systems: Computation and Control.* Springer Verlag, 2000.

[11] F. C. Schweppe. *Uncertain Dynamic Systems.* Prentice-Hall, Englewood Cliffs, 1973.

[12] A. B. Kurzhanski and I. Valyi. *Ellipsoidal calculus for estimation and control.* Birkhauser, Boston, 1997.

[13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.

[14] L. Vandenberghe, S. Boyd, and S.-P. Wu. Determinant maximization with linear matrix inequality constraints. *SIAM Journal on Matrix Analysis and Applications*, 19(2):499–533, 1998.

[15] S. Boyd. Introduction to convex optimization with engineering applications. Lecture Notes for the course at Stanford University, 1998.