A Note on the Nonlinear H^∞ Control for Synchronous Motors

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Abstract

This work deals with the nonlinear H^∞ control of a permanent magnet synchronous motor subject to parameter variations during its operation. The control aim is to track a desired angular trajectory. First, a controller based on the nominal parameter values is determined, while the parameter variations are considered disturbances acting on the nominal plant. Second, a nonlinear H^∞ controller is designed in order to satisfy the sub-optimal attenuation problem in the case of availability of the whole state vector. This second step entails the approximated resolution of a (nonlinear) Hamilton-Jacobi-Isaacs equation. The resulting controller is tested in numerical simulations and its performance is compared with that of a linear H^∞ controller.

Keywords: Synchronous motor, nonlinear H^∞ control, nonlinear systems, parameter uncertainty.

1 Introduction

The permanent magnet (PM) synchronous motors are interesting actuators which can be usefully utilized in all those engineering applications in which their compact structure, reliability and high performances are important factors. Moreover, PM synchronous motors are easily controllable since the state variables can be considered available for measure.

The performances of the PM synchronous motors are affected by the nonlinear nature of their dynamics and by the presence of parameters which may vary during operations. These parameters are the stator winding resistance R and inductance L, the torque load C_l, the inertia J, the viscous friction coefficient f, the motor torque constant k_m. During the motor's operation they may vary due to heating (the resistance R) or to the geometric characteristics of the motor (the inductance L for instance) or finally to unpredictable operative situations (typically C_l and f). For this facts, starting from the works of [2], [22], [3], [14], which are based on the knowledge of the nominal motor parameters, various works have been conducted in the direction of taking into account this problem. The improvements in the adaptive control led to interesting works either in the continuous time setting ([8]; [19] and references therein) or in the discrete time context ([9]; [10]; [20]; [6]) in which the parameters, supposed unknown but constant or slowly varying, are adapted during the operation. On the other hand, these approaches are not applicable when these parameters are rapidly varying, and a more appropriate approach is that in which the control aim is to obtain the best attenuation of their influence on the motor's dynamics, with application of the linear H^∞ control, which demonstrates to be robust against load disturbances and motor parameter variations ([11]; [16]; [12]; [13]; [4]; [5]; [7]).

Considering the results in nonlinear H^∞ control ([21]; [15]), and on the basis of the recalled property of robustness of the H^∞ control versus parameter variations, in this work we design a nonlinear H^∞ control for a PM synchronous motor in the case of tracking of an angular trajectory. The aim is not only to investigate the possible improvements due to a nonlinear control strategy, but also to evaluate the computational complexity and the applicability of such a nonlinear controller in practical control schemes. In fact, since in general the resulting nonlinear controller is not implementable on the available digital signal processors, it is necessary to simplify the designed control law.

We first design a stabilizing controller on the basis of the nominal model of the motor, and then a nonlinear H^∞ control is determined in order to solve the H^∞ sub-optimal control problem in the case of full information. This last controller leads to solve a (nonlinear) Hamilton-Jacobi-Isaacs equation; since it can not be solved analytically, we look for an approximated solution by using a rather standard approach ([1]; [18]), consisting of its series expansion and grouping of the terms of homogeneous powers in the state variables. An infinite series of equations are hence determined; from the practical point of view we compute an approximated nonlinear controller by considering in the controller only the first nonlinear terms involved in the resolution of the Hamilton-Jacobi-Isaacs equation.

The paper is organized as follows. We recall the mathematical model of a PM synchronous motor and we state the control problem in the Section 2. In Section 3 the controller stabilizing the nominal plant is designed. In Section 4 the nonlinear H^∞ controller is determined in order to obtain the best attenuation of the parameter variations on the motor dynamics. Comparative simulation results are presented in Section 5. Some

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observations conclude the paper.

2 Mathematical Model

The dynamical model of a permanent magnet synchronous motor in the (α, β) frame is given by [17]

$$
\begin{aligned}
\frac{di_\alpha}{dt} &= -\frac{R}{L} i_\alpha + \frac{k_m}{L}\omega \sin p\vartheta + \frac{1}{L} v_\alpha \\
\frac{di_\beta}{dt} &= -\frac{R}{L} i_\beta + \frac{k_m}{L}\omega \cos p\vartheta + \frac{1}{L} v_\beta \\
\dot\omega &= \frac{k_m}{J}(-i_\alpha \sin p\vartheta + i_\beta \cos p\vartheta) - \frac{f}{J}\omega - \frac{C_l}{J} \\
\dot\vartheta &= \omega
\end{aligned}
\tag{1}
$$

in which $v = \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix}$, $i = \begin{pmatrix} i_\alpha \\ i_\beta \end{pmatrix}$, $\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix}$ are stator voltage, current and flux vectors respectively; $k_m = p\phi_f$ is the motor torque constant, with p the pole pair number and ϕ_f the rotor flux generated by the the permanent magnets.

An useful representation of a PM synchronous motor is also that in the fixed rotor frame (d, q), in which the dynamics become

$$
\begin{aligned}
\frac{di_d}{dt} &= -\frac{R}{L} i_d + p\omega i_q + \frac{1}{L} v_d = -\alpha_3 i_d + p\omega i_q + \alpha_5 v_d \\
\frac{di_q}{dt} &= -\frac{R}{L} i_q - p\omega i_d - \frac{k_m}{L}\omega + \frac{1}{L} v_q \\
&= -\alpha_3 i_q - p\omega i_d - \alpha_4\omega + \alpha_5 v_q \\
\dot\omega &= \frac{k_m}{J} i_q - \frac{f}{J}\omega - \frac{C_l}{J} = \alpha_1 i_q - \alpha_2\omega - c \\
\dot\vartheta &= \omega
\end{aligned}
\tag{2}
$$

where $c = C_l/J$, $\alpha_1 = k_m/J$, $\alpha_2 = f/J$, $\alpha_3 = R/L$, $\alpha_4 = k_m/L$, $\alpha_5 = 1/L$. Here $k_m i_q$ is the electromagnetic torque generated by the motor.

The *control problem* is to track asymptotically a reference angular trajectory ϑ_r and to impose a reference i_{dr} for the direct component i_d of the current. This must be obtained minimizing the influence of parameter variations

$$
\begin{aligned}
c &= c_0(t) + c_v(t) \\
\alpha_i &= \alpha_{i0} + \alpha_{iv}(t), \quad i = 1, \cdots, 5
\end{aligned}
\tag{3}
$$

on a certain penalty variable z, where α_{i0}, $i = 1, \cdots, 5$, are the nominal values and $c_0(t)$ is the nominal load term C_{l0}/J, and $\alpha_{iv}(t)$, $c_v(t)$ denote their variations.

3 The Stabilizing Control for the Nominal System

Let us derive a control law ensuring the control objectives for the nominal system. Let $\vartheta_e = \vartheta - \vartheta_r$ be the tracking error, and define

$$
\omega_r = \dot\vartheta_r - k_1\vartheta_e = -k_1\vartheta + (\dot\vartheta_r + k_1\vartheta_r) \tag{4}
$$

the angular reference. The tracking error for ω is $\omega_e = \omega - \omega_r$. Hence

$$
\dot\vartheta_e = -k_1\vartheta_e + \omega_e \tag{5}
$$

$$
\begin{aligned}
\dot\omega_e &= \alpha_{10} i_q - \alpha_{20}\omega - c_0 - \dot\omega_r + \alpha_{1v} i_q - \alpha_{2v}\omega - c_v \\
&= -\vartheta_e - k_2\omega_e + \alpha_{10} i_{qe} \\
&\quad + \alpha_{1v}\Big(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\Big) \\
&\quad + \alpha_{2v}(k_1\vartheta_e - \omega_e) + \alpha_{1v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{2v}\dot\vartheta_r - c_v
\end{aligned}
\tag{6}
$$

with $\phi_1 = c_0 + \ddot\vartheta_r + \alpha_{20}\dot\vartheta_r$, $i_{qe} = i_q - i_{qr}$ and $i_{qr} = \psi/\alpha_{10}$ the reference for the i_q current. The desired value ψ for the angular acceleration is given by

$$
\psi = \alpha_{10} i_{qr} = a_1\vartheta_e - a_2\omega_e + \phi_1
$$

where $a_1 = k_1^2 - k_1\alpha_{20} - 1$, $a_2 = k_1 + k_2 - \alpha_{20}$.

Considering the error $i_{de} = i_d - i_{dr}$ one also obtains

$$
\begin{aligned}
\frac{di_{qe}}{dt} &= -\alpha_3 i_q - p\omega i_d - \alpha_4\omega + \alpha_5 v_q - \frac{di_{qr}}{dt} \\
\frac{di_{de}}{dt} &= -\alpha_3 i_d + p\omega i_q + \alpha_5 v_d - \frac{di_{dr}}{dt}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{di_{qr}}{dt} &= \frac{1}{\alpha_{10}}\dot\psi = -\frac{a_1k_1 - a_2}{\alpha_{10}}\vartheta_e + \frac{a_1 + a_2k_2}{\alpha_{10}}\omega_e \\
&\quad - a_2 i_{qe} + \frac{1}{\alpha_{10}}\dot\phi_1 - \alpha_{1v}\frac{a_2}{\alpha_{10}}\Big(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\Big) \\
&\quad - \alpha_{2v}\frac{a_2}{\alpha_{10}}(k_1\vartheta_e - \omega_e) - \lambda_1 \\
\frac{di_{dr}}{dt} &= \frac{di_{dr}(\dot\vartheta_r)}{d\dot\vartheta_r}\ddot\vartheta_r \\
\lambda_1 &= \alpha_{1v}\frac{a_2}{\alpha_{10}^2}\phi_1 - \alpha_{2v}\frac{a_2}{\alpha_{10}}\dot\vartheta_r - c_v\frac{a_2}{\alpha_{10}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{di_{qe}}{dt} &= -\alpha_{30} i_q - p\omega i_d - \alpha_{40}\omega + \alpha_{50} v_q + \frac{a_1k_1 - a_2}{\alpha_{10}}\vartheta_e \\
&\quad - \frac{a_1 + a_2k_2}{\alpha_{10}}\omega_e + a_2 i_{qe} - \frac{1}{\alpha_{10}}\dot\phi_1 + \alpha_{1v}\frac{a_2}{\alpha_{10}}\Big(\frac{a_1}{\alpha_{10}}\vartheta_e \\
&\quad - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\Big) + \alpha_{2v}\frac{a_2}{\alpha_{10}}(k_1\vartheta_e - \omega_e) - \alpha_{3v}\Big(\frac{a_1}{\alpha_{10}}\vartheta_e \\
&\quad - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\Big) - \alpha_{4v}(-k_1\vartheta_e + \omega_e) + \alpha_{5v} v_q \\
&\quad + \lambda_1 - \alpha_{3v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{4v}\dot\vartheta_r \\
\frac{di_{de}}{dt} &= -\alpha_{30} i_d + p\omega i_q + \alpha_{50} v_d - \frac{di_{dr}}{dt} - \alpha_{3v} i_{de} \\
&\quad + \alpha_{5v} v_d - \alpha_{3v} i_{dr}.
\end{aligned}
$$

The controls will be designed so that the nominal part of these dynamics are canceled

$$
\begin{aligned}
v_q &= \frac{1}{\alpha_{50}}\Big[\alpha_{30} i_q + p\omega i_d + \alpha_{40}\omega - \frac{a_1k_1 - a_2}{\alpha_{10}}\vartheta_e \\
&\quad + \frac{a_1 + a_2k_2 - \alpha_{10}^2}{\alpha_{10}}\omega_e - (k_3 + a_2) i_{qe} + \frac{1}{\alpha_{10}}\dot\phi_1 + u_q\Big] \\
v_d &= \frac{1}{\alpha_{50}}\Big[\alpha_{30} i_d - p\omega i_q - k_4 i_{de} + \frac{di_{dr}}{dt} + u_d\Big]
\end{aligned}
\tag{7}
$$

while $u_q = \xi_q(x_e)$, $u_d = \xi_d(x_e)$ are designed in the following Section for the sub-optimal disturbance attenuation.

With the control (7) and rearranging the equations, one eventually obtains the error dynamics

$$\dot{x}_e = A_0 x_e + B_0 u + k(x_e) w \qquad (8)$$

with $x_e = \begin{pmatrix} \vartheta_e & \omega_e & i_{qe} & i_{de} \end{pmatrix}^T$, $u = \begin{pmatrix} u_q & u_d \end{pmatrix}^T$,

$$A_0 = \begin{pmatrix} -k_1 & 1 & 0 & 0 \\ -1 & -k_2 & \alpha_{10} & 0 \\ 0 & -\alpha_{10} & -k_3 & 0 \\ 0 & 0 & 0 & -k_4 \end{pmatrix}, \; B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$w = \begin{pmatrix} \alpha_{1v} & \alpha_{2v} & \alpha_{3v} & \alpha_{4v} & \alpha_{5v} & \alpha_{5v}\dot{\vartheta}_r \\ & \alpha_{5v}\phi_1 & \alpha_{5v} i_{dr} & w_9 & w_{10} & w_{11} \end{pmatrix}^T$$

and

$$\begin{aligned} w_9 &= \alpha_{1v}\phi_1 \frac{1}{\alpha_{10}} - \alpha_{2v}\dot{\vartheta}_r - c_v \\ w_{10} &= \lambda_1 - \alpha_{3v}\phi_1 \frac{1}{\alpha_{10}} - \alpha_{4v}\dot{\vartheta}_r + \alpha_{5v}\lambda_2 \\ w_{11} &= \alpha_{5v}\lambda_3 - \alpha_{3v} i_{dr}. \end{aligned}$$

Finally,

$$k(x_e) = \begin{pmatrix} 0 & \chi_1(x_e) & \frac{a_2}{\alpha_{10}}\chi_1(x_e) & 0 \\ 0 & \chi_2(x_e) & \frac{a_2}{\alpha_{10}}\chi_2(x_e) & 0 \\ 0 & 0 & -\chi_1(x_e) & -i_{de} \\ 0 & 0 & \chi_2(x_e) & 0 \\ 0 & 0 & \chi_3(x_e) & \chi_4(x_e) \\ 0 & 0 & \frac{p}{\alpha_{50}} i_{de} & -\frac{p}{\alpha_{50}}\chi_1(x_e) \\ 0 & 0 & 0 & \frac{p}{\alpha_{10}\alpha_{50}}\chi_2(x_e) \\ 0 & 0 & -\frac{p}{\alpha_{50}}\chi_2(x_e) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T$$

with

$$\begin{aligned} \chi_1(x_e) &= \frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe} \\ \chi_2(x_e) &= k_1\vartheta_e - \omega_e \\ \chi_3(x_e) &= \bar{a}_3\vartheta_e + \bar{a}_4\omega_e + \bar{a}_5 i_{qe} + \xi_q(x_e) - \frac{p}{\alpha_{50}}\chi_2(x_e) i_{de} \\ \chi_4(x_e) &= \frac{\alpha_{30} - k_4}{\alpha_{50}} i_{de} + \xi_d(x_e) + \frac{p}{\alpha_{50}}\chi_1(x_e)\chi_2(x_e) \\ \bar{a}_3 &= \frac{1}{\alpha_{50}}\left(-\frac{1}{\alpha_{10}}(a_1 k_1 - \alpha_{30} a_1 - a_2) - \alpha_{40} k_1\right) \\ \bar{a}_4 &= \frac{1}{\alpha_{50}}\left(\frac{1}{\alpha_{10}}(a_2 k_2 + a_1 - \alpha_{30} a_2) + \alpha_{40} - \alpha_{10}\right) \\ \bar{a}_5 &= \frac{1}{\alpha_{50}}(-k_3 - a_2 + \alpha_{30}) \\ \lambda_2 &= \frac{1}{\alpha_{50}}\left(\frac{1}{\alpha_{10}}\dot{\phi}_1 + \alpha_{30}\frac{1}{\alpha_{10}}\phi_1 + p\dot{\vartheta}_r i_{dr} + \alpha_{40}\dot{\vartheta}_r\right) \\ \lambda_3 &= \frac{1}{\alpha_{50}}\left(\alpha_{30} i_{dr} - p\dot{\vartheta}_r \frac{1}{\alpha_{10}}\phi_1 + \frac{d i_{dr}}{dt}\right). \end{aligned}$$

It is easy to see that $x_e = 0$ is a globally exponentially stable equilibrium point for $w = 0$.

4 Nonlinear H^∞ Control

In this Section a nonlinear H^∞ controller is determined in order to solve the control problem and to obtain the best attenuation of the parameter variations on the motor dynamics, in the case of full information [21], [15]. Hence, we consider the error dynamics (8) and the penalty variable

$$z = \begin{pmatrix} C_0 x_e \\ u \end{pmatrix}, \qquad C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where in z the tracking errors ϑ_e and i_{de} are considered, as well as the control effort u_q, u_d necessary for attenuating the disturbances. Note in (8) the presence of the nonlinear function $k(x_e)$. The control u is designed so that the nonlinear disturbance attenuation problem is solved. This control is given by [21], [15]

$$u = -\frac{1}{2} B_0^T V_{x_e}^T(x_e)$$

with $V_{x_e}(x_e)$ the derivative of the Lyapunov function $V(x_e)$, solving the Hamilton–Jacobi–Isaacs inequality

$$\begin{aligned} & V_{x_e}(x_e)\left[A_0 x_e - \frac{1}{4} B_0 B_0^T(x) V_{x_e}^T(x_e)\right] \\ & \quad + x_e^T C_0^T C_0 x_e + \frac{1}{\gamma^2} V_{x_e} k(x_e) k^T(x_e) V_{x_e}^T(x_e) < 0. \end{aligned} \qquad (9)$$

Since the analitic solution of this inequality is difficult to find, we look for an approximated one. Let us express the functions $V(x_e)$, supposed analytic, and $k(x_e)$, $\xi_q(x_e)$, $\xi_d(x_e)$ as a series of homogeneous polynomials ([1], [18])

$$\begin{aligned} V(x_e) &= \sum_{i=1}^{\infty} V^{[i+1]}(x_e) \\ k(x_e) &= K_0 + k_1(x_e) + k_2(x_e) + \mathcal{O}(x_e^{\otimes 3}) \\ \xi_q(x_e) &= \xi_{q1}(x_e) + \xi_{q2}(x_e) + \mathcal{O}(x_e^{\otimes 3}) \\ \xi_{q1}(x_e) &= F_{q1}\vartheta_e + F_{q2}\omega_e + F_{q3} i_{qe} + F_{q4} i_{de} \\ \xi_d(x_e) &= \xi_{d1}(x_e) + \xi_{d2}(x_e) + \mathcal{O}(x_e^{\otimes 3}) \\ \xi_{d1}(x_e) &= F_{d1}\vartheta_e + F_{d2}\omega_e + F_{d3} i_{qe} + F_{d4} i_{de} \end{aligned}$$

where " $\otimes$ " indicates the tensor product and with $k_i(x_e)$, $\xi_{qi}(x_e)$, $\xi_{di}(x_e)$, $i = 1, 2$ the terms in x_e, $x_e^{\otimes 2}$ respectively, $\mathcal{O}(x_e^{\otimes 3})$ indicating the terms of higher order, and

$$K_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$k_1(x_e) = \begin{pmatrix} 0 & \chi_1(x_e) & \frac{a_2}{\alpha_{10}}\chi_1(x_e) & 0 \\ 0 & \chi_2(x_e) & \frac{a_2}{\alpha_{10}}\chi_2(x_e) & 0 \\ 0 & 0 & -\chi_1(x_e) & -i_{de} \\ 0 & 0 & \chi_2(x_e) & 0 \\ 0 & 0 & \chi_3(x_e) & \chi_4(x_e) \\ 0 & 0 & \frac{p}{\alpha_{50}} i_{de} & -\frac{p}{\alpha_{50}}\chi_1(x_e) \\ 0 & 0 & 0 & \frac{p}{\alpha_{10}\alpha_{50}}\chi_2(x_e) \\ 0 & 0 & -\frac{p}{\alpha_{50}}\chi_2(x_e) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$k_2(x_e)=\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&k_2^1(x_e)&k_2^2(x_e)\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}^T$$

$$k_2^1(x_e)=\xi_{q2}(x_e)-\frac{p}{\alpha_{50}}\chi_2(x_e)i_{de}$$
$$k_2^2(x_e)=\xi_{d2}(x_e)+\frac{p}{\alpha_{50}}\chi_1(x_e)\chi_2(x_e)$$

$$\begin{aligned}
a_3&=\frac{1}{\alpha_{50}}\Big(-\frac{1}{\alpha_{10}}(a_1k_1-\alpha_{30}a_1-a_2)-\alpha_{40}k_1+F_{q1}\Big)\\
a_4&=\frac{1}{\alpha_{50}}\Big(\frac{1}{\alpha_{10}}(a_2k_2+a_1-\alpha_{30}a_2)+\alpha_{40}-\alpha_{10}+F_{q2}\Big)\\
a_5&=\frac{1}{\alpha_{50}}\Big(-k_3-a_2+\alpha_{30}+F_{q3}\Big)\\
a_6&=\frac{1}{\alpha_{50}}F_{q4},\quad a_7=\frac{1}{\alpha_{50}}F_{d1},\quad a_8=\frac{1}{\alpha_{50}}F_{d2}\\
a_9&=\frac{1}{\alpha_{50}}F_{d3},\quad a_{10}=\frac{1}{\alpha_{50}}\left(F_{d4}+\alpha_{30}-k_4\right).
\end{aligned}$$

Hence, it is possible to determine each term $V^{[i+1]}(x_e)$ from the terms of lower order. Moreover, the term $V^{[2]}(x_e)$ appears in a Riccati equation involving the linear approximation of the plant. It is therefore possible to determine recursively a polynomial approximation for $V(x_e)$. Setting

$$V_{x_e}(x_e)=\sum_{i=1}^{\infty}V_{x_e}^{[i]}(x_e)=2x_e^TP_1+\left(x_e^{\otimes 2}\right)^TP_2^T+\left(x_e^{\otimes 3}\right)^TP_3^T+\sum_{i=4}^{\infty}\left(x_e^{\otimes i}\right)^TP_i^T$$

$P_1=P_1^T$, and substituting in (9), one works out

$$\begin{aligned}
&x_e^T\Big(P_1A_0+A_0^TP_1+P_1M_0P_1+C_0^TC_0\Big)x_e\\
&+\Big(x_e^{\otimes 2}\Big)^TP_2^T\Big(A_0+M_0P_1\Big)x_e\\
&+\Big(x_e^{\otimes 3}\Big)^TP_3^T\Big(A_0+M_0P_1\Big)x_e+\frac{1}{4}\Big(x_e^{\otimes 2}\Big)^TP_2^TM_0P_2\Big(x_e^{\otimes 2}\Big)\\
&+\frac{4}{\gamma^2}x_e^TP_1k_1(x_e)k_1^T(x_e)P_1x_e \qquad (10)\\
&+\Big(x_e^{\otimes 4}\Big)^TP_4^T\Big(A_0+M_0P_1\Big)x_e+\cdots+\mathcal{O}\Big(x_e^{\otimes 6}\Big)<0
\end{aligned}$$

since $k_1(x_e)K_0^T=0$, $k_2(x_e)K_0^T=0$, and where it has been set

$$M_0=\frac{4}{\gamma^2}K_0K_0^T-B_0B_0^T=\begin{pmatrix}0&0&0&0\\0&\frac{4}{\gamma^2}&0&0\\0&0&\frac{4-\gamma^2}{\gamma^2}&0\\0&0&0&\frac{4-\gamma^2}{\gamma^2}\end{pmatrix}.$$

Note that in (10) the terms in $x_e^{\otimes(i+1)}$ can be split into a term $\left(x_e^{\otimes i}\right)^TP_i^T\Big(A_0+M_0P_1\Big)x_e$ and terms which are functions of P_j, $j<i$, which can be properly rewritten in the form $\left(x_e^{\otimes i}\right)^TR_i^Tx_e$ with R_i an appropriate matrix. In this way one works out the following equations

$$\begin{aligned}
P_1A_0+A_0^TP_1+P_1M_0P_1+C_0^TC_0&=0\\
P_2^T\Big(A_0+M_0P_1\Big)&=0\\
P_3^T\Big(A_0+M_0P_1\Big)+R_3^T&=0\\
P_i^T\Big(A_0+M_0P_1\Big)+R_i^T&=0,\quad i\geq 4.
\end{aligned}\qquad (11)$$

The first is a Riccati equation which involves, as anticipated, the linear approximation of the plant. The others can be iteratively solved by determining the matrices R_i and under the hypothesis of invertibility of the matrix $(A_0+M_0P_1)$. Therefore,

$$P_2=0,\qquad P_3=-(A_0+M_0P_1)^{-T}R_3$$
$$P_i=-(A_0+M_0P_1)^{-T}R_i,\quad i\geq 4.$$

The control ensuring the sub-optimal disturbance attenuation is hence

$$u=\begin{pmatrix}u_q\\u_d\end{pmatrix}=-B_0^TP_1x_e+\frac{1}{2}B_0^T(A_0+M_0P_1)^{-T}R_3x_e^{\otimes 3}+\frac{1}{2}\sum_{i=4}^{\infty}B_0^T(A_0+M_0P_1)^{-T}R_ix_e^{\otimes i}. \qquad (12)$$

5 Simulation Results

The control designed in Section 4 can be appropriately approximated, so diminuishing the computational complexity; in this way it becomes suitable for on–line implementations. In fact,

$$\begin{pmatrix}v_q\\v_d\end{pmatrix}=v_0+u_L(x_e)+u_{NL}(x_e)$$

with v_0 and u_L given by (7) and

$$u_L(x_e)=\frac{1}{\alpha_{50}}\begin{pmatrix}u_{q,L}(x_e)\\u_{d,L}(x_e)\end{pmatrix}=-\frac{1}{\alpha_{50}}B_0^TP_1x_e$$

while $u_{NL}(x_e)$ can be approximated by the first nonlinear terms in the control (12)

$$u_{NL}(x_e)=\frac{1}{\alpha_{50}}\begin{pmatrix}u_{q,NL}(x_e)\\u_{d,NL}(x_e)\end{pmatrix}=\frac{1}{2\alpha_{50}}B_0^T(A_0+M_0P_1)^{-T}R_3x_e^{\otimes 3}.$$

A further approximation comes from the analysis of the matrix R_3. In fact, while $u_{q,NL}(x_e)$ is negligible, $u_{d,NL}(x_e)$ furnishes an important contribution only by means of the term $\chi_u=R_{3;(4,64)}\ i_{de}^3$, with $R_{3;(4,64)}=4.13$ the entry $(4,64)$ in R_3. Let $u_{NL}^a(x_e)$ be the term in which only χ_u is considered.

The designed control law has been applied to the PM synchronous motor whose nominal parameters are $R_0 = 0.6\ \Omega$, $J_0 = 0.0011$ Kgm^2, $L_0 = 0.0014$ H, $f_0 = 0.0014$ Nms, $p = 4$, $k_{m0} = 0.48$ Wb.

To overcome the problem of strong current excitation consequent to the application of a step reference variation for ϑ_r, a polynomial reference trajectory can be used ([20])

$$\vartheta_r(t)=\begin{cases} (35R_1^4 - 84R_1^5 + 70R_1^6 - 20R_1^6)\vartheta_{r\infty} & \text{if } t < t_{f1} \\ \vartheta_{r\infty} & \text{if } t \in [t_{f1}, t_a) \\ (10R_2^3 - 15R_2^4 + 6R_2^5)\omega_{r\infty} & \text{if } t \in [t_a, t_{f2}) \\ \Delta t_a \omega_{r\infty} - \frac{t_{f2}}{2} + \vartheta_{r\infty} & \text{if } t > t_{f2} \end{cases} \quad (13)$$

with $R_1 = \frac{t}{t_{f1}}$, $R_2 = \frac{t-t_a}{t_{f2}}$, $t_a = 0.6$ s, $\vartheta_{r\infty} = -20$ rad, $\omega_{r\infty} = 250$ rad/s, so that $\vartheta_f(0) = 0$, $\vartheta_r(t_{f1}) = \vartheta_{r\infty}$, $\dot\vartheta_f(0) = 0$, $\ddot\vartheta_f(0) = 0$, $\dddot\vartheta_f(0) = 0$, $\dot\vartheta_r(t_{f1}) = 0$, $\ddot\vartheta_r(t_{f1}) = 0$, $\dddot\vartheta_r(t_{f1}) = 0$, and $\dot\vartheta_f(t_a) = 0$, $\ddot\vartheta_f(t_a) = 0$, $\dddot\vartheta_f(t_a) = 0$, $\dot\vartheta_r(t_{f2}) = \omega_{r\infty}$, $\ddot\vartheta_r(t_{f2}) = 0$, $\dddot\vartheta_r(t_{f2}) = 0$. Here $t_{f1} = 0.25$, $t_{f2} = 0.075$ s, are the response times at 99% of the steady-state values. Moreover, the reference i_{dr} is taken identically zero.

In these simulations we have supposed that the motor parameters and the load torque are subject to variations. More precisely we have considered that the nominal load torque is piecewise constant as follows

$$C_{l0} = \begin{cases} 50\%\, C_{nom} & t < 0.7 \text{ s} \\ 15\%\, C_{nom} & t \geq 0.7 \text{ s} \end{cases}$$

with $C_{nom} = 14.4$ Nm the nominal value. The real value of C_l is subject to a superimposed sinusoidal variation. Moreover, variations of 60% for J, 50% for f, 100% for R, 35% for L have been considered:

$$\begin{aligned} C_l &= C_{l0} + 0.1\, C_{nom} \sin \frac{2\pi}{T} t \\ J &= J_0(1 + 0.6 \sin \frac{\omega}{2\pi} t) \\ f &= f_0(1 + 0.5 \sin 50\omega t) \\ R &= R_0(1 + e^{-t/0.1}) \\ L &= L_0(1 + 0.35 \sin \frac{\omega}{p\pi} t) \end{aligned}$$

with $k_m = k_{m0}$ and $T = 0.14$ s. The gains of the controller (7) have been taken equal to $k_1 = 250$, $k_2 = 250$, $k_3 = 300$, $k_4 = 300$, while a saturation value at 300 V has been taken into account for the actuator.

The first simulation (Figures 1–4, solid line) refers to the application of the controller $v = v_0 + u_L(x_e)$ (i.e. without the nonlinear term). The matrix P_1, solution of the Riccati equation, is given by

$$P_1 = \begin{pmatrix} 2.00\,10^{-3} & 2.36\,10^{-6} & 1.87\,10^{-6} & 0 \\ 2.36\,10^{-6} & 4.07\,10^{-9} & 3.08\,10^{-9} & 0 \\ 1.87\,10^{-6} & 3.08\,10^{-9} & 4.48\,10^{-9} & 0 \\ 0 & 0 & 0 & 3.33\,10^{-3} \end{pmatrix}$$

which ensures an L_2 gain less or equal to $\gamma = 3.33\ 10^{-3}$. A second simulation (Figures 1–4, dash line) regards the approximated nonlinear controller $v = v_0 + u_L(x_e) + u^a_{NL}(x_e)$. The simulation show the better performance of the nonlinear H^∞ controller over the linear one.

Remark 1. Clearly the further approximation made in this Section may vary from case to case and has to be evaluated for each particular application.

Conclusions

In this work we have determined a nonlinear H^∞ controller for a permanent magnet synchronous motor in the case of tracking of an angular trajectory. The motor is subject to parameter variations. The controller design implies the resolution of a Hamilton-Jacobi-Isaacs equation. In general this kind of equation has no analytic solution and, as done in this work and standard in the literature, we have used an iterative procedure in order to derive a solution approximated at the third power of the state variables. Moreover, since in practical on-line implementations even this approximated solution is not workable, we have considered a further approximation, taking into account only the main contributions of the nonlinear terms to the control law. The resulting controller has shown better performances with respect to the linear H^∞ controller.

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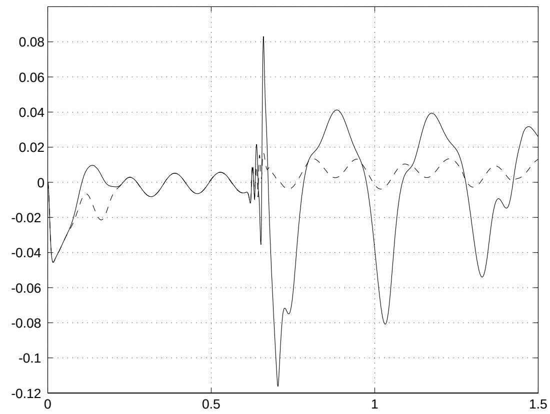

Figure 1 – Error angular position ϑ_e (rad)

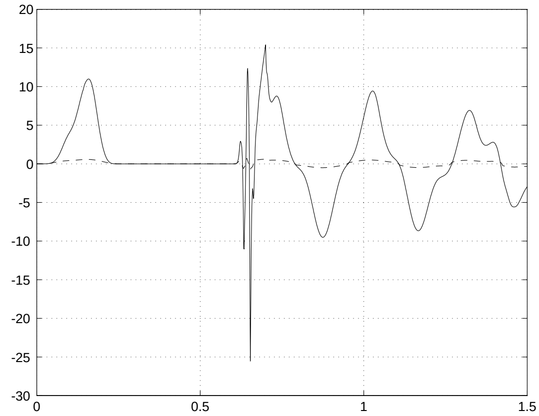

Figure 2 – Error current i_{de} (A)

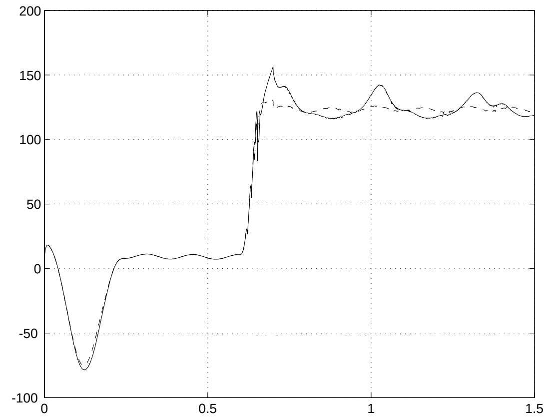

Figure 3 – Voltage v_q (V)

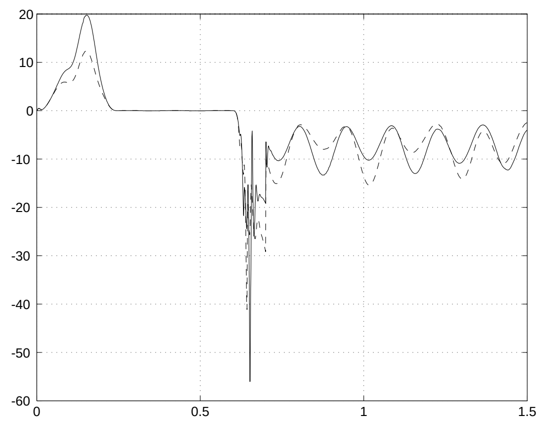

Figure 4 – Voltage v_d (V)