Lyapunov Functions for Impulse and Hybrid Control Systems

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Abstract

This paper characterizes Lyapunov functions for hybrid control systems and, more generally, to impulse differential inclusions. This is useful to govern the evolution of an asymptotically stable solution of a an hybrid system around an equilibrium and can also be used for designing a global optimization algorithm.

1 Introduction

Controlling a solution asymptotically stable around an equilibrium or a viable[1] closed subset can be studied with the century old Lyapunov method. Two questions arise:

1. Characterize such a Lyapunov function and then, find the regulation map (or feedback maps) governing the evolution of asymptotically — or even better, exponentially stable — solutions,
2. Knowing the dynamics and an equilibrium, find such a Lyapunov function, and actually, the best one.

Viability theory and set-valued analysis allows us to answer these two questions for usual control systems. Actually, they are also relevant for hybrid control systems.

A key towards success requires simplification of the problem to use only the relevant properties of the problem, the price to pay is abstraction. The abstraction process amounts to

1. regard control systems as differential inclusions, i.e., differential equations with set-valued right hand sides,
2. regard hybrid control systems as "impulse differential inclusions" defined below,
3. regard dynamical inequalities of the form

$$\forall\, t \geq 0,\ \ \mathbf{v}(x(t)) \leq w(t)$$

as viability (or state) constraints

$$\forall\, t \geq 0,\ \ (x(t), w(t)) \in \mathcal{E}p(\mathbf{v})$$

where the epigraph $\mathcal{E}p(\mathbf{v})$ of the function $\mathbf{v}$ is defined by

$$\mathcal{E}p(\mathbf{v}) := \{(x, w) \in X \times \mathbf{R} \mid w \geq \mathbf{v}(x)\}$$

The standard example is to choose the distance $\mathbf{v}(x) := \|x - c\|$ to an equilibrium c of the system and $w(t) := w_0 e^{-at}$ so that the above inequality boils down to Lyapunov exponential — also called inertial — stability:

$$\forall\, t \geq 0,\ \ \|x(t) - c\| \leq w_0 e^{-at}$$

A first advantage is to summarize the usually protracted description of an hybrid system by only two set-valued maps F — the right-hand side of the differential inclusion governing the continuous evolution of a hybrid system — and R, describing the reset map reinitializing the system when required and a constrained set K inside which the evolution of the "run" or "execution" must remain. Hence, for instance, the existence of a run of an hybrid system for every initial set becomes a viability problem of an adequate auxiliary subset under an impulse differential inclusion, that can be characterized elegantly end efficaciously.

The second advantage of regarding the characterization of a Lyapunov function as a viability problem requires that we do no longer assume the differentiability of the candidate for a Lyapunov function, but just the fact that its epigraph is closed (i.e., that it is lower semicontinuous in mathematical jargon). This is not just a fancy of mathematician, because this allows us to use extended functions taking infinite values, that encapsulate state constraints, and also, when the candidate fails to be a Lyapunov function, to find the smallest Lyapunov function larger than the candidate, that provides in the same time the basin of exponential attraction (also called the inertial manifold). We refer to [4,5, Aubin] for this issue.

And, on top of it, it is so much simpler and handier.

Outline: We begin by giving our definition of hybrid systems, that can be embedded in the framework of hybrid impulse differential inclusions. We then recall the

[1] A subset is said to be viable under a control system if from every initial state starts a solution to the control system "viable" in this subset in the sense that it remains in this subset at each instant.

characterization of viable subsets under an impulse differential inclusion an derive from it a necessary and sufficient condition for the existence of solutions to hybrid differential inclusions. Next, we recall the characterization of Lyapunov functions of usual control systems, that we adapt to the case of impulse differential inclusions. We conclude with an application to an hybrid gradient method for global optimization.

2 Hybrid Differential Inclusions

"Hybrid control systems", as they are called by engineers, or "multiple-phase dynamical economies", as they are called by economists, or "Integrate and Fire" models in neurobiology — may be regarded as hybrid differential inclusions.

First, any solution to a control system with state-dependent constraints on the controls

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in P(x(t)) \end{cases}$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$ where the right hand side is defined by $F(x) := f(x, U(x)) := \{f(x,u)\}_{u \in U(x)}$.

Therefore, from now on, as long as we do not need to implicate explicitly the controls in our study, we shall replace control problems by differential inclusions.

Here, $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ denote finite dimensional vector spaces. We denote by

$$\mathrm{Graph}(F) := \{(x,y) \in X \in Y \mid y \in F(x)\}$$

the graph of a set-valued map $F : X \rightsquigarrow Y$ and $\mathrm{Dom}(F) := \{x \in X | F(x) \neq \emptyset\}$ its domain.

Let us set $x(^-t) := \lim_{\tau \mapsto t-} x(\tau)$ when $x(\cdot)$ is defined on some interval $[t-\eta, t[$ where $\eta > 0$, and, for consistency purposes, $x(s) = x(^-t)$ if $s = t$.

Definition 2.1 *An hybrid differential inclusion (K, F, R) is defined by*

1. *a finite dimensional vector space E of states e called* locations*,*
2. *a set-valued map $K : E \rightsquigarrow X$ associating with any location e a (possibly empty) subset $K(e) \subset X$*
3. *a set-valued map $F : \mathrm{Graph}(K) \rightsquigarrow X$ with which we associate the differential inclusion $x'(t) \in F(e, x(t))$,*
4. *a set-valued map*[2] *(reset map) $R : \mathrm{Graph}(K) \rightsquigarrow E \times X$.*

[2]Usually, R is a single-valued defined on a subset $C \subset X$. We extend it to a set-valued map defined on the whole space X by setting $R(x) := \emptyset$ for any $x \notin C$, so that its extension is a set-valued map.

A run *of the impulse differential inclusion is a map $(e(\cdot), x(\cdot))$ from $[0, T[$ to $X \times E$ which is associated with a non decreasing sequence $\mathcal{T}(e(\cdot), x(\cdot)) := \{t_n\}_{n \geq 0}$ of* impulse or switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots < T$ *such that*

1. *either* $t_{n+1} = t_n$, $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(t_n))$ *and* $x(t_{n+1}) \in K(e(t_{n+1}))$,
2. *or* $t_{n+1} > t_n$, *and, for all* $t \in [t_n, t_{n+1}[$, $x(\cdot)$ *is a solution to the differential inclusion* $x'(t) \in F(e(t_n), x(t))$ *viable in* $K(e(t_n))$ *and we take* $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(^-t_{n+1}))$ *and* $x(t_{n+1}) \in K(e(t_{n+1}))$.

We observe right away that a map $(e(\cdot), x(\cdot))$ is a run of the hybrid differential inclusions if and only if $(e(\cdot), x(\cdot))$ is a run of

$$\begin{cases} i) & e'(t) = 0 \\ ii) & x'(t) \in F(e(t), x(t)) \end{cases}$$

"viable" in $\mathrm{Graph}(K)$ until it reaches the domain of the reset map R.

Indeed the locations remaining constant in the intervals $[t_n, t_{n+1}[$ since their velocities are equal to 0.

This leads us to regard $(e(\cdot), x(\cdot))$ as a run of the an auxiliary system of impulse differential inclusions that we are about to define. We shall replace $E \times X$ by X, F and R by set-valued maps $F : X \rightsquigarrow X$ and $R : X \rightsquigarrow X$ and the graph $\mathrm{Graph}(K) \in E \times X$ of the set-valued map K by a subset $K \subset X$.

3 Impulse Differential Inclusions

Given a control system or a differential game described under the form of a differential inclusion $x' \in F(x)$ and constraints on the states represented by a closed subset K, there are no reasons why an arbitrary subset K should be viable under the differential inclusion $x' \in F(x)$.

Hence, the problem of reestablishing viability arises. One can imagine several mechanisms for this purpose:

1. Change either the dynamics or the set of constraints
   (a) either by changing the controls according to feedbacks or dynamic feedbacks that can be constructed (see for instance [1,2, Aubin]),
   (b) or by changing the dynamics by, for instance, projecting the velocities onto the contingent cones and introducing viability multipliers (see for instance [1,2, Aubin]),

(c) or by restricting the constrained set to its viability kernel, which is by definition the largest subset viable under the dynamics,

(d) or by letting the set of constraints evolve according to mutational equations, as in [3, Aubin].

2. or change the initial conditions by introducing a reset map R mapping any state of K to a (possibly empty) set $R(x) \subset X$ of new "initialized states".

This is the latter strategy we choose to use here: An impulse differential inclusion (and in particular, an impulse control system) is described by a pair (F, R), where the set-valued map $F : X \rightsquigarrow X$ mapping the state space $X := \mathbf{R}^n$ to itself governs the continuous evolution of the system in K and where R, the reset map, governs the discrete switches to new "initial conditions" when the continuous evolution is doomed to leave K.

Such a hybrid evolution, mixing continuous evolution "punctuated" by discontinuous impulses at impulse times is called in the "hybrid system" literature a "run" or an "execution".

Definition 3.1 *Let us consider a finite dimensional vector space X, a closed subset $K \subset X$, a set-valued map $F : X \rightsquigarrow X$ and a set-valued map $R : X \rightsquigarrow X$, regarded as a* reset map. *We denote by $S := R - \mathbf{1}$ the associated* switching map.

We regard the pair (F, R) — or (F, S) — as the dynamics of an impulse differential inclusion.

A run *of the impulse differential inclusion is a map $x(\cdot)$ from $[0, T]$ to X if $T < +\infty$ or from $[0, +\infty[$ to X if $T = +\infty$ which is associated with a non decreasing sequence $\mathcal{T}(x(\cdot)) := \{t_n\}_{n \geq 0}$ of* impulse or switching times *$t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq T$ such that*

1. *$x(t_{n+1}) \in R(x(t_n))$ if $t_{n+1} = t_n$,*
2. *or else, on the interval $[t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ at time t_n until time t_{n+1} at which we take $x(t_{n+1}) \in R(x(^-t_{n+1}))$.*

We say that a run $x(\cdot)$ is viable in K if for any $t \geq 0$, $x(t) \in K$.

We shall denote by $\mathcal{R}_{(F,R)}(x_0)$ the set of runs of the impulse differential inclusion starting from x_0 and by $\mathcal{R}^K_{(F,R)}(x_0)$ the set of runs viable in K.

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of states $x_{n+1} \in R(x_n)$ at a fixed time, or just a (continuous) solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, or an hybrid of these two modes, the discrete and the continuous.

4 The Characterization Theorem

Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud:

Definition 4.1 (Marchaud Map) *We shall say that F is a* Marchaud map *if*

$$\begin{cases} i) & \text{the graph domain of } F \text{ is closed} \\ ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear: } \exists\, c > 0 \mid \forall\, x \in X, \\ & \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

This covers the case of Marchaud control systems where $(x, u) \mapsto f(x, u)$ is continuous, affine with respect to the controls u and with linear growth and when U is Marchaud.

Our purpose is to characterize the viability of a subset K under an impulse differential inclusion:

Definition 4.2 *We shall say that a subset K is viable*[3] *under an impulse differential inclusion (F, R) if from any initial state x of K starts at least one run viable in K.*

The Viability Theorem[4] and its consequences imply the following

Theorem 4.3 *Let (F, R) be an impulse differential inclusion and $K \subset X$ be a closed subset. Assume that F is Marchaud and that R is upper semicontinuous with compact images*[5]*. Then the following statements are equivalent*

1. *K is viable under (F, R),*
2. *The subset*[6] *$K \backslash R^{-1}(K)$ is locally viable under F,*
3. *K, F and R are linked through the tangential condition*[7]

$$\forall\, x \in K \backslash R^{-1}(K), \;\; F(x) \cap T_K(x) \neq \emptyset$$

Since the existence of solutions to hybrid differential inclusions amounts to the viability of the graph of the

[3] Viability issues ("positively invariance") for hybrid systems has been studied in [12, Hespanha & Morse] (Lemma 3).

[4] See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [1, Aubin].

[5] This assumption implies that $R^{-1}(K)$ is closed, which is the property we really need. It remains true when we assume only that the subsets $K \cap (R(x) + B)$ are compact, where B denotes the unit ball.

[6] The subset $K \backslash C$ denotes the intersection of K and the complement of C, i.e., is the set of elements of K which do not belong to C.

[7] The contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n (see for instance [6, Aubin & Frankowska]) or [13, Rockafellar & Wets] for more details).

set-valued map K under an associated auxiliary impulse differential inclusion, we obtain a necessary and condition for the existence of solutions to hybrid differential inclusions thanks to Theorem 4.3.

For that purpose, we need the definition of the contingent derivative $DK(e,x) : E \rightsquigarrow X$ of a set-valued map $K : E \rightsquigarrow X$ at a point (e,x) of its graph: It can be defined by

$$\text{Graph}(DK(e,x)) \; := \; T_{\text{Graph}(K)}(e,x)$$

We also need to introduce the set-valued map $K_1 : E \rightsquigarrow X$ defined by

$$\text{Graph}(K_1) \; := \; \text{Graph}(K)\backslash R^{-1}(\text{Graph}(K))$$

We observe that whenever $R(e,x) := R_E(e) \times R_X(x)$, the map K_1 can be defined through the formulas

$$\forall\, e \in E, \;\; K_1(e) \; = \; K(e) \cap R_X^{-1}(K(R_E(e))$$

Theorem 4.4 *Let (K,F,R) be a hybrid differential inclusion. Assume that F is Marchaud and that R is upper semicontinuous with compact images. Then the hybrid differential inclusion has a solution for every initial state if and only if*

$$\forall\, e \in E, \forall\, x \in K(e)\backslash K_1(e), \;\; F(e,x) \cap DK(e,x)(0) \neq \emptyset$$

5 Lyapunov Functions

5.1 Lyapunov Functions for Differential Inclusions

Let us consider the dynamics (F,R) of an impulse differential inclusion and a time-dependent function $w(\cdot)$ defined as a solution to the differential equation

$$w'(t) \; = \; -\varphi(x(t), w(t)) \qquad (1)$$

where $\varphi : X \times \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth.

This section is devoted to specific viability constraints — called, dynamical inequalities — which can be written in the form

$$\forall\, t \in [0,T], \;\; \mathbf{v}(x(t)) \; \leq \; w(t)$$

where $\mathbf{v} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is a given nontrivial nonnegative extended function.

We recall that the epigraph of the contingent epiderivative $D_\uparrow \mathbf{v}(x)$ of $\mathbf{v}$ at x is the contingent cone to the epigraph of $\mathbf{v}$ at $(x, \mathbf{v}(x))$:

$$\mathcal{E}p(D_\uparrow \mathbf{v}(x)) \; := \; T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x))$$

and that

$$\forall\, u \in X, \;\; D_\uparrow \mathbf{v}(x)(u) \; = \; \liminf_{h \to 0+,\, u' \in u} \frac{\mathbf{v}(x+hu') - \mathbf{v}(x)}{h}$$

See for instance Chapter 6 of [6, Aubin & Frankowska] or [13, Rockafellar & Wets]for more details. One can find in Chapter 9 of [1, Aubin] the following results: When F is Marchaud and φ continuous with linear growth, we know that the two following statements are equivalent:

1. From every $x_0 \in \text{Dom}(\mathbf{v})$ starts a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ and from $\mathbf{v}(x_0)$ starts a solution $w(\cdot)$ to the differential equation $w' = \varphi(x,w)$ such that $\mathbf{v}(x(t)) \leq w(t)$ for all positive times
2. the dynamics F and the extended function $\mathbf{v}$ are linked by the property[8]

$$\forall\, x \in \text{Dom}(\mathbf{v}), \;\; \inf_{v \in F(x)} D_\uparrow \mathbf{v}(x)(v) + \varphi(x, \mathbf{v}(x)) \; \leq \; 0$$

If these equivalent properties do not hold true, there exists an extended lower semicontinuous function $\mathbf{v}_\propto \geq \mathbf{v}$ which is the smallest of the lower semicontinuous functions $\mathbf{v} \geq \mathbf{v}$ satisfying the above equivalent properties.

5.2 Lyapunov Functions for Impulse Differential Inclusions

These results can be also extended to the case of impulse differential inclusions:

Theorem 5.1 *Let $\mathbf{v}$ be a nontrivial nonnegative lower semicontinuous extended function, $F : X \rightsquigarrow X$, be a Marchaud map, $R : X \rightsquigarrow X$ be an upper semicontinuous map with compact images and $\varphi : X \times \mathbf{R}_+ \to \mathbf{R}$ be continuous with linear growth. We set*

1. $\mathbf{v}_R(x) \; := \; \inf_{y \in R(x)} \mathbf{v}(y)$, *the* marginal function,
2. $\mathbf{R}_{\mathbf{v}_R}(x) \; := \; \{y \in R(x) \mid \mathbf{v}(y) = \mathbf{v}_R(x)\}$, *the* marginal map.

Then the two following conditions are equivalent:

1. *for any initial state $x_0 \in \text{Dom}(\mathbf{v})$,* there exist a run *$x(\cdot)$ to the impulse differential inclusion (F,R) and a solution to the differential equation $w(\cdot)$ to (1) satisfying property*

$$\forall\, t \geq 0, \;\; \mathbf{v}(x(t)) \; \leq \; w(t), \quad w(0) = \mathbf{v}(x(0) \qquad (2)$$

2. *$\mathbf{v}$ is a contingent solution to the* Hamilton-Jacobi variational inequalities*:* whenever $\mathbf{v}(x) < \mathbf{v}_R(x)$, *then*

$$\inf_{v \in F(x)} D_\uparrow \mathbf{v}(x)(v) + \varphi(x, \mathbf{v}(x)) \; \leq \; 0 \qquad (3)$$

[8]We recognize the classical definition of one brand of Lyapunov functions because when $\mathbf{v}$ is differentiable, $F \equiv f$ is single-valued, it boils down to

$$\langle \mathbf{v}'(x), f(x) \rangle + \varphi(x, \mathbf{v}(x)) \leq 0$$

Proof — We set $\mathbf{F}(x,w) := F(x) \times \{-\varphi(x,w)\}$ and $\mathbf{R}(x,w) := R(x)\times\{w\}$. Obviously, the impulse differential inclusion $(\mathbf{F},\mathbf{R})$ has a run satisfying (2) if and only if the auxiliary impulse differential inclusion $(\mathbf{F},\mathbf{R})$ has a run starting at $(x_0, \mathbf{v}(x_0))$ viable in $\mathcal{E}p(\mathbf{v})$.

Let us set $\mathbf{v}_R(x) := \inf_{y\in R(x)} \mathbf{v}(y)$. We then note that $\mathbf{R}^{-1}(\mathcal{E}p(\mathbf{v})) = \mathcal{E}p(\mathbf{v}_R)$ and that $(x,w) \in \mathcal{E}p(\mathbf{v})\backslash\mathbf{R}^{-1}\mathcal{E}p(\mathbf{v})$ if and only if

$$\mathbf{v}(x) \leq w < \mathbf{v}_R(x) := \inf_{y\in R(x)} \mathbf{v}(y)$$

Since $\mathbf{F}$ is Marchaud and $\mathbf{R}$ is upper semicontinuous with compact images, this is equivalent to say that for all $(x,w) \in \mathcal{E}p(\mathbf{v})\backslash\mathbf{R}^{-1}\mathcal{E}p(\mathbf{v})$, there exists $u \in F(x)$ such that $(u, -\varphi(x,w)) \in T_{\mathcal{E}p(\mathbf{v})}(x,w)$.

This condition implies (3) because by taking $w = \mathbf{v}((x))$, we infer that

$$(v, -\varphi(x,\mathbf{v}(x))) \in T_{\mathcal{E}p(\mathbf{v})}(x,\mathbf{v}(x)) = \mathcal{E}p(D_\uparrow \mathbf{v}(x))$$

for some $v \in F(x)$. Hence $\mathbf{v}$ is a contingent solution to the Hamilton-Jacobi variational inequality (3).

Conversely, since $F(x)$ is compact and $v \mapsto D_\uparrow \mathbf{v}(x)(v)$ is lower semicontinuous, (3) implies that there exists $v \in F(x)$ such that the pair $(v, -\varphi(x,\mathbf{v}(x)))$ belongs to $T_{\mathcal{E}p(\mathbf{v})}(x,\mathbf{v}(x))$. If $\mathbf{v}(x) < w < \mathbf{v}_R(x)$, we observe that for every $\mu \in \mathbf{R}$, (v,μ) belongs to $T_{\mathcal{E}p(\mathbf{v})}(x,w)$, and in particular, for $\mu := -\varphi(x,w)$. □

Therefore, a run $(x(\cdot), w(\cdot)$ of the auxiliary impulse differential inclusion $(\mathbf{F},\mathbf{R})$ is defined in the following way: At impulse time t_n and initial conditions x_n and $w(t_n)$ such that $\mathbf{v}(x_n) \leq w(t_n)$, then

1. if $\mathbf{v}_R(x_n) \leq w(t_n)$, we take $t_{n+1} = t_n$ and $x_{n+1} \in R(x_n)$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x_n)$, so that $\mathbf{v}(x_{n+1}) \leq w(t_{n+1}) = w(t_n)$,
2. if $w(t_n) < \mathbf{v}_R(x_n)$, there exist a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at time t_n from x_n and a solution $w(\cdot)$ to the differential equation $w' = \varphi(x,w)$ starting at time t_n from $w(t_n)$ satisfying $\mathbf{v}(x(t)) \leq w(t)$ until a time t_{n+1} such that $\mathbf{v}_R(x(^-t_{n+1})) = w(t_{n+1})$ if

$$w(t_n) < \min(\mathbf{v}_\propto(x_n), \mathbf{v}_R(x_n))$$

We thus reset the initialized state by taking $x_{n+1} \in R(x(^-t_{n+1}))$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x(^-t_{n+1}))$ and $w_{n+1} := w(t_{n+1})$, so that $\mathbf{v}(x_{n+1}) = w(t_{n+1})$. □

Example: Exponential Lyapunov Functions of a Impulse Differential Inclusion
Let us assume that the nontrivial lower semicontinuous extended function $\mathbf{v} : X \mapsto \mathbf{R}\cup\{+\infty\}$ is bounded from below: we set

$$v_0 := \inf_{x\in X} \mathbf{v}(x)$$

The function $\mathbf{v} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is said *to enjoy the a-Lyapunov property* if and only if for any initial state x_0, there exists a run $x(\cdot)$ of the impulse differential inclusion (F,R) satisfying

$$\forall\, t \geq 0,\ \ \mathbf{v}(x(t)) \leq w(t) := (\mathbf{v}(x_0) - v_0)e^{-at} + v_0 \tag{4}$$

Such inequalities allow us to deduce many properties on the asymptotic behavior of $\mathbf{v}$ along some runs of the impulsive differential inclusion, such as the fact that $\mathbf{v}(x(t))$ converges to v_0.
We deduce from Theorem 5.1 with $\varphi(x,w) := a(w-v_0)$ that property (4) holds true if and only if whenever $\mathbf{v}(x) < \inf_{y\in R(x)} \mathbf{v}(y)$, then

$$\inf_{v\in F(x)} D_\uparrow \mathbf{v}(x)(v) + a(\mathbf{v}(x) - v_0) \leq 0 \tag{5}$$

In particular, when $\mathbf{v}(x) := \|x-c\|^2$ and $v_0 := 0$:

$$\inf_{v\in F(x)} \langle x-c, v\rangle + 2a\|x-c\|^2 \leq 0$$

We call them *exponential Lyapunov functions of a impulse differential inclusion (with respect to a).*

5.3 Lyapunov Functions for Hybrid Control Systems

Since a hybrid control systems defined by

$$\begin{cases} i) & x'(t) = f(e(t), x(t), u(t)) \\ ii) & x(t) \in U(e(t), x(t)) \end{cases} \tag{6}$$

reset by a set-valued map $R : E\times X \rightsquigarrow E\times X$ is the impulse differential inclusions $\{0\}\times f(K(e), x, U(x))$, we can translate Theorem 5.1 to this case for characterizing an "hybrid" Lyapunov function, defined as an extended functions $\mathbf{v} : E\times X \mapsto \mathbf{R}\cup\{+\infty\}$ possibly depending on the locations (for instance $\mathbf{v}(e,x) := \|x - c(e)\|$ where $c(e) \in K(e)$ is an equilibrium, i.e., a solution to the equation $f(e, c(e), u(e)) = 0$ where $u(e) \in U(c(e))$).

Theorem 5.2 *Let $\mathbf{v} : E\times X \rightsquigarrow \mathbf{R}\cup\{+\infty\}$ be a nontrivial nonnegative lower semicontinuous extended function, $U : E\times X \rightsquigarrow E\times X$, be a Marchaud map, $f : E\times X\times Y$ be a continuous map with linear growth affine with respect to the controls and $R : E\times X \rightsquigarrow E\times X$ be an upper semicontinuous map with compact images and $\varphi : X\times\mathbf{R}_+ \to \mathbf{R}$ be continuous with linear growth. We set*

(a) $\mathbf{v}_R(e,x) := \inf_{(f,y)\in R(e,x)} \mathbf{v}(f,y)$, *the* marginal function,

(b) $\mathbf{R}_{\mathbf{v}_R}(e,x) := \{(f,y) \in R(e,x) \mid \mathbf{v}(f,y) = \mathbf{v}_R(e,x)\}$, *the* marginal map.

Then the two following conditions are equivalent:

(a) *for any initial state* $x_0 \in \mathrm{Dom}(\mathbf{v})$, there exist a run $(e(\cdot), x(\cdot)$ *to the hybrid control* *system (6) and a solution to the differential equation* $w(\cdot)$ *to (1) satisfying property:*

$$\forall\, t \in [t_n, t_{n+1}[,\ \ \mathbf{v}(e(t_n), x(t)) \ \leq\ w(t)$$

(b) $\mathbf{v}$ *is a contingent solution to the* Hamilton-Jacobi variational inequalities*:* whenever $\mathbf{v}(e,x) < \mathbf{v}_R(e,x)$*, then*

$$\inf_{v \in F(e,x)} D_\uparrow \mathbf{v}(e,x)(v) + \varphi(x, \mathbf{v}(e,x)) \ \leq\ 0$$

6 Hybrid Gradient Methods for Global Optimization

Let us consider the minimization problem

$$v_0 \ :=\ \inf_{x \in X} \mathbf{v}(x)$$

where $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is a nontrivial lower semicontinuous extended function assumed to be bounded from below.

A way to introduce the "Gradient Method" is to use the simple differential inclusion

$$\forall\, t \geq 0,\ \ x'(t) \ \in\ B$$

where B denotes the unit ball of the finite dimensional vector space X, leaving open the direction to be chosen by the algorithm. Instead of choosing the velocities at random as in the methods of simulated annealing and being satisfied by convergence in probability, we shall ask whether $\mathbf{v}$ can be an exponential Lyapunov function for the differential inclusion $x' \in B$.

We introduce the function $\mathbf{v}_\propto \geq \mathbf{v}$ which is the smallest of the lower semicontinuous a-Lyapunov functions larger than or equal to $\mathbf{v}$, the minima of which coincide with the global minima of $\mathbf{v}$. The gradient algorithm for $\mathbf{v}_\propto$ converges to global minimal of $\mathbf{v}$ (see [9, Aubin & Najman] and Chapter 8 of [2, Aubin] about the "Montagnes Russes Algorithm").

Instead of using this (costly but efficient) algorithm, we can use a reset map R as a discrete search algorithm which can be paired with the usual gradient method for jumping over local minima.

Theorem 5.1 implies that whenever $\mathbf{v}(x) < \inf_{y \in R(x)} \mathbf{v}(y)$, then

$$\inf_{u \in B} D_\uparrow \mathbf{v}(x)(u) + a(\mathbf{v}(x) - v_0) \ \leq\ 0 \qquad (7)$$

is necessary and sufficient for the existence of one run $x(\cdot)$ to the impulse differential inclusion (B, R) starting from any given initial state $x_0 \in \mathrm{Dom}(\mathbf{v})$.

Therefore, a run $(x(\cdot), (\cdot)$ of the auxiliary impulse differential inclusion $B \times \{-a(\mathbf{v}(x) - v_0)\}, R \times \mathbf{1})$ is defined in the following way: At impulse time t_n and initial conditions x_n and $(\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0$ such that $\mathbf{v}(x_n) \leq (\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0$, then

(a) if $\mathbf{v}_R(x_n) \ \leq\ (\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0$, we take $t_{n+1} = t_n$ and $x_{n+1} \in R(x_n)$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x_n)$, so that $\mathbf{v}(x_{n+1}) \leq (\mathbf{v}(x_0) - v_0)e^{-at_{n+1}} + v_0$,

(b) if $(\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0 < \mathbf{v}_R(x_n)$, there exist a solution $x(\cdot)$ to the gradient method

$$x'(t) \in \{w \in B | D_\uparrow \mathbf{v}(x)(w) + a(\mathbf{v}(x) - v_0) \leq 0\}$$

starting at time t_n from x_n satisfying $\mathbf{v}(x(t)) \leq (\mathbf{v}(x_0) - v_0)e^{-at} + v_0$ until a time t_{n+1} such that $\mathbf{v}_R(x(^-t_{n+1})) = w(t_{n+1})$ if

$$(\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0 < \min(\mathbf{v}_\propto(x_n), \mathbf{v}_R(x_n))$$

We thus reset the initialized state by taking $x_{n+1} \in R(x(^-t_{n+1}))$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x(^-t_{n+1}))$, so that $\mathbf{v}(x_{n+1}) = (\mathbf{v}(x_0) - v_0)e^{-at_{n+1}} + v_0$. □

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