Deadlock-Free Piecewise-Linear Controlled Hybrid Automata

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Abstract

A controlled hybrid automaton is a hybrid automaton whose continuous-state trajectories satisfy inhomogeneous differential equations. This paper presents sufficient conditions for the existence of periodic global solutions of piecewise-linear controlled hybrid automata.

1 Introduction

The hybrid automaton [1] has emerged as an important model for the analysis and design of hybrid dynamical systems. In spite of its popularity, there is a significant technical obstacle associated with verification of such systems. This obstacle is that the verification problem is undecidable for most interesting classes of systems [3]. As a consequence, recent research has focused on developing sophisticated tools that can only approximately verify the system's safety [2] [4].

One way to get around this obstacle is to reframe the verification question. In [8], such an approach was attempted for the class of piecewise-linear *controlled* hybrid automata. Controlled hybrid automata have continuous states that satisfy inhomogeneous differential equations driven by *control* inputs. Rather than asking the traditional question of whether or not the system reaches a specified state set, we ask whether or not there exists a control forcing the system to that set. In [8] it was shown that this question could be answered in a necessary and sufficient manner for periodic solutions of piecewise-linear controlled hybrid automata. This paper summarizes and reports on recent progress in extending that earlier work.

The remainder of this paper is organized as follows. Section 2 states preliminary definitions. Section 3 states and proves the main result of the paper; a sufficient condition for the existence of global non-chattering non-terminating solutions of controlled piecewise-linear hybrid automata. Section 4 shows how we can verify this condition by solving a linear program. If we cannot verify the existence of a solution, then section 5 presents a method that systematically searches for controller parameters enforcing the existence conditions.

2 Preliminary Definitions

Definition 1 *A* controlled hybrid automaton $\mathcal{H}$ *is characterized by the 7-tuple,* $(X, \Sigma, U, A, G, F, q_0)$ *where*

- X *is a smooth manifold in Euclidean* n*-space,* $\Re^n$*.* X *is usually called the* continuous state space *of the system.*
- Σ *is a discrete set of symbols representing the* discrete states *(also called* modes*) of the system.*
- U *is a compact subset of the real line,* $\Re$*, representing the* control inputs *to the system.*
- $A \subset \Sigma \times \Sigma$ *is a set of ordered pairs representing* discrete events *that are generated when the system's discrete state changes.*
- $G : A \to \mathcal{P}(X)$ *maps each event* $(\alpha, \beta) \in A$ *onto a closed set* $G((\alpha, \beta)) \subset X$*. The set* $G((\alpha, \beta))$ *is called the* guard *for event* (α, β)*.*
- $F : X \times U \times \Sigma \to \Re^n$ *maps the current state* $(x, \alpha) \in X \times \Sigma$ *and the current control input* $u \in U$ *onto a vector* $F(x, u, \alpha)$ *in* $\Re^n$*.*
- $q_0 = (x_0, \alpha_0) \in X \times \Sigma$ *is the* initial state *of the system*

For a given $\alpha \in \Sigma$, we let $F_\alpha : X \times U \to \Re^n$ represent the vector field associated with mode α. The differential equation

$$\dot{x}(\tau) = F_\alpha(x(\tau), u(\tau))$$

is called the modal equation associated with mode α.

Consider a controlled hybrid automaton, $\mathcal{H}$, with initial state $q_0 = (x_0, \alpha_0)$. The *state machine* associated with

$\mathcal{H}$ is represented by the ordered triple, (Σ, A, α_0) where Σ is a set of *vertices*, $A \subset \Sigma \times \Sigma$ is a set of *arcs* between vertices, and $\alpha_0 \in \Sigma$ is the initially *marked* vertex of the machine. If $\alpha \in \Sigma$, then the *postset* of α (denoted as $\alpha\bullet$) consists of all $\beta \in \Sigma$ such that $(\alpha, \beta) \in A$. The *preset* of α denoted as $\bullet\alpha$ consists of all $\beta \in \Sigma$ such that $(\beta, \alpha) \in A$. Given a discrete set Σ, we let Σ^* denote the set of all finite length sequences formed by concatenating elements of Σ. An element of Σ^* is called a *string*. A string $\sigma = \{\omega_1, \omega_2, \cdots, \omega_N\}$ is *accepted* by the state machine (Σ, A, α_0) if and only if $\omega_1 = \omega_0$ and $(\omega_i, \omega_{i+1}) \in A$ for $i = 1, \ldots, N-1$.

A *trajectory* is a function $q : I \rightarrow X \times \Sigma$ where I is an interval in $\Re$. The value that a trajectory q takes at time $\tau \in I$ is represented by the ordered pair $(x(\tau), \omega(\tau))$ where $x(\tau) \in X$ and $\omega(\tau) \in \Sigma$. The interval I is called the *interval of existence* for the trajectory.

Given a trajectory q we say a time $\tau \in I$ is *regular* if q is continuous at τ. Otherwise, τ is said to be a *switching* instant. If a trajectory q has an infinite number of switching instants, then q is *non-terminating*. If q has an infinite number of switching instants over a finite subinterval of its interval of existence, then q is said to be *chattering*.

Definition 2 *A trajectory $q : I \rightarrow X \times \Sigma$ is a* solution *of the controlled hybrid automaton $\mathcal{H} = (X, \Sigma, U, A, G, F, q_0)$ if and only if*

- $q(0) = q_0$
- *for all closed interals $[\tau_a, \tau_b] \subset I$ that contain no switching instants, there exists $\alpha \in \Sigma$, an absolutely continuous trajectory $x : [\tau_a, \tau_b] \rightarrow X$, and a measureable control $u; [\tau_a, \tau_b] \rightarrow U$ such that $\omega(\tau) = \alpha$, and $\dot{x}(\tau) = F_\alpha(x(\tau), u(\tau))$ for all $\tau \in [\tau_a, \tau_b]$.*
- *At any switching instant $\tau_s \in I$, there exist $\alpha, \beta \in \Sigma$ such that*

$$\begin{array}{rcl} (\alpha, \beta) & \in & A \\ x(\tau_s) & \in & G((\alpha, \beta)), \\ \beta & = & \lim_{\tau \rightarrow \tau_s^+} \omega(\tau) \end{array}$$

3 Global Non-Terminating Solutions

This section states and proves two results concerning the existence of global non-terminating solutions. The first proposition is a well-known result that recursively calls the Preset-operator to verify the existence of a non-terminating trajectory. The second proposition directly solves the fixed point equation associated with this Preset operator for the class of so-called piecewise-linear controlled hybrid automata.

In order to state and prove the first result we need the following definitions about *reachable* states and the so-called Pre operator.

Definition 3 *Consider a controlled hybrid automaton $\mathcal{H} = (X, \Sigma, U, A, G, F, q_0)$. A state $q_f \in X \times \Sigma$ is* reachable *from q_0 if there exists a finite time $T > 0$ such that the trajectory $q : [0, T] \rightarrow X \times \Sigma$ is a solution of $\mathcal{H}$ where $q_0 = q(0)$ and $q(T) = q_f$.*

Definition 4 *Consider a controlled hybrid automaton $\mathcal{H} = (X, \Sigma, U, A, G, F, q_0)$ and let $K \subset X \times \Sigma$, then $q_0 \in \mathrm{Pre}(K)$ if and only if there exists $q_f \in K$ such that q_f is reachable from q_0.*

The following proposition is a variation on a well known result that has appeared numerous times in the literature [1] [7] for traditional hybrid automata.

Proposition 1 *Let $\mathcal{H} = (X, \Sigma, U, A, G, F, q_0)$ be a controlled hybrid automaton. Let the initial state be $q_0 = (x_0, \omega_0)$. If there exists a $\omega_f \in \Sigma$ such that the recursion*

$$\begin{array}{rcl} K[n+1] & = & \mathrm{Pre}(K[n]) \cap K[n] \\ K[0] & = & G((\omega_f, \omega_0)) \end{array}$$

converges to a nonempty set K^* that contains q_0, then there exists a non-terminating solution q to $\mathcal{H}$.

Proof: Clearly the sequence $\{K[n]\}$ is a monotone non-increasing sequence of closed sets. Therefore $\{K[n]\}$ converges to a closed set K^*. By assumption $q_0 \in K^*$, so K^* is non-empty. Moreover $K^* \subset K[n]$ for all n. In particular, $K^* \subset K[0] = G((\beta, \alpha))$.

Clearly $\mathrm{Pre}(K^*) \cap K^* \subset K^*$. So assume $q \in K^*$ but that $q \notin \mathrm{Pre}(K^*) \cap K^*$. The fact that $q \in K^*$ implies that $q \in K[n]$ for all n. If $q \notin \mathrm{Pre}(K^*) \cap K^*$, then $q \notin \mathrm{Pre}(K^*)$. This set is closed so we can find some N such that for $n > N$, $q \notin \mathrm{Pre}(K[n])$. This means, however, that $q \notin K[n+1] = \mathrm{Pre}(K[n]) \cap K[n]$. Which contracts our earlier assumption. So q must also be in $\mathrm{Pre}(K^*) \cap K^*$ and this implies $K^* = \mathrm{Pre}(K^*) \cap K^* = \mathrm{Pre}(K^*)$.

This final fact is necessary and sufficient for a non-terminating solution to exist. For if $K^* = \mathrm{Pre}(K^*)$ then any point in K^* is reachable from some other point in K^*. We assume $q_0 \in K^*$. The fact that $K^* = \mathrm{Pre}(K^*)$ means we can always return to this set. Moreover since $K^* \subset G((\beta, \alpha))$ we can select a trajectory that fires the discrete event (β, α) infinitely often. The selected trajectory therefore is non-terminating. $\bullet$

Remark: The preceding proposition is the foundation for many algorithmic approaches to the verification problem. To establish that K^* is non-empty, one simply applies the recursion to one of the guards (or some subset of desirable states). If the recursion converges to a non-empty set, K^*, then provided the initial state starts in K^*, we are assured of eventually reaching $K[0]$. The problem with this approach is that the recursion may not converge after a finite number of iterations. If finite termination does not occur, then it is impossible to conclude that K^* is non-empty. This simple observation lies at the heart of the verification problem's undecidability [3].

Instead of attempting to compute K^* through this recursion, we attempt to compute K^* directly from the fixed point equation, $K^* = \text{Pre}(K^*)$. This computation may, in general, be very difficult. So we confine our attention to the case of *piecewise-linear* controlled hybrid automata because we can parameterize the guards as convex combinations of extreme points.

Definition 5 *A controlled hybrid automaton* $\mathcal{H} = (X, \Sigma, U, A, G, F, q_0)$ *is piecewise-linear if and only if*

- *for all* $(\alpha, \beta) \in A$, *there exists an integer* $N > 0$ *and a set of vectors* v_j $(j = 1, \ldots, N)$ *such that*

$$G((\alpha,\beta)) = \Bigg\{ x \in \Re^n \ : \ x = \sum_{k=1}^{N} \lambda_k v_k, \quad 1 = \sum_{k=1}^{N} \lambda_k, \text{ and } \lambda_k \geq 0 \text{ for all } k \Bigg\}$$

- *and the modal equation associate with mode* $\alpha \in \Sigma$ *has the form*

$$F_\alpha(x,u) = A_\alpha x + B_\alpha u$$

where $A_\alpha \in \Re^{n \times n}$ *and* $B_\alpha \in \Re^n$.

We now state the main result of this section. This proposition is a sufficient condition for the existence of a measureable control, u, generating a non-chattering non-terminating global solution of a controlled piecewise-linear hybrid automaton. In the following result, we use the notation $i|N$ to denote i modulo N.

Proposition 2 *Let* $\mathcal{H}$ *be a piecewise-linear controlled hybrid automaton. Assume there exists a string* $\sigma = \omega_0, \omega_1, \cdots, \omega_{N-1}$ *in* Σ^* *that is accepted by the state machine associated with* $\mathcal{H}$. *Let* (A_i, B_i) *denote the pair of system matrices associated with mode* ω_i $(i = 0, \ldots, N-1)$. *Let* $\mathbf{V}_i$ $(i = 0, \ldots, N-1)$ *be an* $n \times p_i$ *matrix whose* j*th column,* $v_{ij} \in \Re^n$ *for* $j = 0, \ldots, p_i - 1$ *, is an extreme point of the guard* $G((\omega_i, \omega_{(i+1)|N}))$. *Let* $\mathcal{C}_i$ *be the controllability matrix*

$$\mathcal{C}_i = \begin{bmatrix} B_i & A_iB_i & A_i^2B_i & \cdots & A_i^{n-1}B_i \end{bmatrix}$$

associated with mode ω_i. *Let* $\mathbf{E}_i$ *be an* $n \times r_i$ *matrix whose* l*th column vector,* $e_{il} \in \Re^n$ *for* $l = 0, \ldots, r_i - 1$, *is a standard basis vector for the subspace* $\text{Im}((\mathcal{C}_i)$.

If there exist real numbers T_i, λ_{ij}, *and* β_{il} *such that for all* $i = 0, \ldots, N-1$,

$$0 = \sum_{l=0}^{r_i-1} \beta_{il} e_{il} + e^{A_iT_i} \sum_{j=0}^{p_i-1} \lambda_{ij} v_{ij} - \sum_{j=0}^{p_i-1} \lambda_{kj} v_{kj} \quad (1)$$

$$1 = \sum_{j=0}^{p_i-1} \lambda_{ij} \quad (2)$$

$$0 \geq \lambda_{ij}, \text{ for all } j = 0, \ldots, p_i - 1 \quad (3)$$

where $k = (i+1)|N$ *in equation 1, then there exists a global non-terminating and non-chattering trajectory* $q : [0, \infty) \to X \times \Sigma$ *that is a solution to* $\mathcal{H}$.

Proof: Note that any point $x \in \text{Im}(\mathcal{C}_i)$ can be represented as

$$x = \sum_{l=0}^{r_i-1} \beta_{il} e_{il}$$

where β_{il} is some set of real numbers. Note that a point x in the guard $G((\omega_i, \omega_{(i+1)|N}))$ can be written as

$$x = \sum_{j=0}^{p_i-1} \lambda_{ij} v_{ij}$$

subject to the constraints that $\lambda_{ij} \geq 0$ and $\sum_{j=0}^{p_i-1} \lambda_{ij} = 1$.

In view of these above observations, we see that equation 1 can be rewritten as

$$e^{A_iT_i} w_i - w_{i+1} \in \text{Im}(\mathcal{C}_i) \quad (4)$$

where w_i is in $G((\omega_{(i-1)|N}, \omega_i))$ and w_{i+1} is in $G((\omega_i, \omega_{i+1|N}))$. From linear systems theory [6], we know that equation 4 is necessary and sufficient for w_{i+1} to be reachable from w_i in time T_i. Substituting the explicit expressions for w_i, w_{i+1} and a vector in $\text{Im}(\mathcal{C}_i)$ into equation 4 yields equation 1 and we can infer that equation 1 is necessary and sufficient for the mode $\omega_{(i+1)|N}$ to be reachable in time T_i from $\omega_{i|N}$. This relation holds for all i and coupling this with the assumption that σ is accepted implies that we have a non-terminating solution to $\mathcal{H}$. This non-terminating solution is global because of the finite time T_i between mode switches and it is obviously non-chattering. •

Remark: While the condition in the preceding proposition is only sufficient for the existence of global non-terminating solutions, it is necessary and sufficient for the existence of a *periodic* solution that has the inter-event switching times T_i.

4 Verifying a Periodic Solution

This section shows how the condition in proposition 2 can be verified by solving a linear program. We begin by reframing equation 1 and 2 as the following matrix vector equations

$$\begin{aligned} \mathbf{c} &= \mathbf{S}\overline{\eta} \\ &= \left[\begin{array}{c|c} \mathbf{G}^T & \mathbf{F}^T \end{array}\right] \left[\begin{array}{c} \mathbf{z} \\ \hline \mathbf{y} \end{array}\right] \end{aligned} \qquad (5)$$

where

$$\begin{aligned} \mathbf{G}^T &= \left[\begin{array}{c} \text{blockDiag}(\mathbf{E}_i) \\ \hline \mathbf{0}_{N\times r} \end{array}\right] \\ \mathbf{F}^T &= \left[\begin{array}{c} \mathbf{\Gamma} \\ \hline \text{blockDiag}(\mathbf{1}_{1\times p_i}) \end{array}\right] \\ \mathbf{\Gamma} &= \left[\begin{array}{cccc} e^{A_0T_0}\mathbf{V}_0 & -\mathbf{V}_1 & \cdots & 0 \\ 0 & e^{A_1T_1}\mathbf{V}_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ -\mathbf{V}_0 & 0 & \cdots & e^{A^{N-1}T_{N-1}}\mathbf{V}_{N-1} \end{array}\right] \end{aligned} \qquad (6)$$

where $r = \sum_{i=0}^{N-1} r_i$ and

$$\mathbf{z} = \left[\begin{array}{c} \beta_{00} \\ \beta_{01} \\ \cdot \\ \beta_{0,r_0-1} \\ \cdot \\ \cdot \\ \beta_{N-1,r_{N-1}-1} \end{array}\right], \quad \mathbf{y} = \left[\begin{array}{c} \lambda_{00} \\ \lambda_{01} \\ \cdot \\ \lambda_{0,p_0-1} \\ \cdot \\ \cdot \\ \lambda_{N-1,p_{N-1}-1} \end{array}\right]$$

Equation 5 is a nonlinear algebraic equation because of the matrix exponential functions in the Γ matrix. If we fix the time intervals T_i, however, then this equation is a linear algebraic equation in variables β_{il} and λ_{ij}. In particular, we simply seek a solution to this linear algebraic equation in which $\lambda_{ij} \geq 0$.

Necessary and sufficient conditions for the existence of an admissible solution to equation 5 subject to $\lambda_{ij} \geq 0$ are given by one of the theorems of the alternative [5]. The theorem of the alternative states that equation 5 has a solution subject to the constraints in equation 3 if and only if there exists no vector $\mathbf{x}$ such that

$$\mathbf{G}\mathbf{x} \leq 0, \mathbf{F}\mathbf{x} = 0, \mathbf{c}^T\mathbf{x} > 0 \qquad (7)$$

Finding such an $\mathbf{x}$ is accomplished by solving the following linear program.

$$\begin{array}{ll} \textbf{maximize:} & \mathbf{c}^T\mathbf{x} \\ \textbf{subject to:} & \mathbf{G}\mathbf{x} \leq 0 \\ & \mathbf{F}\mathbf{x} = 0 \end{array} \qquad (8)$$

Solutions to the above linear program have a special form due to the fact that the vectors $\mathbf{x}$ satisfying $\mathbf{G}\mathbf{x} \leq 0$ form a polytopic cone whose apex is at the origin. In this case, there are two possible solutions. The solution is either unbounded and positive or bounded and equal to zero. Let the solution be $\mathbf{x} = 0$. This means that the original problem has a non-negative solution and we can therefore infer that a global solution exists. If an unbounded solution occurs, then our original algebraic equation has no solution where $\mathbf{y} > 0$ and this means that no global solution exists for the specified switching intervals T_i .

In practice, we actually solve the following linear program,

$$\begin{array}{ll} \textbf{maximize:} & \mathbf{c}^T\mathbf{x} \\ \textbf{subject to:} & \mathbf{G}\mathbf{x} \leq 0 \\ & \mathbf{F}\mathbf{x} = 0 \\ & -100 \leq \mathbf{x} \leq 100 \end{array} \qquad (9)$$

The last constraint assures that the linear program returns a bounded feasible point $\mathbf{x}$. We can then readily check to see if this solution corresponds to an unbounded solution of the original linear program in equation 8 by checking the Lagrange multipliers associated with the upper and lower bound inequality constraints. These constraints are active if the Lagrange multiplier has nonzero components and if this is the case then the original linear program clearly has an unbounded solution. In this case, we can immediately infer that the specified set of switching times T_i does not admit a non-terminating solution.

As an example consider the controlled hybrid automaton shown in figure 1. This is a simple system whose system matrices are

$$\begin{aligned} (A_0, B_0) &= \left(\left[\begin{array}{cc} 0 & 4 \\ 1/4 & 0 \end{array}\right], \left[\begin{array}{c} 4 \\ 1 \end{array}\right] \right), \\ (A_1, B_1) &= \left(\left[\begin{array}{cc} 0 & -9 \\ -1/9 & 0 \end{array}\right], \left[\begin{array}{c} 10 \\ 1 \end{array}\right] \right) \end{aligned}$$

and whose guards are

$$\begin{aligned} V_0 &= \left\{ \left[\begin{array}{c} -2 \\ 0 \end{array}\right], \left[\begin{array}{c} -3 \\ 0 \end{array}\right], \left[\begin{array}{c} -3 \\ 1 \end{array}\right] \right\} \\ V_1 &= \left\{ \left[\begin{array}{c} 2 \\ 0 \end{array}\right], \left[\begin{array}{c} 3 \\ 0 \end{array}\right], \left[\begin{array}{c} 3 \\ 1 \end{array}\right] \right\} \end{aligned}$$

We let $T_0 = 1$ and $T_1 = 1$ and solve the linear program associated with this problem. For the case when $T_0 = 1$ and $T_1 = 1$, the largest Lagrange multipliers associated with the lower bound inequality constraint had a value of 0.4560. This means that one of the lower bounds is active. We can therefore conclude that this choice of T_0 and T_1 does not admit a non-terminating solution. If we change these inter-event times to $T_0 = 1.7$ and $T_1 = 0.29$, then the Lagrange multipliers associated with the bounds are zero, indicating a non-terminating trajectory exists.

The intermediate results of the linear program allow us to characterize the switching sets and from these

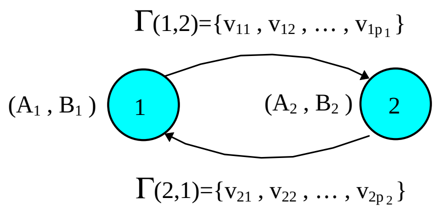

Figure 1: Hybrid Automaton for Example

sets we can identify control trajectories, u that enforce the desired periodic behavior. This additional information is contained in the solution vectors satisfying our original system, $\mathbf{S}\overline{\eta} = \mathbf{c}$. This set of solutions can be parametrized as $\overline{\eta} \in \overline{\eta}_{p0} + \text{null}(\mathbf{S})$ where η_{p0} is a particular solution to the inhomogeneous equation $\mathbf{c} = \mathbf{S}\overline{\eta}$. From these parameterized solutions we can then pick two specific points $x(\tau_1)$ and $x(\tau_0)$ that we know are mutually reachable from each other.

Once a pair of mutually reachable points have been found, there are many choices for the open loop control driving the state between these two points. The minimum energy control $u(t)$ that transfers the first modal system from initial state $x(\tau_0)$ to a target state $x(\tau_1)$ is given by

$$u_0(t) = B_0^T e^{A_0^T(T_0 - t)} \eta_0$$

wehre η_0 is the solution of the equation

$$W_0(0, T_0)\eta_1 = x(\tau_1) - e^{A_0 T_0} x(\tau_0)$$

$W_0(0, T_0)$ is the controllability Gramian of (A_0, B_0). For this specific example, the details were worked out in [8].

5 Searching for Periodic Solutions

If the optimizer of the linear program in equation 9 indicates a periodic solution doesn't exist, then it may be possible to adjust the times T_i to force a solution at $\mathbf{x} = 0$. This approach was originally proposed in [8] where an ad hoc approach of searching for feasible T_i was proposed. This section presents a more systematic search strategy based on the fact that optimal solutions to the linear program in equation 9 are Karush-Kuhn-Tucker (KKT) points [5].

The optimal solution, $\mathbf{x}$, of the linear program in equation 9 will satisfy the Karush-Kuhn-Tucker (KKT) condition

$$\mathbf{G}^T\mathbf{z} + \mathbf{F}^T\mathbf{y} + \lambda_{ul0} = \mathbf{c} \tag{10}$$

where $\mathbf{z}$, $\mathbf{y}$, and λ_{ul0} are Lagrange multipliers associated with the inequality, equality, and bounding constraints, respectively. If any of the bounding Lagrange multipliers, λ_{ul} are nonzero, then we know that the system does not have a non-terminating trajectory with the specified inter-event times T_i.

So let's consider perturbing the times T_i by δT_i. This will change the $\mathbf{F}$ matrix. If the perturbation admits a feasible solution, then the Lagrange multipliers associated with the bounds, δ_{ul} will be zero. We force our perturbation to still satisfy the KKT condition, which means that our perturbed solution will still be an optimal solution to the perturbed linear program. The perturbed KKT condition, therefore becomes

$$\mathbf{G}^T\mathbf{z} + \mathbf{F}^T\mathbf{y} + \sum_{i=1}^{N} \frac{d\mathbf{F}^T\mathbf{y}}{\delta T_i} = \mathbf{c} \tag{11}$$

From equations 10 and 11 we can infer that

$$\left[\frac{d\mathbf{F}^T\mathbf{y}}{d\overline{T}}\right] \delta\overline{T} = -\delta_{ul0} \tag{12}$$

where $\overline{T} = [T_0, T_1, \cdots, T_{N-1}]^T$ and $\delta\overline{T}$ is a vector of the perturbations δT_i. Equation 12 immediately suggests that the "desirable" perturbation $\delta\overline{T}$ will be

$$\delta\overline{T} = \left[\frac{d\mathbf{F}^T\mathbf{y}}{d\overline{T}}\right]^+ \delta_{ul0} \tag{13}$$

where $[A]^+$ is the pseudo-inverse of matrix A.

The perturbation vector in equation 13 is chosen so that the new optimal solution will reduce the non-zero components in δ_{ul0}. So the basic idea is to use the perturbed times $\delta\overline{T}$ to perturb the times T_i and see if this results in an optimizer in which the activity of the upper and lower bounds has been reduced. We simply repeat this until the Lagrange multipliers associated with these bounds are reduced to zero.

A Matlab script was written to test this approach. It was found however, that simply perturbing T_i provided insufficient freedom to null out the Lagrange multipliers λ_{ul0}. We therefore used a slightly different A_i matrix in which a control matrix $\mathbf{K}_i$ is used. The controlled system matrix multiplied by T_i becomes

$$\hat{A}_i T_i = A_i T_i + B_i K_i$$

This means that our Γ matrix in equation 6 becomes

$$\mathbf{\Gamma} = \begin{bmatrix} e^{\hat{A}_0 T_0}\mathbf{V}_0 & -\mathbf{V}_1 & \cdots & 0 \\ 0 & e^{\hat{A}_1 T_1}\mathbf{V}_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ -\mathbf{V}_0 & 0 & \cdots & e^{\hat{A}^{N-1} T_{N-1}}\mathbf{V}_{N-1} \end{bmatrix}$$

We now have more free parameters to play with in attempting to null out the nonzero Lagrange multipliers λ_{ul0}.

We used a simple Matlab script to test this particular idea on the problem shown above. The algorithm converged after a finite number of iterations to a set of gains K_0, K_1, and times T_0 and T_1 that formed a non-terminating solution. Figure 2 shows the largest Lagrange multiplier associated with the active bounding constraint. Note that this mutliplier decreasing in an exponential manner to zero. Most of the simulations we ran exhibited similar behavior.

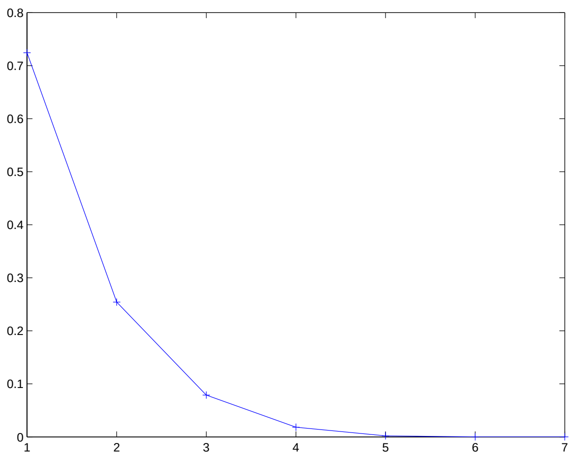

Figure 2: Convergent Behavior of Search

Remark: For some test cases, the search decreased exponentially to a nonzero Lagrange multiplier δ_{ul0}. This means that our search couldn't locate one. Nonetheless, in all cases the performance measure $|\delta_{ul0}|$ decreased in the same manner as shown in figure 2. We suspect that by using a more sophisticated conjugate gradient search we should get better results.

Remark: In originally running these examples we used Matlab's function `linprog`. In some test cases, the optimal points returned by the function were not KKT points. This was due to the fact that `linprog` was running a dual-primal interior point algorithm in which some of the scaling transformation used in computing the Lagrange multipliers suffered from round-off error instabilities. By switching to the traditional simplex algorithm, these problems were avoided.

6 Conclusions

This paper summarized an approach to the verification of hybrid systems in which we verify the existence of measureable controls enforcing a desired periodic behavior. The approach was first presented in [8]. In this paper, we formally state and prove the proposition in a somewhat more general context. Moreover, we present an algorithm to search for controller parameters enforcing a global non-terminating trajectory. This algorithm appears to be a significant improvement over our earlier search algorithm.

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