Hybrid Control Based on Discrete–Event Automata and Receding–Horizon Neural Controllers [1]

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Abstract

A *hybrid* receding–horizon control scheme for nonlinear discrete–time systems is addressed in this work. Such a scheme is composed of two control levels: a *continuous* level characterized by a finite set of neural receding–horizon feedback control laws, and a *discrete–event* level aimed at choosing the best control action to be applied to the plant, depending on the current system conditions and on possible occurred external events. The two–level scheme presents two major innovative aspects: first, a new class of hybrid automata, namely the *discrete–time discrete–event automata*, is used for the modeling of the proposed hybrid control scheme. Moreover, receding–horizon regulators based on neural approximators and off–line determined can be adopted at the continuous level. The stability analysis of the hybrid control system is addressed both in the case in which optimal receding–horizon feedback control functions are used and in the case in which neural approximate regulators are adopted.

1 Introduction

An hybrid system framework in which continuous and discrete dynamics coexist is considered in this paper. The coupling of "time–driven" and "event–driven" components is here related to the existence of a continuous plant whose behavior is supervised by a discrete–event system. Whenever the continuous part of an hybrid system can be further regarded as the combination of finitely many continuous dynamic systems, an interesting class of hybrid systems, namely the class of *switched systems*, is identified. Significant tools for the stability analysis of switched and hybrid systems are provided in [1], in which "multiple Lyapunov functions" are defined and used. A comprehensive study of stability criteria for switched systems is reported in [2], where the authors establish new results on the stability and asymptotic stability of the equilibrium of such systems.

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In this paper, according to the general modeling scheme proposed for instance in [10], [11], an hybrid control scheme is defined in which two control levels are juxtaposed. The first control level, the continuous level, consists of a finite class of receding–horizon nonlinear control functions. The second level, namely the discrete–event one, acts as a supervisor choosing, in the set defined at the continuous level, the best control law to be applied to the plant according to the present system state and eventual external events. The proposed scheme presents two main innovative aspects: first of all, the feedback control functions defined at the continuous level are neural approximations of some optimal receding–horizon control functions. Such approximators are determined off–line thus giving rise to a very small on–line computational effort to generate the control actions at each time instant. Moreover, the overall scheme is represented by means of the new formalism of *discrete–time discrete–event automata* consisting in *hybrid automata* [4] in which the continuous subsystem is a discrete–time dynamic system. The stability properties of the proposed control scheme are analyzed by defining suitable constraints on the switching logic associated with the discrete–event subsystem. Due to space limitations, experimental results showing the effectiveness of the proposed control scheme are not reported in the paper and the reader is referred to [12] where a real case study relevant to traffic control on freeways is addressed in detail.

2 Discrete–time discrete–event automata

A discrete–time discrete–event automaton (DTDEA) is an hybrid automaton [4] in which the continuous–valued dynamical process is a discrete–time dynamic system. By generalizing the hybrid automaton formalism used in [5], a DTDEA is defined as a three-tuple $(\mathcal{G}, \mathcal{B}, \mathcal{L},)$, where $\mathcal{G}$ is a directed graph or network representing the discrete–event subsystem, $\mathcal{B}$ is a set of difference equations representing the discrete–time dynamics of the continuous subsystem, and the relationship between the two subsystems is defined by $\mathcal{L}$, which is a mapping from the vertices and arcs of the network into a propositional logic.

More precisely, $\mathcal{G}$ is a directed graph composed of a set of nodes N and a set of directed arcs A, with $A \subset N \times N$. Thus, $\mathcal{G} = (N, A)$ represents the discrete–event subsystem, since it enumerates all its possible discrete states. The present state of the discrete–event subsystem is denoted by marking the network $\mathcal{G}$, thus transforming it into the triple (N, A, μ), where the final element of the triple is a function $\mu : N \mapsto \{0, 1\}$ associating either zero or one with each node of the network. If $\mu(n) = 1, n \in N$, then node n is marked, which means that the state of the discrete–event subsystem is equal to the discrete state represented by node n. Of course, $\mu(n) = 0$, means that node n is not marked.

The set $\mathcal{B}$ is a finite set of mappings $b_i : \mathbb{R}^n \mapsto \mathbb{R}^n$, $i = 1, ..., \mathrm{card}(\mathcal{B})$ each of which represents a discrete-time dynamical system or mode which generates a state trajectory $x : \mathbb{Z}^+ \mapsto \mathbb{R}^n$ through first-order difference equations of type $x_{k+1} = b_i(x_k)$, $k \in \mathbb{Z}^+$, $b_i \in \mathcal{B}$. The state of the continuous subsystem can be gathered in the four-tuple $z = (\Delta x, x, k_0, x_0)$ where Δx represents one of the mappings in $\mathcal{B}$, $x : \mathbb{Z}^+ \mapsto \mathbb{R}^n$ is a continuous function of $\mathbb{Z}^+$ taking values in $\mathbb{R}^n$, $k_0 \in \mathbb{Z}^+$, and $x_0 \in \mathbb{R}^n$. Then, the following dynamic system is considered

$$x_{k+1} = b_i(x_k) \quad k \in \mathbb{Z}^+, \;\; x_{k_0} = x_0 \qquad (1)$$

The state of the DTDEA can now be defined by combining the four-tuple $z = (\Delta x, x, k_0, x_0)$ representing the state of the continuous subsystem, with the state of the discrete-event subsystem, gathered in the marking vector $\underline{\mu} \triangleq \{\mu(n), n \in N\}$.

The third component of a DTDEA is the mapping $\mathcal{L}$ which represents the interface between the continuous subsystem and the discrete–event one. $\mathcal{L}$ is also called the *event labeling function* and realizes a mapping from the nodes and vertices of graph $\mathcal{G}$ to a propositional logic $\mathcal{P}$. The atomic formulae of such a propositional logic are defined with respect to the continuous state z. Of course, the logical propositions representing the event labels associated with the network elements can be defined in several ways, we choose here to follow the same formalism used in [5]. The set of *atomic equations* is composed of *invariant equations* and *guard equations*. Invariant equations can take on three different forms: the first form is $[\Delta x = b_i]$, with $b_i \in \mathcal{B}$, the second form is $[h(x_0) = 0]$, where $h : \mathbb{R}^n \mapsto \mathbb{R}$ are functions of the initial conditions of the continuous subsystem. The last form of invariant equations is defined to reset the initial time in the continuous state z and, thus, takes on the form $[k_0 = \overline{k}]$, $\overline{k} \in \mathbb{Z}^+$.

As regards guard equations, they have the form $[g(x) > 0]$, where $g : \mathbb{R}^n \mapsto \mathbb{R}$ are functions of the state of the continuous subsystem. An atomic formula p is said to be satisfied by the hybrid state σ iff the equation is true when σ is substituted into the equation. The satisfaction of p by σ is here denoted as $\sigma \models p$. In the DTDEA defined in this paper, the network arcs will be labeled with guard equations, whereas invariant equations will be associated with network nodes.

To define the dynamics of the DTDEA here proposed, it is first of all necessary to define a *switching time* τ as a time instant in which the marking of the network changes, that is, $\underline{\mu}(\tau^-) \neq \underline{\mu}(\tau^+)$. It is to be noted here, that the dynamic evolution of the discrete state $\underline{\mu}$ is due to the fact that the continuous state z satisfies specific constraints or to asynchronous event occurrences. In the first case the switching time is a discrete time instant, whereas in the second case it belongs to the set of real numbers. This makes it necessary to define an overall time space to which switching times belong.

In the present case, it is natural to give the following definition for the metric space $(T, \rho) \triangleq \{(k, \tau) \in \mathbb{Z}^+ \times \mathbb{R}^+ : k = \lceil \tau \rceil\}$, where, for any given pair $t_1 = (k_1, \tau_1), t_2 = (k_2, \tau_2)$, we have $\rho(t_1, t_2) = |\tau_1 - \tau_2|$. The following lemma can be easily proved [13].

Lemma 1. *The metric space (T, ρ) is a time space in the sense of Definition 3.1 in [3].*

Thus, in DTDEA, switching times belong to the time space (T, ρ) above defined. Such a time space will be denoted in the following as $\mathcal{T}$. Moreover, the continuous state associated with a switching time $t \in \mathcal{T}, t = (k, \tau)$ is defined as $z(t) = (\Delta x, x_k, k_0, x_0)$. It is now possible to define an *hybrid trajectory* $\sigma : \mathcal{T} \mapsto \mathcal{H}$ as a function mapping the time space of switching times into the hybrid state space. More precisely, an arbitrary hybrid trajectory $\sigma : \mathcal{T} \mapsto \mathcal{H}$ is generated by the DTDEA if and only if the trajectory satisfies the following conditions: i) for all $t \in [t_1, t_2], : t$ is not a switching time and t_1 is a switching time, there exists a marked node n such that the hybrid state $\sigma(t) \models \mathcal{B}(n)$; ii) if t is a switching time, then there exists an arc (n, m) such that $\sigma(t^-) \models \mathcal{B}((n, m)), \mu(n(t^+)) = \mu(n(t^-)) - 1 = 0, \mu(m(t^+)) = \mu(m(t^-)) + 1 = 1$.

Given the abstract time space $\mathcal{T}$, equipped with the metric ρ, it is now possible to embed the hybrid dynamical system corresponding to the DTDEA into a general dynamical system defined on $\mathbb{R}^+$ and, above all, by following the same approach as in [13], it will be possible to adopt the results about the stability properties of hybrid systems defined in [3].

3 The hybrid control scheme

The proposed hybrid–control scheme is structured as follows. Let us consider the discrete-time dynamic system (in general, nonlinear)

$$x_{k+1} = f(x_k, u_k)\,, \quad k = 0, 1, \ldots \qquad (2)$$

where $x_k \in X \subset \mathbb{R}^n$ and $u_k \in U \subset \mathbb{R}^m$ are the state and control vectors, respectively. We assume that the constraint regions X and U belong to the class $\mathcal{Z} \triangleq \bigcup_{i=1}^{\infty} \mathcal{Z}_i,\ \mathcal{Z}_i \subset \mathbb{R}^i$, where $\mathcal{Z}_i$ is a compact set containing the origin as an *internal point*. Moreover, let $f \in \mathcal{C}^1\left[\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n\right]$, with $f(0,0) = 0$. Furthermore, a class of control functions

$$\Gamma \triangleq \{\, \gamma_i : \mathcal{X}_i \mapsto \mathbb{R}^m, \, i \in I \,\} \qquad (3)$$

is given, where $\mathcal{X}_i \subseteq X$, $\mathcal{X}_i \in \mathcal{Z}$, $\forall\, i \in I$; I is a finite set of integers allowing the choice of a single $\gamma_i \in \Gamma$. The adopted control functions are discrete–time *receding–horizon* (RH) regulators determined as follows. For any

$i \in I$, we define the *finite-horizon* (FH) cost function

$$J^i_{FH}(x_k, u_{k,k+N^i-1}, N^i, a^i, P^i) = \sum_{j=k}^{k+N^i-1} h^i(x_j, u_j) + a^i \|x_{k+N^i}\|^2_{P^i}, \; k \geq 0 \quad (4)$$

where $u_{k\tau} \triangleq \text{col}\,(u_k, \ldots, u_\tau)$, $h^i \in \mathcal{C}^1\ [\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^+]$, is a transition cost function with $h^i(0,0) = 0$, $\|x\|^2_{P^i} \triangleq x^T P^i x$, $a^i \in \mathbb{R}$ is a positive scalar, $P^i \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix, and N^i is a positive integer corresponding to the length of the control horizon. Then we can state the following

Problem 1. *For any $i \in I$ and at every time instant $k \geq 0$, find the RH optimal control law $u_k^{RH_i^\circ} = \gamma^{i^\circ}_{RH}(x_k) \in U$, where $u_k^{RH_i^\circ}$ is the first vector of the control sequence $u_k^{FH_i^\circ}, \ldots, u_{k+N^i-1}^{FH_i^\circ}$ (i.e., $u_k^{RH_i^\circ} \triangleq u_k^{FH_i^\circ}$), which minimizes the cost (4) for the state $x_k \in X$.*

Then, the definition of set Γ, for a generic $i \in I$ requires the solution of Problem 1. In this respect, it can be noticed that the statement of Problem 1 does not impose any particular way of computing the control vector $u_k^{RH_i^\circ}$ as a function of x_k . The optimal control law can be determined numerically in two ways: 1) *on-line computation*: Problem 1 is an open-loop optimal control problem and may be regarded as a nonlinear programming one to be solved on–line; 2) *off-line computation*: this approach implies that the control law $\gamma^{i^\circ}_{RH}(x_k)$ has to be "a priori" computed thus enabling the i–th RH regulator to generate "instantaneously" the control vector $u_k^{RH_i^\circ}$ for *any* vector x_k belonging to the set X of admissible states.

In the proposed hybrid control scheme, the off–line computation of the i–th control law turns out to be the natural choice as this allows the a–priori definition of the class Γ . In this connection, we need to derive ("a priori") an FH closed-loop optimal control law $u_j^{FH_i^\circ} = \gamma^{i^\circ}_{FH}(x_j, j),\, k \geq 0,\, j = k, \ldots, k + N^i - 1$ that minimizes the cost (4) for any $x_k \in X$. Because of the time-invariances of the dynamic system (2) and of the cost function (4), from now on we consider the control functions $u_j^{FH_i^\circ} = \gamma^{i^\circ}_{FH}(x_j, j-k),\, k \geq 0,\, j = k, \ldots, k + N^i - 1$ instead of $u_j^{FH_i^\circ} = \gamma^{i^\circ}_{FH}(x_j, j)$, and state the following

Problem 2. *Find the FH optimal feedback control law $\left\{u_j^{FH_i^\circ} = \gamma^{i^\circ}_{FH}(x_j, j{-}k) \in U,\, k \geq 0,\, j = k, \ldots, k{+}N^i{-}1\right\}$ that minimizes the cost (4) for any $x_k \in X$.*

Then, once the solution of Problem 2 has been achieved, we consider only the first optimal control function (i.e., the one corresponding to $j = k$) and write

$$u_k^{RH_i^\circ} = \gamma^{i^\circ}_{RH}(x_k) \triangleq \gamma^{i^\circ}_{FH}(x_k, 0),\, \forall\, x_k \in X,\, k \geq 0 \quad (5)$$

Hence, we are able to specify the following finite class Γ° of optimal RH control functions: $\Gamma^\circ \triangleq \left\{\gamma^{i^\circ}_{RH} : X \mapsto \mathbb{R}^m,\, i \in I\right\}$ Unfortunately, the off–line determination of class Γ° is in general a very difficult task.

To address the stability analysis for the hybrid control scheme (see Section 4), it is of customary importance to characterize the stabilizing properties of the individual control functions $\gamma^{i^\circ}_{RH}$, $i \in I$. To this end, the following basic assumptions are needed:

(i) The linear system $x_{k+1} = Ax_k + Bu_k$, obtained via the linearization of the system (2) in a neighborhood of the origin (i.e., $A \triangleq \left.\dfrac{\partial f}{\partial x_k}\right|_{x_k=0,\, u_k=0}$ and $B \triangleq \left.\dfrac{\partial f}{\partial u_k}\right|_{x_k=0,\, u_k=0}$), is stabilizable.

(ii) The transition cost functions $h^i(x, u)$ depend on both x and u , and for each $i \in I$ there exist two strictly increasing functions $r^i, s^i \in \mathcal{C}[\mathbb{R}^+, \mathbb{R}^+]$, with $r^i(0) = s^i(0) = 0$, such that $r^i(\|(x,u)\|) \leq h(x,u) \leq s^i(\|(x,u)\|^2)$, $\forall\, x \in X, \forall\, u \in U$, where $(x,u) \triangleq \text{col}\,(x,u)$.

(iii) There exists a compact set $X_0 \subseteq X$, $X_0 \in \mathcal{Z}$, with the property that there exists a control horizon $M \geq 1$ such that there exists a sequence of admissible control vectors $\{u_i \in U, i = k, \ldots, k+M-1\}$ that yield an admissible state trajectory $x_i \in X$, $i = k, \ldots, k+M$ ending in the origin of the state space (i.e., $x_{k+M} = 0$) for any initial state $x_k \in X_0$.

(iv) The optimal FH feedback control functions $\gamma^{i^\circ}_{FH}(x_l, l)$, $l = k, \ldots, k + N^i - 1$, which minimize the i-th cost (4), are $\mathcal{C}^1$ functions with respect to x_l , for any $x_l \in X$ and any finite integer $N^i \geq 1$.

Let us next denote by

$$J^{i^\circ}_{FH}(x_k, N^i, a^i, P^i) \triangleq J^i_{FH}(x_k, u^\circ_{k,k+N^i-1}, N^i, a^i, P^i) = \sum_{l=k}^{k+N^i-1} h^i(x_l^{FH^\circ}, u_l^{FH^\circ}) + h^i_F(x^{FH^\circ}_{k+N^i})$$

the cost corresponding to the i-th optimal N^i-stage trajectory starting from x_k (to simplify the notation, we let $h^i_F(x) \triangleq a^i\|x\|^2_{P^i}$, without any ambiguity whenever a^i and P^i need not be rendered explicit). Then, we have the following

Theorem 1.[6] *Consider a generic $i \in I$. If assumptions (i) to (iv) are verified, there exist a positive scalar $\tilde{a}^i$ and a positive definite symmetric matrix $P^i \in \mathbb{R}^{n\times n}$ such that, for any $N^i \geq M$ and any $a^i \in \mathbb{R}$, $a^i \geq \tilde{a}^i$, the following properties hold: 1) the RH control law asymptotically stabilizes the origin, which is an equilibrium point of the resulting closed-loop system; 2) there exists a positive scalar β^i such that the set $\mathcal{W}(N^i, a^i, P^i) \in \mathcal{Z}$, $\mathcal{W}(N^i, a^i, P^i) \triangleq \{x \in X :\ J^{i^\circ}_{FH}(x, N^i, a^i, P^i) \leq \beta^i\}$, is an invariant subset of*

X_0 and a domain of attraction for the origin, i.e., for any $x_k \in \mathcal{W}(N^i, a^i, P^i)$, the state trajectory generated by the RH regulator remains entirely contained in $\mathcal{W}(N^i, a^i, P^i)$ and converges to the origin.

From Theorem 1 above, it follows that, for any given $i \in I$, a key role is played by the final cost $a^i \|x_{k+N^i}\|^2_{P^i}$. More details on the above theorem can be found in [6].

As is well known, the off–line computation of the RH control functions $\gamma^{i^\circ}_{RH}$ is in general possible only in *approximate way*. The approach here proposed consists in approximating $\gamma^{i^\circ}_{RH}(x_k)$ by a control function of the form $\hat{\gamma}^{i}_{RH}(x_k, w^i)$, where $\hat{\gamma}^{i}_{RH}$ is the input/output mapping of a multilayer feedforward neural network and w^i is a vector of parameters to be tuned. A feedforward network is composed of L layers, and, in the generic layer s, n_s neural units are active. The input/output mapping of the q-th neural unit of the s-th layer is given by $y_q(s) = g\left[\sum_{p=1}^{n_{s-1}} w_{pq}(s) y_p(s-1) + w_{0q}(s)\right]$, $s = 1, \ldots, L$; $q = 1, \ldots, n_s$, where $y_q(s)$ is the output variable of the neural unit, and $w_{pq}(s)$ and $w_{0q}(s)$ are the weight and bias coefficients, respectively. We use the "activation function" $g(x) = \tanh(x)$. All these coefficients are the components of the vector w; the variables $y_q(0)$ are the components of x_k, and the variables $y_q(L)$ are the components of control vector u_k.

As, for any given $i \in I$, we have the possibility of computing (off line) any number of open-loop optimal control sequences $u_k^{FH^{i^\circ}}, \ldots, u_{k+N-1}^{FH^{i^\circ}}$ (see Problem 1) for different vectors $x_k \in X$, then, for every $i \in I$, we design an approximate RH neural control law by determining a vector w^{i° that minimizes a suitable functional measuring the approximation error made on the set X.

The stability properties of the proposed hybrid control scheme will be based on the fact that the control errors generated by the neural control law $\hat{\gamma}^{i}_{RH}(x_k, w^{i^\circ})$ are uniformly bounded on X by a suitable scalar ε^i. This may be obtained by solving the minimax problem stated below. Specifically, the network is the parallel of m single-output neural networks containing a single hidden layer and linear output activation units. Each network generates one of the m components of the control vector. We denote by $\hat{\gamma}^{(\nu^i_j)}_{RHj}(x_k, w^i_j)$ the input-output mapping of the j-th of such networks, where ν^i_j is the number of neural units in the hidden layer and w^i_j is the weight vector, and we denote by $\gamma^{i^\circ}_{RHj}(x_k)$ the j-th component of the vector function $\gamma^{i^\circ}_{RH}$. Now, we can state the following

Problem 3. *Let us consider a given positive scalar ε^i. For each function $\hat{\gamma}^{(\nu^i_j)}_{RHj}(x_k, w^i_j)$, $i \in I$, $j = 1, \ldots, m$, find the numbers $\nu^{i^*}_1, \ldots, \nu^{i^*}_m$ of neural units such that*

$$\min_{w^i_j} \max_{x_k \in X} \left| \gamma^{i^\circ}_{RHj}(x_k) - \hat{\gamma}^{(\nu^i_j)}_{RHj}(x_k, w^i_j) \right| \leq \frac{\varepsilon^i}{\sqrt{m}}, \quad j = 1, \ldots, m \qquad (6)$$

In Section 4, we shall address the computation of the scalars ε^i in the context of stability analysis of the overall hybrid scheme. As to the numbers $\nu^{i^*}_j$, rather a naive trial–and–error procedure for determining them has to be devised: for each i, j, increase ν^i_j until the term on the left–hand side of (6) is less than or equal to $\frac{\varepsilon^i}{\sqrt{m}}$. This procedure stems from the recent results on the approximation properties of neural approximators here briefly reported. Following [7], we assume that each of the optimal control functions $\gamma^{i^\circ}_{RHj}$ to be approximated has a bound to the average of the norm of the frequency vector weighted by its Fourier transform (see (7) below). However, the functions $\gamma^{i^\circ}_{RHj}$ have been defined on the compact set X, not on the space $\mathbb{R}^n$. Then, in order to compute the Fourier transforms, we need "to extend" the functions $\gamma^{i^\circ}_{RHj}(x_k)$ from X to $\mathbb{R}^n$. Toward this end, we define the functions $\overline{\gamma}^{i^\circ}_{RHj} : \mathbb{R}^n \to \mathbb{R}$ that coincide with $\gamma^{i^\circ}_{RHj}(x_k)$ on X. We also define the class of functions

$$G_{c^i_j} \triangleq \left\{ \overline{\gamma}^{i^\circ}_{RHj} \quad \text{such that} \quad \int_{\mathbb{R}^n} |\omega|_X \left|\Gamma^i_j(\omega)\right| \mathrm{d}\omega \leq c^i_j \right\} \qquad (7)$$

where $|\omega|_X \triangleq \max_{x \in X} |\omega \cdot x|$, $\Gamma^i_j(\omega)$ is the Fourier transform of $\overline{\gamma}^{i^\circ}_{RHj}$, and c^i_j is any finite positive scalar. We can now state the following

Theorem 2. *Assume that, in the solution of Problem 2, the first control function $\gamma^{i^\circ}_{RH}(x_k) = \gamma^{i^\circ}_{FH}(x_k, 0)$ (see (5)) is unique, and that $\overline{\gamma}^{i^\circ}_{RHj} \in G_{\tilde{c}^i_j}$ for some finite positive scalar $\tilde{c}^i_j$, for every j with $1 \leq j \leq m$. Then, for every j with $1 \leq j \leq m$, for every probability measure σ, and for every $\nu^i_j \geq 1$, there exist a weight vector w^i_j (i.e., a neural control function $\hat{\gamma}^{(\nu^i_j)}_{RHj}(x_k, w^i_j)$) and a positive scalar $c^{i'}_j$ such that*

$$\int_X \left| \gamma^{i^\circ}_{RHj}(x_k) - \hat{\gamma}^{(\nu^i_j)}_{RHj}(x_k, w^i_j) \right|^2 \sigma\left[\mathrm{d}(x_k)\right] \leq \frac{c^{i'}_j}{\nu^i_j}$$

where $c^{i'}_j = (2\tilde{c}^i_j)^2$. Moreover, there exists a positive scalar k^i_j such that

$$\max_{x_k \in X} \left| \gamma^{i^\circ}_{RHj}(x_k) - \hat{\gamma}^{(\nu^i_j)}_{RHj}(x_k, w^i_j) \right|^2 \leq k^i_j \frac{c^{i'}_j}{\nu^i_j}$$

Theorem 2 derives straight from two theorems by Barron [7],[8]. It states that, for any control function $\gamma^{i^\circ}_{RHj}(x_k)$, the number of parameters required to achieve an L_2 or an L_∞ approximation error of order

$O(1/\nu_j^i)$ is $O(\nu_j^i n)$, which grows linearly with the dimension n of the state vector. For a further discussion of this property of feedforward neural approximators, in comparison with traditional linear approximators and other classes of nonlinear ones, see for instance [9].

Summing up, at the continuous level, the following set of RH approximate neural regulators is available:

$$\hat{\Gamma} \triangleq \left\{ \hat{\gamma}_{RH}^i : X \mapsto \mathbb{R}^m, \, i \in I \right\} \tag{8}$$

where, for the sake of notational simplicity, the superscripts (ν_j^i) have been dropped (this simplified notation will be used in the rest of the paper).

The discrete–event part of the considered class of hybrid systems represents the different operating conditions of the continuous plant by associating with each of such conditions a different control law (among the class of control functions $\hat{\Gamma}$). This means that each discrete state models the continuous plant working under the application of a single $\hat{\gamma}_{RH}^i \in \hat{\Gamma}$.

Owing to the description of the continuous subsystem, the set $\mathcal{B}$ includes difference equations characterized by different control laws to be applied to the plant. Then, each node of graph $\mathcal{G}$ is labelled with invariant equations of the form

$$\left[x_{k+1} = f[x_k, \hat{\gamma}_{RH}^i(x_k, w^i)] \right], \quad [x_0 = \overline{x}], \quad [k_0 = \overline{k}]$$

Each node of the network is thus representing the system working under the application of a specific RH regulator. As regards the *guard equations* associated with arcs in $\mathcal{G}$, two main kinds of guard equations can be considered. The first kind is relevant to the definition of specific conditions on the continuous state that make the process require a different control action (e.g., the current operation mode has to be changed in order to meet some specific requirement or as an effect of some unexpected event). Such guard equations are clearly strictly dependent on the specific application. A second kind of guard equations regard the introduction of suitable constraints guaranteeing stability properties of the overall hybrid control scheme.

4 Stability analysis

In this section, the stability properties of the hybrid control scheme will be investigated (due to space limitations, the proofs of the following stability results are not reported and the reader is referred to [12]). Specifically, the case in which the optimal RH control laws are used in the continuous level will be first addressed in order to establish the main stability property in this case. Subsequently, the stability analysis will be generalized to the case in which neural approximate RH control laws are adopted at the continuous level.

Firstly, by recalling Theorem 1 stated in Section 3, we are able to introduce the compact sets $\mathcal{X}_i \triangleq \mathcal{W}(N^i, a^i, P^i)$, $i \in I$ Owing to Theorem 1, the sets $\mathcal{X}_i$ are domains of attraction for the origin of the state space under the action of the i–th optimal control law γ^{i°. Moreover, they are also controlled left–invariant sets and the function $V_i(x_k) \triangleq J_{FH}^{i^\circ}(x_k, N^i, a^i, P^i)$ is the corresponding Lyapunov function (see [6]). Now, let us introduce the set $\mathcal{I}(x_k) \triangleq \{ i \in I : x_k \in \mathcal{X}_i \}$ characterizing the subset of indexes j such that the corresponding set $\mathcal{X}_j$ contains the current state x_k. Moreover, let

$$\delta V_i(x) \triangleq V_i(x) - V_i(f[x, \gamma^{i^\circ}(x)]), \quad \forall x \in \mathcal{X}_i, \forall i \in I$$

$$\delta V_i \triangleq \min_{x \in \mathcal{X}_i} \delta V_i(x), \quad \forall i \in I \tag{9}$$

For the reader's convenience, some notations used in [2] are introduced. Let us consider a *switching sequence* indexed by an initial state x_0 and defined as:

$$\Xi \triangleq \{x_0; (i_0, t_0), (i_1, t_1), \ldots, (i_r, t_r), \ldots; \\ i_r \in I, t_r \in \mathcal{T}, r \in \mathbb{N}\} \tag{10}$$

where the pair (i_r, t_r) means that $x_{k+1} = f\left[x_k, \gamma^{i_r^\circ}(x_k)\right]$, $\forall k \in [t_r, t_{r+1})$ Thus, the switching sequence, which can be finite or not, denotes the indexes of the switching times and associates such times with the indexes of the activated control actions. With reference to Ξ, it is now necessary to identify the indexes of the switching times at which a specific control strategy γ^{i°, $i \in I$ is switched on and off. This is done by defining the set $\Xi|i \triangleq \{t_{r_1}, t_{r_1+1}, t_{r_2}, t_{r_2+1}, \ldots, t_{r_m}, t_{r_m+1}, \ldots : i_{r_m} = i \text{ in } \Xi, m \in \mathbb{N}\}$

For the sake of notational simplicity, we express the switching sequence $\Xi|i$ associated with the i-th control strategy, as $\Xi_i \triangleq \{t_0^i, t_1^i, \cdots\}$, where $t_{2r}^i \in \Xi|i$, $r \in \mathbb{N}$, are the time instants at which the control strategy γ^{i° is switched on, and time instants $t_{2r+1}^i \in \Xi|i, r \in \mathbb{N}$, correspond to the switchings off of the same control function. To prove the stability result the following constraint is needed:

Constraint A. *Consider the switching time t at which a generic control law $\gamma^{i^\circ}, i \in I$ is switched off and a new control law $\gamma^{j^\circ}, j \in I$ is switched on; index j should satisfy the constraint*

$$j \in \mathcal{J}(x_t) \triangleq \{j \in \mathcal{I}(x_t) : V_j(x_t) - V_i(x_t) \le \delta V_i\} \tag{11}$$

It is now possible to prove the following stability result for the overall hybrid control scheme when the optimal RH control laws γ^{i° are used at the continuous level.

Proposition 1 [12]. *Assume that Theorem 1 holds true $\forall i \in I$. Moreover, assume that Constraint A is fulfilled for each trajectory $x_\Xi(\cdot)$ of the hybrid system determined by the switching sequence (10) and $\forall i \in I$. Then the equilibrium $x = 0$ of the hybrid system is stable.*

Remark 1. The above constraints are *guard equations* associated with each arc of the network representing the switching between two different control modes. It is to be noted here, that constraint (11) is actually an atomic formula in the defined propositional logic thanks to the expression of invariant sets $\mathcal{X}_i$ in Section 3.

Now, we are interested in proving stability results for the case where the class of neural approximate RH control laws $\hat{\Gamma} = \left\{ \hat{\gamma}^i_{RH} : X \mapsto \mathbb{R}^m, \, i \in I \right\}$ is used at the continuous level. Again, for notational simplicity, in the following we drop the index "RH" from $\hat{\gamma}^i_{RH}$. Let

$$\Delta\hat{V}_i(x) \stackrel{\triangle}{=} V_i(x) - V_i(f[x, \hat{\gamma}^i(x)]) \,, \quad \forall\, x \in \mathcal{X}_i, \, \forall\, i \in I$$

denote the variation in the Lyapunov function V_i under the action of the neural approximate control law $\hat{\gamma}^i$. Then, we have the following

Lemma 2 [12]. *Assume that Theorem 1 holds true* $\forall\, i \in I$. *Then,* $\forall\, i \in I$ *there exists a positive scalar* $\varepsilon^i \in \mathbb{R}$, $\varepsilon^i > 0$ *and an approximate neural control law* $\hat{\gamma}^i$ *solving Problem 3 such that*

$$\Delta\hat{V}_i(x) = V_i(x) - V_i(f[x, \hat{\gamma}^i(x)]) > 0, \, \forall\, x \in \mathcal{X}_i \quad (12)$$

Moreover, Constraint A has to be modified as follows:

Constraint A′. *Consider the switching time t at which a generic neural RH control law* $\hat{\gamma}^i, i \in I$ *is switched off and a new control law* $\hat{\gamma}^j, j \in I$ *is switched on; index j should satisfy the constraint*

$$j \in \hat{\mathcal{J}}(x_t) \stackrel{\triangle}{=} \{ j \in \mathcal{I}(x_t) : \, V_i(x_t) - V_j(x_t) \geq \delta V_i \} \quad (13)$$

Clearly, the switching sequence (10) now refers to the neural approximate RH control laws. In this respect, the following lemma can be proved.

Lemma 3 [12]. *Assume that Theorem 1 holds true* $\forall\, i \in I$. *Moreover, consider a class* $\hat{\Gamma}$ *of neural approximate RH control laws solving Problem 3 in such a way that inequality (12) in Lemma 2 is satisfied. Assume also that Constraint A′ is fulfilled for each trajectory* $x_\Xi(\cdot)$ *of the hybrid system determined by the switching sequence (10) and* $\forall\, i \in I$. *Then, for each trajectory* $x_\Xi(\cdot)$ *of the hybrid system determined by the switching sequence (10) and* $\forall\, i \in I$, *the functions* $V_i(x_k) \stackrel{\triangle}{=} J^{i\circ}_{FH}(x_k, N^i, a^i, P^i)$ *are weak Lyapunov–like functions for* f *and* $x_\Xi(\cdot)$ *over* Ξ_i.

Finally, using Lemma 3 above, it is possible to prove the following stability result for the overall hybrid control scheme.

Proposition 2 [12]. *Assume that Theorem 1 holds true* $\forall\, i \in I$. *Moreover, consider a class* $\hat{\Gamma}$ *of neural approximate RH control laws solving Problem 3 in such a way that inequality (12) in Lemma 2 is satisfied. Assume also that Constraint A′ is fulfilled for each trajectory* $x_\Xi(\cdot)$ *of the hybrid system determined by the switching sequence (10) and* $\forall\, i \in I$. *Then the equilibrium* $x = 0$ *of the hybrid system is stable.*

Remark 2. It is important to stress the constructive nature of Problem 3 and Lemma 3. Specifically, numerical procedures analogous to the ones presented in [6] can be devised to design off–line the class $\hat{\Gamma}$ of neural RH regulators guaranteeing the stability of the overall hybrid control scheme owing to Proposition 2 above.

As a final comment about the stability analysis developed so far, it is worth noting the meaning of the guard equations related to Constraints A and A′. Actually, the sets of indexes $\mathcal{J}(x_k)$ and $\hat{\mathcal{J}}(x_k)$ include the RH control functions for which, in connection with the present state x_k, the switching does not perturb the hybrid state trajectory too much, thus allowing the definition of the weak Lyapunov function considered in Lemma 3.

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