Controllability of Quantum Mechanical Systems with Continuous Spectra

T. J. Tarn † J. W. Clark ‡ D. G. Lucarelli †
†*Department of Systems Science and Mathematics, Washington University, St. Louis, Missouri 63130*
‡*Department of Physics, Washington University, St. Louis, Missouri 63130*

Abstract

The property of controllability of quantum systems is revisited. In the case of a system having this property, couplings to external agents are available whose adjustment can guide the state to any chosen target on a suitably defined manifold (or arbitrarily close to any such target) at any chosen time. With due consideration to unbounded operators corresponding to physical observables possessing continuous spectra, sufficient conditions for controllability based on Lie-algebraic arguments are obtained. The results are not limited to finite-dimensional systems, nor to infinite-dimensional systems with discrete spectra. The applicability of the results to systems with both bound and scattering states is demonstrated for the case of the one-dimensional Pöschl–Teller potential. Attention is also directed to transitivity of the pertinent Lie algebra of a system without drift as a necessary and sufficient condition for controllability.

1 Introduction

Over the last few decades, a silent revolution has been taking place. The exponential growth in microelectronic processing power has been achieved by astonishing decreases in the size of integrated circuits. These circuits, which integrate electrical, mechanical, and sometimes optical devices, have evolved out of the silicon revolution, which has replaced bulky, complex, costly electronics with compact, affordable, high-performance microsystems. Future designs of such microsystems are expected to enable combinations of silicon chips to sense, "think," act, and communicate – in essence, to become intelligent machines.

The structures of current microsystems are approaching fundamental limits of classical description, and the next generation of devices may well display exotic properties due to quantum fluctuations and other quantum effects. A new research field is developing with the goal of understanding the basic physics associated with such quantum structures, exploring their controllability, and engineering a new species of useful devices.

Interest in the interplay between quantum mechanics, information, and computation began with recognition of the theoretical possibility of quantum computation and dates back to ideas of Benioff, Feynman, and Deutsch [1]–[3]. Recently there has been renewed and intense research into the feasibility and the unique powers of quantum computation. The latter effort has produced a series of important results, most prominently Shor's algorithm [4] for finding prime factors in polynomial time rather than exponential time and Grover's algorithm for searching an unsorted database [5].

In this paper we demonstrate the applicability of an earlier controllability result of two of us [6] to quantum systems characterized by continuous (or partially discrete, partially continuous) spectra. While the efficacy of the original result is well established for systems with discrete spectra, its relevance to the continuous case has been questioned [7]. However, in a recent review article [8] the authors recognize that the theorem may indeed be valid for the class of systems considered here.

2 Problem Formulation

We consider a quantum system whose state $\psi(t)$ evolves from $\psi(t=0)=\psi_0$ according to the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)=\left(H_0+\sum_{l=1}^{r}u_l(t)H_l\right)\psi(t)\,, \qquad (1)$$

where H_0, H_1, ..., H_r are linear, Hermitian operators in the state space $\mathcal{H}$ of the system and the u_l are real functions of t. Imposing $||\psi(t)||=1$, the system evolves on the unit sphere $S_{\mathcal{H}}$ in $\mathcal{H}$.

Eq. (1) provides the basis for a rather general control problem. The operator H_0 is the Hamiltonian for the free evolution of the quantum system to be controlled (*e.g.* a molecule), while the terms $u_l(t)H_l$, $l=1,...,r$, represent its interactions with certain external (classical) fields, with controls $u_l(t)$. Through the adjustment of these control functions, the scientist seeks to guide the time development of the state $\psi(t)$ so as to attain a specified objective. We may make the immediate observation that the control $u(t)=\{u_l(t)\}$ needed to realize a posed behavior of $\psi(t)$ will depend on $\psi(t)$ and the problem becomes nonlinear.

To address the existence issue systematically, we need precise definitions of *reachable sets* and *controllability*. The state $\psi(t) \in S_{\mathcal{H}}$ of the controlled system (1) evolves from ψ_o on a set M which forms a differentiable manifold, finite or infinite-dimensional. (*N.B.* $S_{\mathcal{H}}$ may itself be endowed with manifold structure.)

Def. Given ψ_o , $\psi_f \in M$, we say that the state ψ_f is *reachable* from ψ_o at time t_f if there exists an admissible control $u(t)$ such that $\psi(t = t_f|u, \psi_o) = \psi_f$. The set of states reachable from ψ_o at time t_f is denoted $R_{t_f}(\psi_o)$. The set of states reachable from ψ_o at some positive time is $R(\psi_o) = \cup_{s>0} R_s(\psi_o)$.

Def. The control system is said to be *strongly completely controllable* if $R_t(\psi_o) = M$ holds for all times $t > 0$ and all $\psi_o \in M$. The system is *completely controllable* if $R(\psi_o) = M$ holds for all $\psi_o \in M$.

In down-to-earth terms, strongly complete controllability means that, using the available controls, you can get there from here at *any* later time you chose, while complete controllability means you can get there from here at *some* later time.

More specialized definitions of controllability come into play when the initial and target states are restricted to a subset of M. In particular, if we are concerned with states in an analytic domain.

Def. Let ψ_o be an *analytic vector* belonging to an *analytic domain* $\mathcal{D}_\omega$ that is dense in the state space. Then the system is *strongly analytically controllable* [respectively, *analytically controllable*] on $M \subseteq S_{\mathcal{H}}$ if $R_t(\psi_o) = M \cap \mathcal{D}_\omega$ holds for all $t > 0$ and all $\psi_o \in M \cap \mathcal{D}_\omega$ [respectively, if $R(\psi_o) = M \cap \mathcal{D}_\omega$ holds for all $\psi_o \in M \cap \mathcal{D}_\omega$].

Next, we narrow the problem somewhat, by assuming that the coupling operators H_1, ..., H_r are *time-independent* and the corresponding control amplitudes u_l are *piecewise constant* functions of t.

One would like to adapt to the quantum problem classical controllability results for bilinear control systems. However, these results deal with finite-dimensional systems and so there are nontrivial complications obstructing the extension.

The difficulties faced are twofold: First, in quantum mechanics, the state space $\mathcal{H}$ and the unit sphere $S_{\mathcal{H}}$ are typically *infinite-dimensional*. (The manifold M on which the system evolves from ψ_o can also be infinite-dimensional.) Secondly, the operators H_l are often *unbounded* operators (as for example x, $-i\mathrm{d}/\mathrm{d}x$, $-\mathrm{d}^2/\mathrm{d}x^2$).

These domain problems can be overcome if one assumes the existence of an *analytic domain* $\mathcal{D}_\omega$, a set of state vectors having these three properties: (i) it is dense in the Hilbert space $\mathcal{H}$, (ii) it is invariant under the given operators H_l (with $l = 0, \ldots, r$) and (iii) on it, the solution of the Schrödinger equation (1) can be expressed globally in exponential form. In simple terms, the availability of an analytic domain, in the context of piecewise-constant controls $u_l(t)$, allows one to write the evolution operator $\mathcal{U}_t$ corresponding to a Hamiltonian H in the familiar way, $\mathcal{U}_t = \mathrm{e}^{-iHt/\hbar}$.

The existence of $\mathcal{D}_\omega$ is guaranteed by *Nelson's Theorem* [9], if we choose to impose the associated conditions, which, as will now be revealed, are not especially strenuous. (From this point till the end of the section it is more convenient to work with the *skew*-Hermitian operators $H_l' = H_l/i\hbar$ (and drop the prime).)

Def. An element ω of $\mathcal{H}$ is called an *analytic vector of the operator* A in $\mathcal{H}$ if the series expansion of $(\exp sA)\omega$ in the real parameter s has a positive radius of absolute convergence,

$$\sum_{n=0}^{\infty} ||A^n \omega|| \frac{s^n}{n!} < \infty \, .$$

Note: If A is bounded, all vectors of $\mathcal{H}$ are trivially analytic vectors.

Def. An element ω of $\mathcal{H}$ is an *analytic vector for the Lie algebra* $\mathcal{L}$ if, for some $s > 0$ and some linear basis $\{H_{(1)}, ..., H_{(d)}\}$ of $\mathcal{L}$, the series

$$\sum_{n=0}^{\infty} \sum_{\substack{0 \le n_1, n_2, \cdots, n_d \le \infty \\ n_1 + \cdots + n_d = n}} \sum_{1 \le i_1 < \cdots < i_d \le d} ||H_{(i_1)}^{n_1} \cdots H_{(i_d)}^{n_d} \omega|| \frac{s^n}{n!}$$

is absolutely convergent.

Under certain conditions, it is possible to build – with analytic vectors – an analytic domain $\mathcal{D}_\omega$ that is invariant under the elements of the Lie algebra $\mathcal{A}$ of the control problem. One set of criteria is furnished by *Nelson's Theorem.*

Theorem (Nelson). Let $\mathcal{L}$ be a Lie algebra of skew-Hermitian operators in a Hilbert space $\mathcal{H}$, the operator basis $\{H_{(1)}, \cdots, H_{(d)}\}$, $d < \infty$, of $\mathcal{L}$ having a common invariant dense domain. If the operator $T = H_{(1)}^2 + \cdots + H_{(d)}^2$ is essentially self-adjoint, then there exists a unitary group Γ on $\mathcal{H}$ with Lie algebra $\mathcal{L}$. Let $\bar{T}$ denote the unique self-adjoint extension of T. Then it furthermore follows that the analytic vectors of $\bar{T}$ (i) are analytic vectors for the whole lie algebra $\mathcal{L}$ and (ii) form a set invariant under Γ and dense in $\mathcal{H}$.

In essence, this theorem establishes the existence, under reasonable conditions, of a dense domain $\mathcal{D}_\omega$ of analytic vectors that affords a foothold for the extension of controllability results for finite-dimensional spaces to the quantum problem. The set of analytic vectors of $\bar{T}$ provides such a subspace $\mathcal{D}_\omega$ of $\mathcal{H}$. With the identification

$$\mathcal{L} = \mathcal{A} \doteq \{H_0, H_1, ..., H_r\}_{\rm LA} ,$$

the elements of $\mathcal{A}$ become densely defined vector fields on $\mathcal{D}_\omega \cap M$, where M is a finite-dimensional manifold on which the system point evolves with time. We take $\dim M \cap \mathcal{D}_\omega = d < \infty$, and for a starting point ψ_o we may choose M to be the closure of the set

$$\{ e^{s_0 H_0} e^{s_1 H_1} \cdots e^{s_r H_r} \psi_o | s_k \in \mathbb{R}^1 , k = 0, 1, ..., r \} .$$

Assuming the existence of an analytic domain $\mathcal{D}_\omega$ (which in general need not entail satisfaction of the requirements of Nelson's Theorem), Huang, Tarn, and Clark (HTC) were able to derive sufficient conditions for controllability, characterizing the reachable sets $R_t(\psi_o)$ and $R(\psi_o)$ in terms of the three Lie algebras

$$\mathcal{A} = \{H_0, H_1, ..., H_r\}_{\rm LA} , \quad \mathcal{B} = \{H_1, H_2, ..., H_r\}_{\rm LA} ,$$

$$\mathcal{C} = \{{\rm ad}^j_{H_0} H_l | l = 1, ..., r; j = 0, 1, ...\}_{\rm LA} ,$$

where ${\rm ad}^j_X Y = [X, {\rm ad}^{j-1}_X Y]$, $j \geq 1$, with ${\rm ad}^0_X Y = Y$.

Of special significance are the dimensionalities of the tangent subspaces $\mathcal{A}(\phi)$, $\mathcal{B}(\phi)$, and $\mathcal{C}(\phi)$ of $M \cap \mathcal{D}_\omega$ at $\phi \in M \cap \mathcal{D}_\omega$ defined by the vector fields associated with these Lie algebras.

The following key result appears as a corollary to the HTC theorem proven in [6]. (Its application is less cumbersome than that of the theorem itself.)

HTC Corollary. Let $\mathcal{C} = \{{\rm ad}^j_{H_0} H_l | l = 1, ..., r; j = 0, 1, ...\}_{\rm LA}$ be the ideal in the Lie Algebra $\mathcal{A} = \{H_0, H_1, ..., H_r\}_{\rm LA}$ generated by $H_1, ..., H_r$. The system, with piecewise constant controls, is *strongly analytically controllable* on M provided that (i) $[\mathcal{C}, \mathcal{B}] \subset \mathcal{B}$ and (ii) $\dim \mathcal{C}(\phi) = d < \infty$ for all $\phi \in M \cap \mathcal{D}_\omega$.

Condition (i) of the HTC Corollary is that the Lie algebra $\mathcal{B}$ is an ideal in $\mathcal{C}$, while condition (ii) requires that the tangent space associated with $\mathcal{C}$ at ϕ have constant, finite dimension d for all points ϕ on the intersection of $\mathcal{D}_\omega$ and M.

The HTC results have these practical implications: Assuming the delineated conditions are met, we can always control the system so that the state $\psi(t)$, starting at any point $\psi_o \in M \cap \mathcal{D}_\omega$, arrives arbitrarily close to any desired point in the (finite-dimensional) manifold M after any desired time interval t. As a consequence, the expectation value of an observable quantity can be made to approach arbitrarily closely the expectation value of that quantity for any prescribed state vector in $M \subset \mathcal{H}$, at any time $t > 0$.

On the other hand, *all* we might desire cannot be fulfilled:

HTC No-Go Theorem. If the set $\{H_0, H_1, ..., H_r\}$ generates a d-dimensional Lie algebra $\mathcal{A}$ which admits an analytic domain $\mathcal{D}_\omega$, the quantum system is *not* analytically controllable on the *full* unit sphere $S_{\mathcal{H}}$ if d is finite.

3 The Pöschl–Teller Potential

The one-dimensional Pöschl–Teller potential $V(\rho) = -V_0/\cosh^2\rho$, where V_0 is a positive constant, is one of the few exactly solvable potentials in quantum mechanics. The spectrum of a particle moving in this potential is characterized by a finite number of bound states, plus a continuum of scattering states. There exist two well-developed approaches to the scattering problem for this potential [10]-[12], corresponding to two different realizations of the dynamical symmetry group $SU(1,1)$. We refer to these as the *potential group* approach and the *scattering group* approach. Here we shall explore both approaches within a control-theoretic context and contrast the physical interpretations of the two treatments. The purpose of this exercise is to show that the original HTC theorem does indeed apply to systems having a non-compact dynamical symmetry group, despite some objections voiced in the literature.

Consider the dynamical algebra $su(1,1)$. The generators J_x, J_y, J_z satisfy the commutation relations

$$[J_x, J_y] = -iJ_z , \quad [J_y, J_z] = iJ_x , \quad [J_z, J_x] = iJ_y .$$

The $su(1,1)$ transformations leave the hyperboloid $x^2 + y^2 - z^2 = 1$ invariant, and a realization of the commutation relations on this hyperboloid yields the differential operators

$$J_x = -i\left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right) , \quad J_y = i\left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) ,$$

$$J_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) .$$

The Casimir invariant is given by

$$C = -J_x^2 - J_y^2 + J_z^2 .$$

We define the ladder operators

$$J_+ = J_x + iJ_y , \quad J_- = J_x - iJ_y$$

and write C in terms of the $J_\pm$, J_z generators as

$$C = -J_+J_- + J_z^2 - J_z .$$

Introducing the hyperbolic coordinates

$$x = r\cosh\rho\cos\phi , \quad y = r\cosh\rho\sin\phi , \quad z = r\sinh\rho$$

and performing a similarity transformation $U = \cosh^{1/2}\rho$, the generators can be expressed as $J_z = -i\partial/\partial\phi$ and

$$J_\pm = e^{\pm i\phi}\left(\mp\frac{\partial}{\partial\rho} - i\tanh\rho\left(\pm\frac{1}{2} - i\frac{\partial}{\partial\phi}\right)\right) ,$$

with Casimir invariant

$$C = \frac{\partial^2}{\partial\rho^2} - \frac{\partial^2/\partial\phi^2 + 1/4}{\cosh^2\rho} - \frac{1}{4}\,.$$

To solve the eigenvalue problem

$$C|j,m\rangle = j(j+1)|j,m\rangle\,,$$
$$J_z|j,m\rangle = m|j,m\rangle\,,$$

we seek solutions of the form

$$\chi(\rho,\phi) = \langle\rho,\phi|j,m\rangle = u_j^m(\rho)e^{im\phi}\,.$$

This separable ansatz easily solves the eigenvalue problem for J_z, yielding the differential equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{m^2-1/4}{\cosh^2\rho}u_j^m(\rho)\right) = -\left(j(j+1)+\frac{1}{4}\right)u_j^m(\rho)$$

for the factor $u_j^m(\rho)$ in the wave function $\chi(\rho,\phi)$. This equation is just the Schrödinger equation for a particle in one dimension with coordinate ρ, subject to a Pöschl–Teller potential with strength proportional to $m^2 - 1/4$. It ensues immediately that the Hamiltonian governing the autonomous dynamics of the system is trivially related to the Casimir invariant:

$$H_0 = -(C + 1/4)\,.$$

We now construct the control system

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \Big[-(C+1/4) + u_1(t)(J_+ - J_-) + u_2(t)i(J_+ + J_-)\Big]\psi(t)\,,$$
$$\psi(0) = \psi_0\,,$$

where for generality the quantities $\psi(t)$ and ψ_0 – as in Eq. (1) – are to be interpreted as abstract state vectors rather than wave functions. A dense set of analytic vectors can be constructed for $SU(1,1)$ representations [13]. The controllability analysis then rests on the properties of the three Lie algebras

$$\mathcal{A} = \{-(C+1/4), J_+ - J_-, i(J_+ + J_-)\}_{\mathrm{LA}}\,,$$
$$\mathcal{B} = \{J_+ - J_-, i(J_+ + J_-)\}_{\mathrm{LA}}\,,$$
$$\mathcal{C} = \{\mathrm{ad}^j_{C+1/4}H_i | i = 1,\, j = 0,1,2\ldots\}_{\mathrm{LA}}\,,$$

where $H_1 = J_+ - J_-$ and $H_2 = i(J_+ + J_-)$. Since C is a Casimir invariant, it commutes with the generators $J_\pm$. Therefore $\mathrm{ad}^j_{C+1/4}J_\pm = 0$ for $j > 0$, and the ideal generated by $\mathcal{C}$ in $\mathcal{A}$ coincides with $\mathcal{B}$. Clearly $[\mathcal{C},\mathcal{B}] \subset \mathcal{B}$ and we may conclude from the HTC corollary that the system is strongly analytically controllable.

This example reveals a more general property of quantum mechanical control systems. That is, if the free or autonomous evolution is driven by a Casimir invariant (a dynamical symmetry), and we can physically realize control Hamiltonians in terms of the group generators, the system is guaranteed to be strongly analytically controllable. This result agrees with one's intuition, since for such a system it is possible to solve for its complete spectrum by the application of a combination of creation and annhilation operators to the ground state. The controllability result confirms that such a combination exists and provides a framework in which "classical" control results can be extended to the quantum realm.

To proceed further, we must choose a representation of the group generators. The unitary nonexceptional representations of $SU(1,1)$ may be divided into a continuous series $\widehat{C}$ and a discrete series $\widehat{D}$. The Bargmann index k and the $SO(2)$ content m follow:

$$\widehat{C} : k = \frac{1}{2} + i\kappa, \begin{cases} \text{in } C^0_{1/4-\kappa^2} : \kappa \geq 0,\ m = 0, \pm 1, \pm 2, \ldots \\ \text{in } C^{1/2}_{1/4-\kappa^2} : \kappa > 0,\ m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots \end{cases}$$

$$\widehat{D} : k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \begin{cases} \text{in } D^+_k : m = k, k+1, k+2, \ldots \\ \text{in } D^-_k : m = -k, -k-1, \ldots\,. \end{cases}$$

The scattering states of the potential correspond to the continuous representations $\widehat{C}$, while the bound states correspond to the discrete representations $\widehat{D}$.

It may be observed that in the above *potential group* approach, the presence of a Casimir operator as the autonomous Hamiltonian, while ensuring controllability, constrains the system to evolve on a submanifold, such that the accessible scattering states of the system correspond to a fixed energy but different potential strengths.

This unusual situation motivates a shift to the alternative *scattering group* approach developed in [10]-[12]. We reformulate the control problem so that the system evolves on a manifold that gives access to scattering states of different energies, all corresponding to the same fixed potential strength. The essential difference between the two treatments lies in the realization adopted for $su(1,1)$ and the choice of basis, which is specified by simultaneous diagonalization of the Casimir of the group and a non-compact generator.

Consider the $su(1,1)$ algebra written in a basis denoted (I_x, I_y, I_z). The commutation relations now read

$$[I_x, I_y] = iI_z\,, \quad [I_y, I_z] = iI_x\,, \quad [I_z, I_x] = -iI_y\,.$$

Realizing this algebra on a hyperboloid oriented along the y-axis, we are led to the differential operators

$$I_x = i\left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)\,, \quad I_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\,,$$

$$I_z = -i\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)\,.$$

The Casimir operator associated with this set of generators is

$$C = I_y^2 - I_x^2 - I_z^2 \, .$$

Next we simultaneously diagonalize the non-compact generator I_z and the Casimir C,

$$C|l,k\rangle = l(l+1)|l,k\rangle$$
$$I_z|l,k\rangle = k|l,k\rangle \, .$$

Note that since I_z is non-compact, k is a continuous variable. Again let us seek separable solutions of the form

$$\zeta(\rho,\phi) = \langle \rho,\phi|l,k\rangle = v_l^k(\rho)e^{ik\phi} \, .$$

After changing to hyperbolic coordinates and performing the similarity transformation $U = \cosh^{1/2}\rho$, we are again led to an equation for the ρ dependence of the wave function that is identical with the Schrödinger equation for a particle moving in one dimension in a Pöschl–Teller well:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{l(l+1)}{\cosh^2\rho} v_l^k(\rho) \right) = k^2 v_l^k(\rho) \, .$$

The free Hamiltonian is identified as $H_0 = aI_z^2$, where a is a positive constant. It has been argued [14] that raising and lowering operators can be defined by $I_\pm = -I_y \pm I_x$. These operators satisfy the usual commutation relations for raising and lowering operators, with the insertion of an imaginary coefficient:

$$[I_z, I_+] = iI_+ \, , \quad [I_z, I_-] = iI_- \, .$$

Interpreting the operators I_+ and I_- as control vector fields and computing the Lie algebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$, we arrive at a system with the same algebraic structure as in our first example. Accordingly, the equivalence $\mathcal{C} = \mathcal{B}$ continues to hold. Thus the HTC corollary once again implies that the system is strongly analytically controllable.

4 Transitivity and Universality

Finally, let us consider a scenario which is frequently encountered in quantum information processing, where one sets the Hamiltonian governing the free dynamics to zero, i.e. $H_0 = 0$, and the Hilbert space is taken to be finite-dimensional. Finite spin systems provide the most concrete example. Our quantum control system reduces to

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \left(\sum_{l=1}^{r} u_l(t) H_l \right) \psi(t) \, , \qquad (2)$$

with $\psi(0) = \psi_0$. A Lie algebra $\mathcal{G}$ is said to be *transitive* on a set E if its corresponding Lie group G is *transitive* in the sense that $Gx = E$ for all $x \neq 0$. For a finite-dimensional bilinear system evolving without drift on $M = \mathbb{R}^n - \{0\}$, transitivity of the Lie algebra generated by the vector fields H_i in Eq. (2) is a necessary and sufficient condition for controllability [15]. The deciding issue of controllability for arbitrary members of this class of systems then becomes equivalent to the problem of characterizing all Lie groups whose representation is transitive on $\mathbb{R}^n - \{0\}$. This classification problem has been treated in [15]–[16]. Under the identification $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$, transitivity of the H_i is equivalent to controllability for finite-dimensional quantum systems. Moreover, any transitive Lie algebra may be generated under commutation by two suitably chosen matrices M_1 and M_2 [17]. The set of pairs of matrices (M_1, M_2) which generate a transitive Lie algebra form an open dense set in the set of pairs of matrices.

In the quantum computation literature, a set of unitary transformations is called *universal* if any desired unitary transformation can be achieved by applications of transformations of the set. But in effect this is just the property of controllability, and the aforementioned results on the denseness of matrices generating a transitive Lie algebra can be translated into the language of quantum computation as the dictum "almost every quantum logic gate is universal" [18]–[19].

Returning to the infinite-dimensional case, but with $H_0 = 0$, the conditions for controllability are in a sense less demanding, since we do not require that the system reach all of $S_{\mathcal{H}}$ but only the constraint manifold characterized by $M \cap D_\omega$. In our framework we still require that the Lie algebra $\mathcal{A}$ satisfies Nelson's Theorem. A necessary condition for this to hold is that the Lie algebra be finite-dimensional. Consequently, for the quantum system without drift, the system is strongly analytically controllable if and only if $\mathcal{A}$ is finite-dimensional.

5 Conclusions and Future Work

These examples show that we can apply the HTC theorem (or its corollary) to systems with continuous spectra and, in principle, exercise strong analytic control – as defined in Sec. 2 – over a one-dimensional quantum system whose autonomous motion is governed by a Pöschl–Teller potential. We note, however, that within the framework of this theorem, the manifold on which the system can be controlled depends on the choice of basis and the representation of the group generators that determine the dynamics of the system.

If one wishes to provide for transitions between the discrete and continuum, a unified approach in which the bound state group $SU(2)$ and the scattering state group $SU(1,1)$ are embedded in a larger group, say $Sp(4,\mathbb{R})$ may be appropriate. The generators of the group $Sp(4,\mathbb{R})$ can connect all states, bound and scattering, in the same potential, as well as all states with the same energy but corresponding to potentials with different strengths. This more ambitious prospect is left for future exploration.

Research support was provided in part by the U. S. National Science Foundation under Grant No. PHY–99–00713 (JWC) and BES–98–00829 (TJT).

References

[1] P. A. Benioff, J. Stat. Phys. **22**, 563 (1980); Phys. Rev. Lett. **48**, 1581 (1982); J. Stat. Phys. **29**, 515 (1982); Ann. N. Y. Acad. Sci. **480**, 475 (1986).

[2] R. P. Feynman, Opt. News **11**, 11 (1985); Found. Phys. **16**, 507 (1986); Int. J. Theor. Phys. **21**, 467 (1982).

[3] D. Deutsch. *Proc. Roy. Soc. London A* **400**, 97 (1985).

[4] P. W. Shor, in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science, Sante Fe., NM, 1994*, edited by Shafi Godwasser (IEEE Computer Society Press, Los Alamitos, CA, 1994) p. 124.

[5] L. Grover, in *Proceedings of the 28th Annual ACM Symposium on the Theory of Computation* (ACM Press, New York, 1996) p. 212.

[6] G. M. Huang, T. J. Tarn and J. W. Clark, J. Math. Phys. **24**, 2608 (1983).

[7] M. Zhao and S. Rice, J. Chem. Phys. **95**, 2465 (1991).

[8] R. Gordon and S. Rice, Ann. Rev. Phys. Chem. **48**, 601 (1997).

[9] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (Polish Scientific Publishers, Warsaw, 1977).

[10] Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. **148**, 346 (1983).

[11] Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. **167** 181 (1986).

[12] Y. Alhassid, in *Group Theoretical Methods in Physics*, edited by W. W. Zachary (World Scientific, Singapore) p. 337.

[13] G. Lindblad and B. Nagel, Ann. Inst. Henri Poincaré **13**, 27 (1970).

[14] N. Mukunda, J. Math. Phys. **8**, 2210 (1967).

[15] D. Elliot and T. J. Tarn (unpublished).

[16] W. Boothby and E. Wilson, Siam J. Cont. Opt. **17**, 212 (1979).

[17] W. Boothby, J. Diff. Eq. **17**, 296 (1975).

[18] S. Lloyd, Phys. Rev. Lett. **75**, 346 (1995).

[19] D. Deutsch, A. Barenco, and A. Ekert, Proc. R. Soc. London A, **449** 669 (1995).