Operational techniques for the Floquet Hamiltonians

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Abstract

The control operations on a charged non-relativistic particle by variable external fields are examined. Contrasting with the static theory, the effective Hamiltonians for the sequencies of field pulses have not necessarily lower bounds. In particular, the Floquet systems of rotating fields can generate the harmonic oscillators turned upside down, with infinite ladders of negative levels. The Floquet Hamiltonians reduced to pure negative kinetic energy can be also achieved; they permit to invert the free evolution of the Schrödinger's particle, opening new perspectives in the control techniques. The manipulation Hamiltonians without lower bounds imply instability at the level of the quantum field theory

1 The manipulation problem

In its long past the theoretical physics was dominated by the scheme of a dynamical system in static external conditions, represented by a fixed, time independent Hamiltonian. The time dependent external fields and/or conditions appear now and then, as a secondary aspect. Yet, if any physical theory was at all created, this is only since we live in a variable universe, where the physical conditions can be changed and the experiments can be performed. As the matter of fact, the information contained in a fixed Hamiltonian (and in the associated one parameter dynamical group) is extremely poor. It tells only how the system *actually behaves*, but it does not tell how *it could behave* in different external conditions. Thus, e.g., the information encoded in the free (empty space) Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi \tag{1}$$

is almost void. To verify (1), the empty space is simply insufficient: the space must contain at least some obstacles on which the wave packet could diffract and some screens on which it could be detected. The same concerns all other cases of fixed Hamiltonians. From the operational point of view they are not self-sufficient: no physical law can be checked without elements of the manipulation theory [1, 2]. This turns quite explicit in particle trapping and mass spectroscopy [3]-[13], in trends of quantum control [14]-[49]; for microsystems driven by pulses of time dependent external fields [14, 30, 48, 49].

Having this in mind, we propose that the dynamics of a quantum system should be represented not by a single Hamiltonian (or a single one-parameter group) but by a wider collection G of *manipulation operators* $U_1, U_2, \ldots$ which express not so much the actual evolution of the system, but rather its entire *capacity to evolve* [50]. Assuming that the manipulation operations $U_1, U_2, \ldots$ can be superposed, G should be a *semigroup*, and assuming that the collection of the control (manipulation) parameters is nontrivial, G is wider than the traditional one-parameter groups of unitary operators. Let $U \in G$ be one of the accessible operations, performed in a given time interval $[t_1, t_2]$, $\tau = t_2 - t_1 > 0$. Obviously, U can be represented as:

$$U = e^{-i\tau H} \tag{2}$$

where H is a self-adjoint generator, called further the *effective Hamiltonian* for the operation (2). In the laboratory, each 'manipulated evolution' (2) is induced by some time-dependent *instantaneous* hamiltonians $h(t), t \in [t_1, t_2]$; yet, the structure of H may be very different than that of $h(t)$. The point is that H obeys the Baker- Campbell- Hausdorff (BCH) composition [16]. Thus, e.g., if U in (2) is achieved as the result of n manipulation steps, $U = U_n \ldots U_1$, where $U_k = \exp(-i\tau_k H_k)$ are generated by the Hamiltonians H_k in the time lapses τ_k, then H is given by the discrete BCH formula:

$$e^{-i\tau H} = e^{-i\tau_n H_n} \ldots e^{-i\tau_1 H_1} \tag{3}$$

where $\tau = \tau_1 + \ldots + \tau_n$. Notice that neither H and $H_k, (k = 1, \ldots, n)$ are uniquely defined, nor the spectrum of H must be similar to the spectra of $H_1, \ldots, H_n$. The manipulation scheme which shows the closest analogy with the traditional

1-parameter dynamical groups concerns the periodic system driven by a sequence of identical manipulation steps:

$$U(\tau) = e^{-i\tau H}, \ldots, \; U(n\tau) = U(\tau)^n = e^{-in\tau H} \qquad (4)$$

If this is the case, (2) generates a *Floquet process* and H becomes a *Floquet Hamiltonian*. If U differs very little from 1, then the Floquet process might be a good strategy to approach a desired manipulation effect by a sequence of little, careful steps resembling the approximation processes of the traditional theory. The practical value of the method, of course, depends on our ability to control the Floquet Hamiltonian and this is precisely the subject of our talk.

Below, we shall consider the simple case of a Schrödinger's particle in $\mathbf{R}^3$ controlled by electromagnetic field pulses. We are interested in answering the following questions: (1) can any unitary operation $U : \mathcal{H} \to \mathcal{H}$ be induced as a manipulation operation? (2) is the manipulation process stable, or can it show some resonance sensitivity to the external radiation of definite frequencies?

2 The positivity breaking: inverted oscillators

In the axiomatic quantum theories it is quite customary to believe that the Hamiltonians must be positive. The intense effort to assure the positivity property has lead some authors to pay a high price, accepting even the negative probabilities. Yet, the localization problems might inspire some doubts [43]. The purely negative Hamiltonian (inverted oscillator) have been proposed by Glauber and coworkers as the model for the superfluorescence phenomena [44]. As subsequently proved, the positivity fails for the *effective* Hamiltonians, even if the *instantaneous* Hamiltonians $H(t)$ are positive. A simple case when this happens was described in 1989 [27, 28]. The charged, nonrelativistic particle of mass m and charge e is placed in a homogeneous, rotating magnetic field $\mathbf{B}(t) = B(\mathbf{n}\cos\omega t + \mathbf{m}\sin\omega t)$, where $\mathbf{n}$, $\mathbf{m}$ are two orthogonal unit vectors; the vector potential (in the laboratory approximation) reads:

$$\mathbf{A}(t) = -\frac{1}{2}\mathbf{x} \times \mathbf{B}(t) \qquad (5)$$

The instantaneous Hamiltonians $H(t)$ in the Schrödinger's representation are defined by:

$$\begin{aligned} H(t) &= \tfrac{1}{2m}\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 \\ &= \tfrac{1}{2m}\left[\mathbf{p}^2 + \left(\tfrac{e\mathbf{B}(t)}{2c}\right)^2 \mathbf{x}_\perp^2\right] - \tfrac{e\mathbf{B}(t)\mathbf{M}}{2mc} \end{aligned} \qquad (6)$$

where $\mathbf{x}_\perp$ is the projection of $\mathbf{x}$ onto the plane orthogonal to $\mathbf{B}(t)$ and $\mathbf{M}$ is the angular momentum. The evolution operator obeys:

$$\frac{dU(t)}{dt} = -iH(t)\,U(t) \qquad (7)$$

For each fixed t, $H(t)$ is manifestly positive, but this is no longer true for the Floquet Hamiltonian. Indeed, after an elementary transformation, using the simplified units and assuming that z is the rotation axis, one has:

$$U(t) = e^{it\omega M_z}\, e^{-itF} \qquad (8)$$

$$F = \frac{1}{2}\left[\mathbf{p}^2 + a^2(y^2 + z^2)\right] - aM_x + \omega M_z \qquad (9)$$

where the algebraic type of (9) depends on one dimensionless constant $\alpha = a/\omega = eB/2mc\omega$ (B = the constant norm of the rotating magnetic field). If $|\alpha| < 0.579982655..$, the system is stable on quantum mechanical level (no parametric resonance); yet:

$$F = \omega_0 A_0^\dagger A_0 - \omega_1 A_1^\dagger A_1 + \omega_2 A_2^\dagger A_2 \qquad (10)$$

where $A_j^\dagger A_j$ are three commuting 1-dimensional oscillators, $\omega_j = \omega\sqrt{|\sigma_j|}$, with σ_j defined as 3 real roots of the 3-rd order algebraic equation:

$$\Delta(\sigma) = \sigma^3 + 2(1+2\alpha^2)\sigma^2 + (1+3\alpha^2)\sigma + \alpha^2 = 0 \qquad (11)$$

The first term in (8) is just the rotation and reduces to 1 for any $t = nT$ ($T = 2\pi/\omega$). Henceforth, F is a natural Floquet Hamiltonian for the sequence of the time intervals $[nT, (n+1)T], n = 1, 2.3, \ldots$ Yet, F is not positive and has no lower bound. Can it participate in a conservative energy balance with the rest of the universe? The similar problem was also detected in [51, 52, 53]. Before answering, let us report more cases of the manipulation Hamiltonians.

3 Evolution loops: free evolution inverted

Some curious manipulation effects correspond to the exact cases of the (BCH)-formula. An elementary example is the following sequence of 12 unitary operations which closes to unity [19]:

$$e^{-i\frac{1}{\tau}\frac{q^2}{2}}\, e^{-i\tau\frac{p^2}{2}} \ldots e^{-i\frac{1}{\tau}\frac{q^2}{2}}\, e^{-i\tau\frac{p^2}{2}} \equiv 1 \qquad (12)$$

(the equivalence $\equiv$ means the operator proportionality up to a phase factor). Note that all operations in (12) have a dynamical sense: the 6 exponentials $\exp(-i\tau p^2/2)$ are just the incidents of the free evolution, while the $\exp(-iq^2/2\tau)$ represent limiting cases of sharp kicks of the oscillator potential. The entire sequence of the 12 dynamical events produces an *echo phenomenon* for the non-spin states: at the end of the process the Schrödinger's particle always returns to its initial state (compare the recent ideas about cooling the particle clouds by potential kicks [47]). Since 6 of the operators in (12) represent the free evolution incidents, the identity implies that the free evolution can be inverted by the sequence of 11 remaining operations:

$$e^{-i\frac{1}{\tau}\frac{q^2}{2}}\, e^{-i\tau\frac{p^2}{2}} \ldots e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}} \equiv e^{i\tau\frac{p^2}{2}} \qquad (13)$$

Note, that the *echo formula* (12) is just a particular case in an ample family of the circular identities permitting to invert the free evolution process (cf. [24], [29]). Some other examples are:

$$\left(e^{-3i\tau\frac{p^2}{2}}e^{-i\frac{1}{\tau}\frac{q^2}{2}}\right)^3 \equiv \left(e^{-2i\tau\frac{p^2}{2}}e^{-i\frac{1}{\tau}\frac{q^2}{2}}\right)^4 \equiv 1 \qquad (14)$$

implying again inversion formulae, e.g.:

$$e^{3i\tau\frac{p^2}{2}} \equiv e^{-\frac{i}{\tau}\frac{q^2}{2}}e^{-3i\tau\frac{p^2}{2}}e^{-\frac{i}{\tau}\frac{q^2}{2}}e^{-3i\tau\frac{p^2}{2}}e^{-\frac{i}{\tau}\frac{q^2}{2}} \qquad (15)$$

with the effective Hamiltonians proportional to $-p^2/2$. The geometric phases of Aharonov-Anandan type for the loop processes (12, 14) are a separate problem [54, 55]; one of possible techniques to determine them is based on the observation of the Gaussian packets (see [56]).

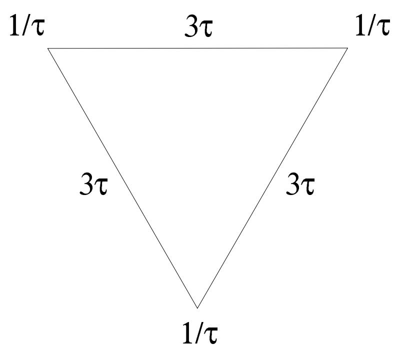

Figure 1. The triangular diagram of a simple evolution loop. The vertices symbolize the kicks of the oscillator potential, while the sides stand for the intervals of the free evolution.

Due to the identities (12-15) an ample class of dynamical manipulations can be engineered just by deforming the loop process [24, 45]. An elementary example is the two kick operation:

$$e^{-i\sigma\frac{p^2}{2}}e^{-i(1+\frac{1}{\sigma})\frac{q^2}{2}}e^{-i\frac{p^2}{2}}e^{-i(1+\sigma)\frac{q^2}{2}} \equiv e^{i\ln\sigma\frac{qp+pq}{2}} \qquad (16)$$

permitting to squeeze the wave packets in the coordinate space [20, 46]. Note that the formulae analogous to (15-16) are quite easy to implement inside of the ion traps [7].

It is quite essential that all these operations, i.e., the exponential kicks in (12-16), the time manipulations (13-15) and the squeezing (16), can be alternatively produced by softly varying fields, without any abrupt potential shocks. The operational key lies in a simple case of the Sturm Liouville problem, known as the Helmholtz eigenvalue problem [57]. The Helmholtz differential equation is:

$$\frac{d^2\xi}{dt^2} + \lambda\phi(t)\xi(t) = 0 \qquad (17)$$

and the Helmholtz *spectral problem* consists in selecting the λ values for which (17) has non-trivial solutions obeying the boundary condition:

$$\xi(a) = \xi(b) = 0 \qquad (18)$$

where $[a,b]$ is a certain fixed interval. Suppose $\phi(t)$ in (17) is non-negative in $[a,b]$. Then, consider the quantum, time-dependent oscillator piloted by $\lambda\phi(t)$:

$$H(t) = \frac{p^2}{2} + \lambda\phi(t)\frac{q^2}{2} \qquad (19)$$

Let $U(t,a)$ be the corresponding evolution operator given by (7). It turns out, that if $\lambda = \lambda_n$ is one of the spectral values $\lambda_1, \lambda_2, \ldots$ for (17-18), then the evolution operator $U(b,a)$ is the product of parity, squeezing, and a q-dependent phase factor:

$$U(b,a) = P^n e^{i\eta\frac{q^2}{2}} e^{i\ln\sigma\frac{qp+pq}{2}} \qquad (20)$$

(see [41, 46]). If moreover the ‘manipulation function’ $\phi(t)$ is symmetric in the operation interval $[a,b]$ with respect to the center $(a+b)/2$, then the squeezing turns trivial ($\sigma = 1$) and the evolution operation (20) yields a ‘soft imitation’ of a sharp oscillator kick. Henceforth, the ‘kick formulae’ (12-16) can be implemented by continuously varying fields, without any δ-like pulses.

The parameters for generating the free evolution inversion can be read as well from the Helmholtz equation (17), though the boundary condition now must be of Neumann type:

$$\xi'(a) = \xi'(b) = 0 \qquad (21)$$

If $\lambda = \lambda_n$ is the n-th spectral value for the Sturm-Liouville problem (17) and (21), and if $\phi(t)$ is symmetric in the operation interval $[a,b]$, then the Hamiltonian (19) generates in $[a,b]$ a ‘distorted free evolution’:

$$U(b,a) = P^n e^{i\tau'\frac{p^2}{2}} \qquad (22)$$

where the ‘effective time’ τ', in general, does not coincide with the real operation time $\tau = b - a$. (The value of τ' for each particular manipulation function $\phi(t)$ can be very easily determined by applying Prüfer method to solve the 1-dimensional spectral problems [58, 41]).

Note, that the manipulation operations (20), (22) can be quite easily implemented in 2 space dimensions by placing a charged microparticle in a homogeneous, time dependent magnetic field inside of an ideal cylindrical solenoid [41, 45].

The analogous magnetic operations in 3 space dimensions require just some more involved algebra. Thus, the magnetically induced evolution loop (i.e., an *echo phenomenon* for non-spin variables) in 3 dimensions is an elementary phenomenon [24, 45]. The simplest such effect occurs for a charged Schrödinger’s particle affected by a sequence of 6 rectangular, homogeneous magnetic pulses in 6 time lapses $T/2$ along 3 orthogonal axes: $B\mathbf{n}, -B\mathbf{n}, B\mathbf{m}, -B\mathbf{m}, B\mathbf{s}, -B\mathbf{s}, \ldots$, where $\mathbf{n}, \mathbf{m}, \mathbf{s}$ are three orthogonal unit vectors. If the process is periodically repeated, the resulting Floquet operator depends just on one dimensionless constant $\alpha = eBT/2mc$. If:

$$r(\alpha) = 2\cos\alpha - \alpha\sin\alpha = 0 \qquad (23)$$

then the 4 repetitions of our 6-pulse pattern closes the evolution loop, creating a *magnetic equivalent* of the 3-dimensional quantum oscillator. Some more general *magnetic oscillators* in $\mathbf{R}^3$ are created by n repetitions of the 6-pattern whenever $r(\alpha) = 2\cos\frac{2\pi l}{n}, n = 3, 4, \ldots; 0 < 2l < n$ [24]. The deformations of this process opens the door to new manipulation techniques useful for the 'quantum engineering'. One of most interesting cases is the deformation described in [45], which instead of closing to unity, produces an operation inverse to the free evolution in $\mathbf{R}^3$

$$U(\tau) \equiv e^{-i\tau' \frac{\mathbf{p}^2}{2}} \tag{24}$$

The existence of such a phenomenon in 3-space dimensions implies that the evolution operations in the interaction frame (normally superposed with the unavoidable free evolution) may now appear alone, as the exact manipulation effects. In particular, (24) makes possible a process composed of n evolution steps $U_1, \ldots, U_n$ of the form $U_j = e^{-i\tau_j H_j}$ where $\tau_1 H_1 + \cdots + \tau_n H_n = 0$. If V_j are small, the mechanism of the BCH formula must then produce a precession effect:

$$U = U_n \ldots U_1 = e^{-i\tau H_{ef}}, \tag{25}$$

$(\tau = \tau_1 + \cdots + \tau_n)$ where the main part of the Hamiltonian

$$H_{ef} = \left[-i\frac{p^2}{2}, \phi(q)\right] = \frac{1}{2}\left[\phi'(q)p + p\phi'(q)\right] \tag{26}$$

generates a 'nonlinear squeezing'. By applying the same rule many times, one can approximate the manipulation operations $\exp(-i\tau H_{ef})$ with:

$$H_{ef} = \left[-i\frac{p^2}{2}, \ldots \left[-\frac{ip^2}{2}, \phi(q)\right] \ldots\right] \tag{27}$$

Note, that the operators of the form (27) are a natural basis in the subspace of the hermitian polynomials $f(q, p)$ (with Weyl's ordering). Though the topological problems (strong convergence) remain open, this indicates that there is no algebraic barrier, which could forbid us to generate dynamically quite an arbitrary unitary operation [19, 24].

4 Energy conversion

While to engineer any unitary operation might be only a matter of patience, the stability and the energy exchange of the manipulated system with rest of the universe are open problems. Tentatively, two types of instability might be expected.

(1) The instability of the Floquet (periodic) systems due to the *parametric resonance*. The phenomenon does not invalidate the manipulation designs; inversely, it can be a necessary ingredient. An example is the generation of the squeezing for the charged particle states, an effect (from the definition) belonging to the parametric resonance area.

(2) A different kind of instability occurs if the manipulation Hamiltonian has no lower bound. It is quite surprising that the problem was almost overlooked in the literature. Thus, e.g., Zeldovich himself, in his pioneer work of 1967 (where the hypothesis of the quasi-energy was formulated) seemed convinced that the Floquet Hamiltonians must be always bounded from below. He writes about: "... the transitions from the lowest quasienergy state (d) into an excited state (2) with $E_d - E_2 < 0$" [59]. As if the quasienergy had to possess always the 'ground state'! This is not the case, however; an elementary counter-example is the inverted harmonic oscillator $H_- = -(p^2/2 + \omega^2 q^2/2)$ persistently entering into the Floquet Hamiltonians of the rotating systems [27, 28, 51, 52].

The 'polemic incidents' around this phenomenon seem quite creative. Thus, Ford and O'Connel [53] notice the existence of the "upside down oscillator" in the Floquet Hamiltonian of the rotating systems but take it as a "serious defect" of the rotating wave approximation, adding also that "... the reservoir is not passive (...) and the second law of thermodynamics is not satisfied"! Glauber comments his own model almost ironically: "To be in actual possession of an inverted oscillator would be a fantastic thing: it would solve the world energy problem" (R. Glauber, in Proc. of VI International school in Ustroń, September 1985 [44]). Bialynicki-Birula, Eberly and Kalinski construct a quantum analogue of Trojan asteroids as the wave packets sustained by the rotating electric field [51]. They notice the existence of an inverted oscillator but limit their discussion to the bound states and quasienergy spectrum. Their work has met an objection that the well in the rotating field is too shallow to sustain a bound state [60, 61]. In the exchange of polemic notes in Phys. Rev. Lett. both sides (the authors and the adversaries), seem to forget that the inverted oscillator cannot be discussed on purely quantum mechanical ground. Indeed, if one is interested only in quantum mechanical level, the 'top state' of the inverted oscillator is as stable as the 'ground state' of the traditional one. It is the level of quantum field theory, which makes difference. While the ground state is stationary, the "top state" is not: it is our hypothesis that the particle in the rotating field, obeying the 'negative oscillator' will radiate spontaneously, sinking down into an infinite ladder of the negative energy values. (The reservoir indeed is not passive!). Yet, one has to remember that the system in time dependent fields is not conservative: so, it can sink *ad infinitum*, emitting endlessly, not as a *perpetuum mobile* but at the cost of the external source maintaining the rotating field. If so, the Floquet Hamiltonian would work as a 'conversion machine', realistic case of Glauber's model, which absorbs the energy of the 'mother field' to reemit it as a radiation of Floquet frequencies.

To check our hypothesis, we propose to observe the emission spectra of a charged particle gas distributed in the nodal points of a coherent, standing electromagnetic wave. Normally, the charged particles in nodal points of a quickly oscillating electromagnetic field are supposed to be stable due to the Kapitza-Landau ponderomotric force [28, 30]; but if we are right, the gas will be unstable and will radiate spontaneously the Floquet frequencies.

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References

[1] N Wiener, *Cybernetics or control and communication in the animal and the machine*, Wiley & Sons, New York (1961)

[2] R Bellman, *Introduction to the mathematical theory of control process*, vol.I, II, Academic Press, New York (1987)

[3] W Paul, in *Nobel Lectures in Physics 1981-1990*, World Scientific, Singapore (1993), p. 610; Rev. Mod. Phys. **58**, 233 (1986)

[4] W Paul, M Raether, Z. Phys. **140**, 262 (1955); W Paul, HP Reinhard, U Von Zahn, Z. Phys. **152**, 143 (1958); F Von Bush, W Paul, Z. Phys. **164**, 581 (1961); Z. Phys. **164**, 588 (1961)

[5] LS Brown, G Gabrielse, Rev. Mod. Phys. **58**, 233 (1986)

[6] H Dehmelt, Rev. Mod. Phys. **62**, 525 (1990)

[7] DJ Fernández, Nuov. Cimento B **107**, 885 (1992); Acta Phys. Polon. B **21**, 589 (1990)

[8] CE Berry, J. Appl. Phys. **25**, 28 (1954)

[9] IW McLafferty, Annal. Chem. **28**, 306 (1956)

[10] MJ O'Neal, *Applied Mass Spectroscopy*, Inst. of Petroleum, London (1954)

[11] PH Davson, *Quadrupole Mass Spectrometry and its Applications*, Elsevier, Amsterdam (1976)

[12] RE March, RJ Hughes, *Quadrupole Storage Mass Spectrometry*, Wiley, New York (1989)

[13] F Dietrich, E Chen, JW Quint, H Walther, Phys. Rev. Lett. **59**, 2931 (1987); F Dietrich, E Peik, M Chen, H Walther, Phys. Blätter **44**, 12 (1988)

[14] EL Hahn, Phys. Rev. **80**, 580 (1950)

[15] WE Lamb, Phys. Today **22** *No.4*, 23 (1969)

[16] B Mielnik, J Plebański, Ann. Inst. Henri Poincaré **XII**, 215 (1970)

[17] E Lubkin, J. Math. Phys. **15**, 663, 673 (1974)

[18] HP Yuen, Phys. Rev. A **13**, 2226 (1976)

[19] B Mielnik, Rep. Math. Phys. **12**, 331 (1977)

[20] LS Brown, LJ Carson, Phys. Rev. A **20**, 2486 (1979)

[21] GM Huang, TJ Tarn, JW Clark, J. Math. Phys. **24**, 2608 (1983)

[22] RG Brewer, EL Hahn, Sci. Am. **251** *(12)*, 50 (1984)

[23] A Royer, Phys. Rev. A **37**, 2460 (1985)

[24] B Mielnik, J. Math. Phys. **27**, 2290 (1986)

[25] X Ma, C Rhodes, Phys. Rev. A **39**, 1941 (1988)

[26] G Grübl, J. Phys. A **37**, 2985 (1988)

[27] DJ Fernández, *Electrón de Schrödinger en campos electromagnéticos oscilantes*, Ph.D. thesis, Physics Department, CINVESTAV, México (1988)

[28] B Mielnik, DJ Fernández, J. Math. Phys. **30**, 537 (1989)

[29] Y Aharonov, J Anandan, S Popescu, L Vaidman, Phys. Rev. Lett. **64**, 2965 (1990)

[30] DJ Fernández, B Mielnik, Phys. Rev. A **41**, 5788 (1990)

[31] TJ Tarn, JW Clark, GM Huang, in *Modeling and Control of Systems*, ed. A Blaquiere, Springer, Berlin (1989)

[32] B Baseia, SS Mizrahi, MHY Moussa, Phys. Rev. A **46**, 5885 (1992)

[33] P Brumer, M Shapiro, Anu. Rev. Phys. Chem. **43**, 257 (1992) and the literature therein

[34] P Gross, H Singh, H Rabitz, K Mease, GM Huang, Phys. Rev. A **47**, 4593 (1993)

[35] P Gross, V Ramakrishna, E Villalonga, H Rabitz, M Littman, S Lyon, M Shayegan, Phys. Rev. B **49**, 100 (1994)

[36] P Brumer, M Shapiro, Sci. Am. **272**, 34 (March 1994)

[37] Y Chen, P Gross, V Ramakrishna, H Rabitz, J. Chem. Phys. **102**, 8001 (1995)

[38] P Bala, P Grochowski, B Lesyng, JA McCammon, in *Quantum Mechanical Simulation Methods for Studying Biological Systems*, Springer Verlag, Berlin (1996), pp. 119-156; J. Phys. Chem. **100**, 2535 (1996)

[39] V Ramakrishna, R Ober, X Sun, O Steuernagel, J Botina, H Rabitz, Phys. Rev. A **61**, 032106 (2000)

[40] F Delgado, B Mielnik, J. Phys. A **31**, 309 (1998)

[41] F Delgado, B Mielnik, MA Reyes, Phys. Lett. A **237**, 359 (1998)

[42] F Delgado, B Mielnik, Phys. Lett. A **249**, 369 (1998)

[43] GC Hegerfeld, in *Irreversibility and Causality in Quantum Theory - Semigroups and Rigged Hilbert Spaces*, ed. A Bohm et al, Springer Lecture Notes in Physics **504**, 283 (1998); Ann. Phys. (Leipzig) **7**, 116 (1998)

[44] RJ Glauber, *Simple models for linear damping and amplification* in *Quantum Optics*, Proceedings of the VI Int. School of Coherent Optics, Ustroń, Poland (1985), ed. A Kujawski, M Lewenstein, Ossolineum, Wroclaw 1986, pp. 9-30; F Haake, JW Hauss, H King, G Schröderand R Glauber, Phys. Rev. **A 23**, 1322 (1981); F Haake, JW Hauss, R Glauber, Phys. Rev. **A 23**, 3255 (1981)

[45] DJ Fernández, B Mielnik, J. Math. Phys. **35**, 2083 (1994)

[46] B Mielnik, DJ Fernández, in *IV ICSSUR*, NASA Conf. Publ. **3322**, 241 (1996)

[47] JF Poyatos, JI Cirac, P Zoller, Phys. Rev. Lett. **77**, 4728 (1996)

[48] TC Weinacht, J Ahn, PH Bucksbaum, Nature **397**, 233 (1999)

[49] C Strout, M Noel, Optics and Photonic News, April (1999)

[50] B Mielnik, J. Math. Phys. **21**, 44 (1980)

[51] I Bialynicki-Birula, M Kalinski, JH Eberly, Phys. Rev. Lett. **73**, 1777 (1994)

[52] S Hacyan, Phys. Rev. A **53**, 4481 (1996)

[53] GW Ford, RF O’Connel, Physica A **243**, 377 (1997)

[54] Y Aharonov, J Anandan, Phys. Rev. Lett. **58**, 1593 (1987)

[55] DJ Fernández, MA del Olmo, M Santander, J. Phys. A **25**, 6409 (1992); DJ Fernández, O Rosas-Ortiz, Phys. Lett. A **236**, 275 (1997)

[56] YC Ge, MS Child, Phys. Rev. Lett. **78**, 2507 (1997)

[57] CM Bender, SA Orzag, *Advanced Mathematical Methods for Scientists and Engineers*, Mc. Graw-Hill, New York (1978)

[58] H Prüfer, Math. Annalen **95**, 409 (1926)

[59] Ya B Zeldovich, JETP **24**, 1006 (1967)

[60] D Farrelly, E Lee, T Uzer, Phys. Rev. Lett. **75**, 972 (1995)

[61] I Bialynicki-Birula, M Kaliński, JH Eberly, Phys. Rev. Lett. **75**, 973 (1995)