Neural Network Enhanced Output Regulation in Uncertain Nonlinear Systems

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Abstract— *The problem of designing a control law to achieve asymptotic tracking and disturbance rejection in a nonlinear plant where both the reference and disturbance signals are generated by an exosystem is called nonlinear output regulation problem. It is known that solvability of this problem relies on the existence of a feedforward function defined by a set of mixed nonlinear partial and algebraic equations called regulator equations. Previous approaches to solving the output regulation problem call for the solution of the regulator equations. However, solving the regulator equations is difficult due to the nonlinearity and complexity. This paper proposes a novel approximation approach to solving the output regulation problem by directly approximating the feedforward function using a class of artificial neural networks. Further, a control configuration is developed that allows the reduction of the tracking error by the on-line adjustment of the parameters of the neural networks.*

Keywords: Output regulation problem, regulator equations, neural networks.

I. Introduction

A central control problem is to design a feedback control law for a plant such that the output of the plant can asymptotically track a class of reference inputs and reject a class of disturbances while maintaining the closed-loop stability. Both the reference inputs and disturbances are generated by an autonomous differential equation called exosystem. This problem is called output regulation problem or, alternatively, servomechanism problem or asymptotic tracking and disturbance rejection. For the class of linear systems, the solvability of the output regulation problem was thoroughly studied in the 1970s by many researchers including Davison, Francis and Wonham, and Desoer, to name just a few [3],[5],[6] and [4]. For the class of nonlinear systems, research had long been limited to the special case where the reference inputs and disturbances are constant [6],[7] and [10] until 1990 when Isidori and Byrnes published their award-winning paper on the output regulation problem for nonlinear systems with time-varying reference inputs and disturbances [13]. A celebrated result in [13] is that the solvability of the nonlinear output regulation problem is tied to the solvability of a set of partial differential and algebraic equations known as regulator equations or Isidori and Byrnes equations.

Nonlinear output regulation theory can potentially solve the asymptotic tracking and disturbance rejection problem for a large class of nonlinear systems that cannot be handled by the popular inversion based approaches since, as will be seen in Section 2, the solvability of this problem can be formulated for general nonlinear systems including nonminimum phase nonlinear systems. Nevertheless, this potential capability can be realized only if the solution of the regulator equations are available. But for many typical nonlinear systems, the exact solution of the regulator equations are not available because of the nonlinearity of the equations. Recently, various approximation methods have been proposed to solve the regulator equations. These include the Taylor series approach in [11], [12], [15], the recurrent neural network approach in [2], and the feedforward neural network approach in [16]. While the Taylor series approach may lead to very accurate approximation of the solution to the regulator equations in the neighborhood of the origin of the Euclidean space under consideration, the strength of the neural network approach lies in the fact that neural networks can approximate the solution of the regulator equations up to an arbitrarily small error in any given compact subset.

However, the approaches as reported in [11], [12], [15], [2], and [16] have a fundamental limitation in that solving the regulator equations requires the precise knowledge of the plant and incurs tedious computation. In this paper, we will proposed an approximation approach to the output regulation problem from a completely different point of view that eliminates the need to explicitly solve the regulator equations. This approach is based on the observation that the solution of the regulator equations simply provides the information on the feedforward control necessary for guaranteeing the zero steady state tracking error, and this feedforward control is the function of the exogenous signal. Exploiting this observation and using the Universal Approximation Theorem [9], we can further represent this function by an artificial neural network, and cast the problem of finding this feedforward control into a parameter minimization problem. Moreover, this parameter minimization process can be carried out either offline or online based on the measurement of the tracking error of the closed-loop system. The major advantages of our proposed approach over the previous approaches include 1) the precise knowledge of the plant is not needed, and 2) computational complexity is significantly reduced. The first advantage offers effective solution of asymptotic tracking and disturbance rejection problem in sophisticated *uncertain* nonlinear systems as evidenced by the successful application of our approach to two well known nonlinear systems, namely, the inverted pendulum on a cart system and the ball and beam system. To elaborate the second advantage, we note that for a $n-$ dimensional plant with m inputs, our previous approach in [16] calls for $n+m$ neural networks whereas our current approach only ask for m neural networks.

Due to the space limit, all proofs as well as examples are omitted. A complete version of the paper is available from the second author upon request.

II. Problem Description

We will consider a nonlinear plant described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), v(t)), \quad x(0) = x_0,\ t \geq 0 \\ y(t) &= h(x(t), u(t), v(t)) \end{aligned} \tag{2.1}$$

where $x(t)$ the n-dimensional plant state, $u(t)$ the m-dimensional plant input, $y(t)$ the p-dimensional plant output representing tracking error, and $v(t)$ a q-dimensional exogenous signal whose components represents reference inputs to be tracked and/or external disturbances to be rejected. It is assumed that $v(t)$ is generated by a q-dimensional autonomous differential equation as follows

$$\dot{v}(t) = Sv(t),\ \ v(0) = v_0,\ t \geq 0 \tag{2.2}$$

where the matrix S satisfies the following assumption:

A1: All the eigenvalues of S are simple, and have zero real parts.

For convenience, we assume f and h are globally defined and sufficiently smooth, i.e., $f \in C^k(R^n \times R^m \times R^q)$, and $h \in C^k(R^n \times R^m \times R^q)$, with $k \geq 2$. Also, we assume $f(0,0,0) = 0$, and $h(0,0,0) = 0$.

We will consider two classes of control laws, namely,

1. State feedback:

$$u(t) = \psi(x(t), v(t)) \tag{2.3}$$

where the function $\psi(\cdot,\cdot)$ is required to be sufficiently smooth and satisfy $\psi(0,0) = 0$.

2. Output feedback with feedforward:

$$\begin{aligned} u(t) &= \psi(z(t), v(t)) \\ \dot{z}(t) &= g(z(t), y(t), v(t)) \end{aligned} \tag{2.4}$$

where $z(t)$ is the compensator state of dimension n, and the function $g(\cdot,\cdot,\cdot)$ is also required to be sufficiently smooth and satisfy $g(0,0,0) = 0$.

To formulate the requirements on the closed-loop system, we denote the closed-loop system consisting of the plant (2.1), exosystem (2.2), and the controller (2.3) or (2.4) as follows

$$\begin{aligned} \dot{x}_c(t) &= f_c(x_c(t), v(t)),\ \ x_c(0) = x_{c0} \\ \dot{v}(t) &= Sv(t) \\ y(t) &= h_c(x_c(t), v(t))),\ \ t \geq 0 \end{aligned} \tag{2.5}$$

where for the state feedback control, $x_c = x$, and $h_c(\cdot,\cdot)$ and $f_c(\cdot,\cdot)$ are described as follows:

$$\begin{aligned} f_c(x_c(t), v(t)) &= f(x(t), \psi(x(t), v(t)), v(t)) \\ h_c(x_c(t), v(t)) &= h(x(t), \psi(x(t), v(t)), v(t)) \end{aligned}$$

and for the output feedback with feedforward control, $x_c = (x, z)$, and $h_c(\cdot,\cdot)$ and $f_c(\cdot,\cdot)$ are described as follows:

$$\begin{aligned} h_c(x_c(t), v(t)) &= h(x(t), \psi(z(t), v(t)), v(t)) \\ f_c(x_c(t), v(t)) &= \begin{bmatrix} f(x(t), \psi(z(t), v(t)), v(t)) \\ g(z(t), h_c(x_c(t), v(t)), v(t)) \end{bmatrix} \end{aligned}$$

In terms of the closed-loop system, we can describe the output regulation problem as follows [13]: *Output Regulation Problem:* Designing a feedback control law of the form (2.3) or (2.4) such that the closed-loop system satisfies

R1: The eigenvalues of $\frac{\partial f_c}{\partial x}(0,0)$ have negative real part.

R2: For sufficiently small x_{c0} and v_0, the solution of the closed-loop system exists for all $t \geq 0$, and

$$\lim_{t\to\infty} ||h_c(x(t), v(t)|| = 0 \tag{2.6}$$

Remark 1: It is shown in [13] that the first requirement ensures the (local) asymptotic stability of the closed-loop system, which in turn ensures the (local) bounded-input bounded state property for the closed-loop system (2.5).

To state the solvability conditions of the above problem, we list the following two assumptions below.

A2: The pair $\{\frac{\partial f}{\partial x}(0,0,0),\ \frac{\partial f}{\partial u}(0,0,0)\}$ is stabilizable.

A3: The pair $\{\frac{\partial f}{\partial x}(0,0,0),\ \frac{\partial h}{\partial x}(0,0,0)\}$ is detectable.

Now we can state the solvability of the above problem as follows which is adapted from the work in [13]

Theorem 1: Under assumptions A1 and A2 (A1, A2 and A3), the state feedback (output feedback with feedforward) output regulation problem is solvable if and only if

A4: There exist two sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$ defined in an open neighborhood V of the origin of R^q such that $\mathbf{x}(0) = 0$, $\mathbf{u}(0) = 0$, and, for all $v \in V$,

$$\begin{aligned} \frac{\partial \mathbf{x}(v)}{\partial v} Sv &= f(\mathbf{x}(v), \mathbf{u}(v), v) \quad (2.7) \\ 0 &= h(\mathbf{x}(v), \mathbf{u}(v), v) \quad (2.8) \end{aligned}$$

Remark 2: Under the assumptions A2 and A3, there exist $K \in R^{m\times n}$ and $L \times R^{n\times p}$ such that all the eigenvalues of the following two matrices $\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial u}(0,0,0)K$ and $\frac{\partial f}{\partial x}(0,0,0) + L\frac{\partial h}{\partial x}(0,0,0)$ have negative real parts. With K, L, $\mathbf{x}(v)$, and $\mathbf{u}(v)$ ready, either the state feedback or output feedback with feedforward control law that solves the output regulation problem can be given as follows [13]

$$u = \psi(x, v) = \mathbf{u}(v) + K[x - \mathbf{x}(v)] \quad (2.9)$$

or

$$\begin{aligned} \dot{z} &= f(z, \psi(z, x), v) + L(y - h(z, \psi(z, v), v)) \\ u &= \mathbf{u}(v) + K[z - \mathbf{x}(v)] \quad (2.10) \end{aligned}$$

Remark 3: It may be interesting to take a look at the role played by the solution of (2.7) and (2.8). Consider the closed-loop system resulting from the state feedback control law (2.9):

$$\begin{aligned} \dot{x}(t) &= f(x(t), \mathbf{u}(v(t)) + K(x(t) - \mathbf{x}(v(t)), v(t)) \\ &= f_c(x(t), v(t)) \quad (2.11) \\ y(t) &= h(x(t), \mathbf{u}(v(t)) + K(x(t) - \mathbf{x}(v(t)), v(t)) \\ &= h_c(x(t), v(t)) \quad (2.12) \end{aligned}$$

It is clear that equations (2.7) and (2.8) can be equivalently written as follows

$$\begin{aligned} \frac{\partial \mathbf{x}(v)}{\partial v} Sv &= f_c(\mathbf{x}(v), v) \\ 0 &= h_c(\mathbf{x}(v), v) \end{aligned}$$

which shows $\mathbf{x}(v)$ is an invariant manifold of the closed-loop system (2.11) on which the output is identically zero. What is more, since all the eigenvalues of $\frac{\partial f_c}{\partial x}(0,0)$ have negative real part, and all the eigenvalues of the matrix S have zero real parts by assumption A1, this invariant manifold is actually a center manifold of the augmented system consisting of (2.11) and (2.2) [13]. Thus by the center manifold theorem [1], [13], for all sufficiently small x_0 and v_0, the solution of (2.11) starting from x_0 denoted by $x(t, x_0, v_0)$ exists for all $t \geq 0$, and there exist $M > 0$ and $\lambda > 0$ such that

$$\begin{aligned} &||x(t, x_0, v_0) - \mathbf{x}(v(t))|| \leq \\ &Mexp(-\lambda t)||x(0, x_0, v_0) - \mathbf{x}(v(0))|| \quad (2.13) \end{aligned}$$

which together with the sufficient smoothness of $h(x, u, v)$ shows that the closed-loop system described by (2.11) and (2.12) satisfies R2.

It can be seen that the control law relies on the solution of Equations (2.7) and (2.8) which are known as the regulator equations. Since the regulator equations consist of a set of nonlinear partial and differential equations, it is usually impossible to obtain the exact solution of the regulator equations. Thus, as mentioned in Introduction, various approximation methods including the Taylor series approach [11], and neural network approach [2], and [16] have been proposed to solve the regulator equations. On the basis of the approximate solution of the regulator equations, a more practical requirement than R2 can be proposed as follows

R3: For any given $\epsilon > 0$, design a control law such that for all sufficiently small initial conditions x_{c0} and v_0, the closed-loop system has a bounded solution for all $t \geq 0$, and

$$\lim_{t\to\infty} \sup ||h_c(x(t), v(t))|| \leq \epsilon \quad (2.14)$$

We will call the problem of designing control law so that the closed-loop system can achieve R1 and R3 as *Approximate Output Regulation Problem.* Though this problem can be solved based on the approximation solution of the regulator equation as shown in [11] and [16], it has a limitation in that solving the regulator equations requires the precise knowledge of the plant and incurs tedious computation. In the following, we will propose a completely different approach to approximately solving the output regulation problem which does not need to explicitly solve the regulator equations. To bring our idea across, let us first concentrate on the state feedback case and rewrite the control law (2.3) into the following form

$$u(t) = Kx(t) + \mathbf{c}(v(t)) \quad (2.15)$$

where

$$\mathbf{c}(v) = \mathbf{u}(v) - K\mathbf{x}(v) \quad (2.16)$$

Equation (2.15) clearly shows that the control law consists of two parts the first of which is a state feedback which aims to stabilize the closed-loop system while the second of which can be viewed as the precise feedforward control needed to achieve zero steady state tracking error. Thus it suffices to obtain $\mathbf{c}(v)$ in order to obtain (2.15). Of course $\mathbf{c}(v)$ is available if the solution of the regulator equations is available. However what makes our idea challenging thus interesting is that we try to obtain $\mathbf{c}(v)$ without solving the regulator equations. To this end, we note that, once the feedback gain K is predesigned, the feedforward control is a sufficiently smooth function of the exogenous input v only. Thus it is possible to approximate the feedforward control (2.16) by a feedforward neural network. In fact, recalling the well known universal approximation theorem [9], given a real valued function $\mu \in C^k$ defined on a compact subset Γ, and any $\epsilon > 0$, there exists an integer N, and real numbers w_i^O, and w_{ij}^I, $i = 1, \cdots, N$, and $j = 0, 1, \cdots, n$ such that

$$\hat{\mu}(W, x) = \Sigma_{i=1}^N w_i^O \phi(\sum_{j=1}^n w_{ij}^I x_j + w_{i0}^I) \quad (2.17)$$

satisfies $\sup_{x\in\Gamma} |(\hat{\mu}(W, x) - \mu(x))| < \epsilon$ where $\phi \in C^k(R)$ is any non-constant and bounded real valued function. The mapping described by the right hand side of (2.17) is called a three layer feedforward neural network, and the integer N is the number of the hidden neurons. As a result of the universal approximation theorem, given any $\epsilon > 0$, and any compact subset Γ, there exists a $m-$dimensional vector valued function $\hat{\mathbf{c}}(W, v)$ whose components denoted by $\hat{\mathbf{c}}_i(W, v)$, $i = 1, \cdots, m$, take the form (2.17) such that

$$\max_{v\in\Gamma} ||\mathbf{c}(v) - \hat{\mathbf{c}}(W, v)|| < \epsilon \quad (2.18)$$

Replacing $\mathbf{c}(v)$ in (2.15) by $\hat{\mathbf{c}}(W, v)$ leads to the following approximate state feedback neural network control law:

$$u(t) = Kx(t) + \hat{\mathbf{c}}(W, v(t)) \quad (2.19)$$

We need to further address the following issues:

- What is the property of the closed-loop system resulting from the approximate control law (2.19)?
- How is the maximal steady state tracking error of the closed-loop system related to the neural network approximation accuracy as measured by the error bound ϵ in (2.18)?
- How to find a desirable neural networks so that (2.14) is satisfied for a given ϵ?

These issues will be addressed in the next two sections, respectively.

III. Performance Analysis

In this section, we will consider the property of the closed-loop system under the approximate control laws. Let us first focus on the state feedback control law (2.19) which results in the following closed-loop system:

$$\dot{x}(t) = f(x(t), Kx(t) + \hat{\mathbf{c}}(W, v(t)), v(t)) \quad (3.1)$$
$$y(t) = h(x(t), Kx(t) + \hat{\mathbf{c}}(W, v(t)), v(t)) \quad (3.2)$$

Clearly, this system still satisfies R1 regardless of W. Thus again using the center manifold theorem [1], [13], the augmented system consisting of (3.1) and (2.2) has a stable center manifold passing through the origin, or what is the same, there exists a sufficiently smooth function $\mathbf{x}_w(v)$ defined in an open neighborhood V of the origin of R^q such that $\mathbf{x}_w(0) = 0$, and, for all $v \in V$,

$$\frac{\partial \mathbf{x}_w(v)}{\partial v} Sv = f(\mathbf{x}_w(v), K\mathbf{x}_w(v) + \hat{\mathbf{c}}(W, v), v) \quad (3.3)$$

Moreover, denote the solution of the closed-loop system (3.1) starting from $x(0) = x_0$ and $v(0) = v_0$ by $x_w(t, x_0, v_0)$, then for all sufficiently small x_0 and v_0, $x_w(t, x_0, v_0)$ exists for all $t \geq 0$, and there exist $M > 0$ and $\lambda > 0$ such that

$$||x_w(t, x_0, v_0) - \mathbf{x}_w(v(t))|| \leq$$
$$Mexp(-\lambda t)||x(0, x_0, v_0) - \mathbf{x}_w(v(0))||, \quad t \geq 0 \quad (3.4)$$

Theorem 2: Under assumptions A1, A2 and A4, consider the state feedback control law (2.19) where K is such that the requirement R1 is satisfied, and $\hat{\mathbf{c}}(W, v)$ satisfies (2.18) for some compact set $\Gamma \subset R^q$ and some sufficiently small ϵ. Then the closed-loop system described by (3.1) and (3.2) has the property that, for all sufficiently small $x(0)$ and $v(0)$,

(i) The solution $x_w(t, x_0, v_0)$ exists and is bounded for all $t \geq 0$.

(ii) There exists $M_1 > 0$ such that

$$\lim_{t\to\infty} \sup ||y(t)|| \leq M_1 \epsilon \quad (3.5)$$

Next we consider the output feedback with feedforward case. Let us first rewrite the output feedback with feedforward control law given by (2.10) as follows

$$\begin{aligned} u &= Kz + \mathbf{c}(v) \\ \dot{z} &= f(z, Kz + \mathbf{c}(v), v) \\ &+ L(y - h(z, Kz + \mathbf{c}(v), v)) \end{aligned} \quad (3.6)$$

Replacing $\mathbf{c}(v)$ by $\hat{\mathbf{c}}(W, v)$ gives the approximate output feedback with feedforward control law

$$\begin{aligned} u &= Kz + \hat{\mathbf{c}}(W, v) \\ \dot{z} &= f(z, Kz + \hat{\mathbf{c}}(W, v), v) \\ &+ L(y - h(z, Kz + \hat{\mathbf{c}}(W, v), v)) \end{aligned} \quad (3.7)$$

We have the following result for the closed-loop system resulting from the output feedback with feedforward control.

Theorem 3: Under assumptions A1 to A4, consider the control law (3.7) where K and L is such that the requirement R1 is satisfied, and $\hat{\mathbf{c}}(W, v)$ satisfies (2.18) for some compact set $\Gamma \subset R^q$ and some sufficiently small ϵ. Then the closed-loop system described by (2.1), (2.2) and (3.7) has the property that, for all sufficiently small $x_c(0)$ and $v(0)$,

(i) The solution $x_w(t, x_{c0}, v_0)$ exists and is bounded for all $t \geq 0$.

(ii) There exists $M_2 > 0$ such that

$$\lim_{t\to\infty} \sup ||y(t)|| \quad \leq \quad M_2\epsilon \quad (3.8)$$

IV. Algorithm

In the last section, we have shown that a good approximation of the feedforward function by a neural network can lead to a guaranteed steady state tracking error as indicated in Theorems 2 and 3. In this section, we will further consider the problem of developing an algorithm to find the desired neural network. The solution for this problem is not so obvious since $\mathbf{c}(v)$ is of course assumed to be unknown. Therefore, we need to take a deeper look at our problem. Let us note that, under assumption A1, the exogenous signal $v(t)$ is either a constant or a multitone sinusoidal function. Assume $v(t)$ hence $q_w(t) = \mathbf{x}_w(v(t))$ is a periodical function of t with its period denoted by T. Then $h(q_w(t), Kq_w(t) + \hat{\mathbf{c}}(W, v(t)), v(t))$ is also a periodic function with period T. Therefore, we can always achieve (3.5) by making $\int_0^T ||h(q_w(t), Kq_w(t) + \hat{\mathbf{c}}(W, v(t)), v(t))||^2 dt$ sufficiently small. Again $q_w(t)$ is unknown. But, since K is stabilizing, by (3.4), $x_w(t, x_0, v_0)$ approaches $q_w(t)$ exponentially. Therefore, there exists a sufficiently large $T_0 > 0$ such that

$$\begin{aligned} &\int_0^T ||h(q_w(t), Kq_w(t) + \hat{\mathbf{c}}(W, v(t)), v(t))||^2 dt \\ \approx &\int_{T_0}^{T_0+T} ||h(x_w(t, x_0, v_0), Kx_w(t, x_0, v_0) \\ + &\hat{\mathbf{c}}(W, v(t)), v(t))||^2 dt \end{aligned} \quad (4.1)$$

Now let

$$J(W) = \frac{1}{T} \int_{T_0}^{T_0+T} ||y(W, t)||^2 dt \quad (4.2)$$

where

$$y(W, t) = h(x_w(t, x_0, v_0), Kx_w(t, x_0, v_0) + \hat{\mathbf{c}}(W, v(t)), v(t))$$

Then, for any weight vector W, $J(W)$ is well defined. Therefore, we can convert the problem of finding a desirable neural network as the following parameter minimization problem

$$\min_W J(W) \quad (4.3)$$

By the universal approximation theorem, there exists $\hat{W} \in R^M$, where M is the total number of the weights used to represent $\hat{\mathbf{c}}(W, v)$, such that $J(\hat{W})$ is arbitrarily small.

It is noted that the gradient of $J(W)$ is not available since the way $x_w(t, x_0, v_0)$ relies on W is unknown. Nevertheless, it is possible to calculate $J(W)$ by using the following approximation

$$J(W) \approx \sum_{j=1}^{N_s} \frac{1}{N_s} ||y(T_0 + \frac{j}{N_s} T)||^2 \quad (4.4)$$

where N_s is the number of samples in one period. The instantaneous values of $y(W, t)$ at $T_0 + \frac{j}{N_s}$ can be obtained through a simulation scheme shown in Figure 1. Since the feedback gain K is predesigned to make the requirement R1 be satisfied, the closed-loop system described in Figure 1 is always stable regardless of the value of W as described in Remark 2. Moreover, it is clear that $J(W)$ relies only on the error measurement of the output $y(W, t)$. Therefore, precise knowledge of $f(\cdot, \cdot, \cdot)$ and $h(\cdot, \cdot, \cdot)$ is not needed. All we need to know is that the requirement R1 is satisfied by properly designed gain K.

Since $J(W)$ can be calculated for any W, we can use any direct search based approaches to solve (4.3) such as the Hooke-Jeeves method or the more current genetic algorithm.

Remark 5: The configuration shown in Figure 1 can be used for both simulation and real time implementation. When it is used for real time implementation, the state x is assumed to be available for feedback. If the state is not available, then another configuration based on the output feedback with feedforward as shown in Figure 2 can be adopted.

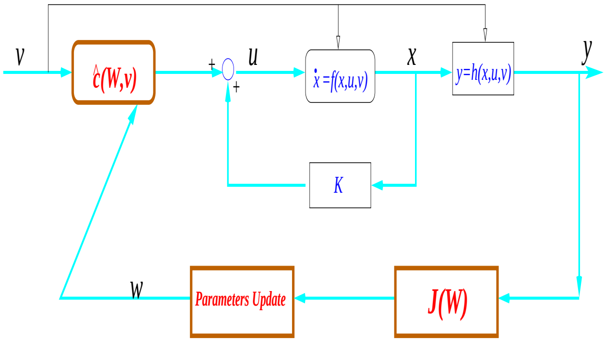

Fig. 1. State Feedback Control Architecture

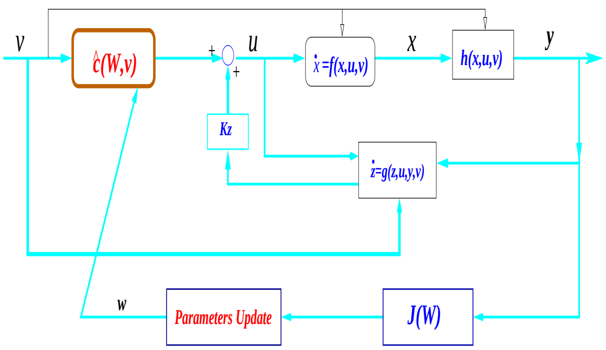

Fig. 2. Output Feedback with Feedforward Control Architecture

V. Conclusions

This paper has proposed a new idea of solving the output regulation problem without solving the regulator equations. This idea coupled with the approximation capability of the neural networks has led to a completely different approach to solving the approximate output regulation problem. Two major advantages of this approach over the previous approach studied in [16] have been observed. That is, the current approach needs much less neural networks, and it does not require precise knowledge of the plant and the exosystem. To be fair, the previous approach also has some advantage over the approach proposed here. That is, the approximate solution of the regulator equations does not rely on the feedback gain K. Further, once an approximate solution of the regulator equations are available in a compact set Γ, a single controller can be synthesized that applies to all exogenous signal $v(t)$ generated by (2.2) as long as $v(t) \in \Gamma$ for all $t \geq 0$.

We have also tested our approach on two well known nonlinear systems, namely, the inverted pendulum on a cart system, and the ball and beam system. Our approach shows significant advantage over other methods such as the Jacobian linearization or approximate input-output linearization in terms of the tracking accuracy.

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