The Chattering Limit of Singularly Perturbed Optimal Control Problems

Zvi Artstein[1]

Abstract

The paper examines the structure of the variational limits that may arise in connection with singularly perturbed optimal control problems. Once the notion of a variational limit is displayed, we analyze the roles of the fast and slow components of the singularly perturbed system in the generation of the variational limits. Comparing a general form of variational limits for singularly perturbed systems to the method of order reduction, which is largely used — yet a special case, is a goal of the paper.

1 Variational limits in optimal control

We start by describing what is meant by a variational limit of an optimal control problem. The context in which the notion is needed is when the optimal control problem is parameterized and what we are interested in is the behaviour of the system when the parameter approaches a prescribed limit. Consider for example the following system.

$$\begin{aligned} &\text{minimize } \int_a^b c(x,t,u)dt \\ &\text{subject to } \frac{dx}{dt} = f_\varepsilon(x,t,u) \qquad (1.1)_\varepsilon \\ &x(a) = x_0 \ . \end{aligned}$$

Here ε is a real valued parameter, typically thought of as $\varepsilon \to 0$. As is customary, we consider $x \in R^n$ and $u \in R^m$, the n- and m-dimensional euclidean spaces, with, possibly, a constraint of the form $u \in U$. Given the system $(1.1)_\varepsilon$, we are interested in identifying its variational limit as $\varepsilon \to 0$, namely, an optimal control system from which we can draw information about the behaviour of the original system when ε is small. In some cases one can simply plug $\varepsilon = 0$ into the original system and get a candidate for the variational limit. Note, however, that the limit $\varepsilon = 0$ may not be provided in $(1.1)_\varepsilon$; moreover, we shall see that even if $\varepsilon = 0$ may be employed, plugging $\varepsilon = 0$ may not result in an appropriate variational limit.

We do not display here a formal definition of a variational limit, but simply state what is expected of it. A variational limit as $\varepsilon \to 0$ of the family $(1.1)_\varepsilon$ is an optimal control problem from which we can draw information about the characteristics of $(1.1)_\varepsilon$ for small ε. We denote it by $(1.1)_0$. Natural requirements from the variational limit $(1.1)_0$ are as follows.

(i) Denote by val(ε) the optimal value of the problem $(1.1)_\varepsilon$. Then val(ε) converges to val(0) as $\varepsilon \to 0$.

(ii) Suppose that an optimal solution $u_0(\cdot)$ of the variational limit $(1.1)_0$ is identified. Then when the control $u_0(\cdot)$ is applied to the problem $(1.1)_\varepsilon$ for small ε, it constitutes an approximate solution.

(iii) For small ε, optimal solutions of $(1.1)_\varepsilon$ are close to optimal solutions of $(1.1)_0$.

Note that in item (iii) we leave open the meaning associated with the notion of closeness. Indeed, the sense in which the closeness is taken, say weak or strong, is part of the quality characteristics of a variational limit.

A natural variational limit arises when the right hand side of the constraint differential equation in $(1.1)_\varepsilon$, namely $f_\varepsilon(x,t,u)$, converges, say to $f_0(x,t,u)$, uniformly on compact sets. Under general conditions the control problem determined by the limit constraint $f_0(x,t,u)$ is indeed a variational limit. Furthermore, in addition to properties (i)–(iii) of the variational limit, one can derive relations between other characteristics of the original problem and its variational limit; e.g., relations between the multipliers, the Hamiltonians, etc. In singularly perturbed systems, however, the situation is not as simple, as we plan to demonstrate in the sequel. In this paper we provide a description of a general approach to the variational limit problem for singularly perturbed optimal control problems, and a comparison to the order reduction method. Some concrete examples are displayed as a demonstration of the approach, and the need for it. References to available partial results in this direction are also given; further developments are in the works, and will be published elsewhere.

2 The order reduction method

In this section we display very briefly the order reduction method from the point of view of the notion of

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[1] Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel (zvika@wisdom.weizmann.ac.il)

variational limits. We state up front that the method is very useful and has led to important applications. Yet, it might be relevant to point out its shortcomings, as we try to do in this paper.

Consider the optimal control problem of coupled slow and fast motions given as follows.

$$\begin{aligned} \text{minimize } & \textstyle\int_a^b c(x,t,z,u)dt \\ \text{subject to } \tfrac{dx}{dt} &= f(x,t,z,u) \\ \varepsilon\tfrac{dz}{dt} &= g(x,t,z,u) \\ x(a) &= x_0 \\ z(a) &= z_0 \ . \end{aligned} \tag{2.1$_\varepsilon$}$$

Here again $x \in R^n$ and $u \in R^m$, and assume that $z \in R^k$. It is clear why the state variable z is referred to as the fast state; indeed, the ordinary differential equation governing the variable z should be written as

$$\frac{dz}{dt} = \frac{1}{\varepsilon} g(x,t,z,u) \ ,$$

hence, the velocity of the solution $z(t)$ is of order $\frac{1}{\varepsilon}$.

A common approach to the analysis of the system $(2.1)_\varepsilon$ for small ε is to consider the system when $\varepsilon = 0$, as follows.

$$\begin{aligned} \text{minimize } & \textstyle\int_a^b c(x,t,z,u)dt \\ \text{subject to } \tfrac{dx}{dt} &= f(x,t,z,u) \\ 0 &= g(x,t,z,u) \\ x(a) &= x_0 \\ z(a) &= z_0 \ . \end{aligned} \tag{2.1$_0$}$$

Under general conditions, the optimal control problem $(2.1)_0$ is indeed a variational limit of $(2.1)_\varepsilon$; an exposition of the approach and a list of references that follow this approach can be found in the monograph by Kokotovic, Khalil and O'Reilly [11], and through the collection of papers in Kokotovic and Khalil [10]. See also O'Malley [12] and references therein.

The system $(2.1)_0$ is referred to as an order reduction of the system $(2.1)_\varepsilon$. Indeed, the fast flow enters into the system via the algebraic equation. This reduces the dimension of the ordinary differential equation that has to be addressed.

The manner in which the system $(2.1)_0$ is treated is as follows. First, for every fixed (x,t) one has to determine the set $V(x,t)$ of feasible controls of the reduced system. This set consists of the solutions of the algebraic equation for the fast flow plus another property that we display shortly. For further reference we write

$$\begin{aligned} V(x,t) = \{(z,u) : 0 = g(x,t,z,u) \ \text{ and} \\ (z,u) \ \text{ satisfies Property 2.1}\} \end{aligned} \tag{2.2}$$

where

Property 2.1. We say that the pair (z_0,u_0) satisfies Property 2.1 for (x,t) fixed, if it can be made, by an appropriate choice of a control $u = u(z)$ or $u = u(z,s)$, a uniformly asymptotically stable equilibrium of the fast differential equation on the fast time scale, namely, a uniformly asymptotically stable point of

$$\frac{dz}{ds} = g(x,t,z,u(z,s)) \ . \tag{2.3}$$

(Notice that the derivative of the fast variable here is with respect to the fast time s).

Although the set $V(x,t)$ comprises of pairs of fast states and controls, it serves as the control set for the slow dynamics. Once the control set $V(x,t)$ is determined, we are left with a standard optimal control problem for the slow variable as follows.

$$\begin{aligned} \text{minimize } & \textstyle\int_a^b c(x,t,v)dt \\ \text{subject to } \tfrac{dx}{dt} &= f(x,t,v) \\ x(a) &= x_0 \\ v &\in V(x,t) \end{aligned} \tag{2.1$_R$}$$

(recall that V is a set of elements of the form (z,u)).

The solution of $(2.1)_R$ is associated first with $(2.1)_0$ and then with the original problem $(2.1)_\varepsilon$ for small ε as follows. First, the optimal control $v(t) = (z(t),u(t))$ is determined for $(2.1)_R$. Coupling this optimal control with the mechanism which, for each fixed $v(t)$, stabilizes the differential equation (2.3) around $z(t)$, we get a solution to $(2.1)_0$. In many applications (in fact, in most of the applications worked out so far) the coupling can be done by a feedback of the form $u = u(x,t,z)$, which for a fixed t stabilizes (2.3) around $z(t)$ (recall that the time variable in (2.3) is s). The resulting $u(x,t,z)$ is then a variational limit solution to the original equation for ε small. Namely, for small ε, applying $u(x,t,z)$ in the original equation results in a near optimal solution, as called for in requirement (ii) in the first section. Requirement (i) is then satisfied automatically, and in some cases one can establish the third requirement as well. In some cases the stabilization in (2.3) needs to be carried out with $u = u(x,t,z,s)$. Then the relation to the variational limit solution is more involved. Let us just mention that it can be obtained by fitting the fast variable s with the small parameter.

What we wish to point out in this paper is the shortcoming which results when assuming Property 2.1. Indeed, the near optimal solutions obtained via the described method, lead necessarily to a fast flow which converges to a fixed point on the fast time scale; namely, if x and t are kept fixed, the optimal fast trajectory must tend to an equilibrium. Although this is indeed a property which holds in many applications, it does not hold in general. The same drawback arises when order reduction is used to analyze singularly perturbed dynamics. The observation was made a long time ago and attempts to cope with the general case of, both, the controlled and the uncontrolled dynamics are available. See Artstein [1], Artstein and Gaitsgory [2, 3], Artstein and Vigodner [4], Gaitsgory [6, 7], Gaitsgory and Leizarowitz [8], Grammel [9]. These works are in the background of the form of a variational limit offered in the present paper.

3 The case of fast flow

To make our plan clear, we concentrate in this section on the case where, in the optimal control $(2.1)_\varepsilon$, only the fast flow is exhibited. Namely,

$$\begin{aligned} &\text{minimize } \int_a^b c(t,z,u)dt \\ &\text{subject to } \varepsilon\tfrac{dz}{dt} = g(t,z,u) \\ &z(a) = z_0 \ . \end{aligned} \tag*{$(3.1)_\varepsilon$}$$

(Note, however, that the time t is a slow time; indeed, it can be viewed as satisfying the equation $\frac{dt}{dt} = 1$.)

Applying the order reduction method to the system $(3.1)_\varepsilon$ would amount to considering the problem

$$\begin{aligned} &\text{minimize } \int_a^b c(t,z,u)dt \\ &\text{subject to } 0 = g(t,z,u) \\ &z(a) = z_0 \ . \end{aligned} \tag*{$(3.1)_0$}$$

In particular, for each t a solution $(z(t), u(t))$ of the algebraic equation has to be identified. It should satisfy Property 2.1 and must be picked in accordance with the optimization requirement as described in the previous section. The lack of a slow differential equation leads to a trivial optimization requirement, namely, for each fixed t

$$\begin{aligned} &\text{minimize } c(t,z,u) \\ &\text{subject to } 0 = g(t,z,u) \\ &(z,u) \in V(t) \ . \end{aligned} \tag*{$(3.1)_R$}$$

(Note that here the constraint set $V(x,t)$ as defined in (2.2) is a function of t only.) The point-wise minimization yields an optimal trajectory

$$(\bar{z}(t), \bar{u}(t)) \ ; \tag{3.2}$$

note, however, that the constraint on the initial condition in $(3.1)_\varepsilon$ may not be satisfied. Nevertheless, with Property 2.1 the trajectory (3.2) can be interpreted as a limit solution of $(3.1)_\varepsilon$ as described in the previous section. Indeed, at the initial time, a boundary layer phenomenon is exhibited, which drives the fast flow from the initial state z_0 to the initial state $\bar{z}(a)$ of the optimal trajectory, and then the uniform asymptotic stability keeps it around the desired optimal trajectory.

Although in most of the applications worked out so far the optimal trajectory induced by the static optimization procedure turns out continuous, it may not be continuous in the general case. Still, the description in the preceding paragraph holds, if interpreted properly. We leave out the details.

The drawback of the procedure as described, is the passage to the point-wise static problem $(3.1)_0$. The convergence to an equilibrium is an assumption which is not derived from optimality considerations. A close look at the interplay between the fast flow on the fast time scale and the optimization criterion, reveals the following. On a short time interval around a fixed slow time t, the contribution to the cost, as $\varepsilon \to 0$, is an integral on a long fast time interval. In the limit, optimality considerations would lead to an infinite horizon problem on the fast time scale as follows.

$$\begin{aligned} &\text{minimize } \lim_{T\to\infty} \tfrac{1}{T} \int_0^T c(t, z(s), u(s))ds \\ &\text{subject to } \tfrac{dz}{ds} = g(t, z(s), u(s)) \ . \end{aligned} \tag{3.3}$$

Notice that t is fixed in the problem (3.3) while the fast time is again denoted by s. The way the static problem together with Property 2.1 enter into the variational limit solution of the original problem, reflects the outcome of solving such infinite horizon problems continuously as the time t progresses. Adopting $(3.1)_0$, and consequently $(3.1)_R$, as a variational limit of $(3.1)_\varepsilon$ is equivalent to the assumption that for each fixed t the optimal solution $(z(s), u(s))$ of (3.3) tends to the equilibrium $(\bar{z}(t), \bar{u}(t))$ as $s \to \infty$. It is well known, however, that converging to an equilibrium may not be a property of an optimal solution of the infinite horizon problem (3.3).

What we show next is a way to exploit the structure of the infinite horizon problem (3.3) in order to get a general form of a variational limit for $(3.1)_\varepsilon$.

Recall the differential equation of the fast flow in (3.3), namely

$$\frac{dz}{ds} = g(t, z(s), u(s)) \ . \tag{3.4}$$

Definition 3.1. A pair $(z(\cdot), u(\cdot))$ is admissible for (3.4) if it solves the differential equation.

Definition 3.2. A probability measure μ on $R^k \times R^m$ is a limit occupational measure of (3.4) if there is an admissible pair $(z(\cdot), u(\cdot))$ of (3.4) whose distribution on $[0,T]$ converges, as $T \in \infty$, weakly to μ. Namely, for each measurable $B \subseteq R^k \times R^m$,

$$\begin{aligned} &\lim_{T\to\infty} \tfrac{1}{T} \ \lambda\{s \in [0,T] : (z(s), u(s)) \in B\} \\ &\quad \to \mu(B) \end{aligned} \tag{3.5}$$

where λ is the Lebesgue measure.

More on occupational measures and their relations to the controlled dynamics can be found in Artstein [1], Colonius and Klienman [5], Gaitsgory and Leizarowitz [8], Vigodner [13].

Note that the collection of occupational measures associated with (3.4) may vary from one slow time t to another. The rational for the following definition is that the limit occupational measure reflects the limit distribution of the fast flow, hence affects the cost via its average.

Definition 3.3. Let μ be a limit occupational measure of (3.4). The cost associated with μ is defined by

$$c(t,\mu) = \int c(t,z,u)\mu(dz \times du) \ . \tag{3.6}$$

Property 3.4. The limit occupational measure μ satisfies Property 3.4 if for an appropriate choice of a control $u = u(z)$ or $u = u(z,s)$ when plugged into (3.4), the distributions of any solution $(z(s), u(s))$ of the resulting differential equation, namely of

$$\frac{dz}{ds} = g(t, z(s), u(z,s)) \tag{3.7}$$

converge weakly (see Definition 3.2) to μ.

Observe that Property 2.1, which characterizes points in the space, reflects a particular case of Property 3.4, namely, when the point in space is identified with the measure supported on that point.

At this stage we are ready to display our generalization to the order reduction special case $(3.1)_0$-$(3.1)_R$, as follows. First, denote by $M(t)$ the limit occupational measures of (3.4), and then let

$$\begin{aligned} V_C(t) = \{\mu : \mu \in M(t) \ \text{ and} \\ \mu \ \text{ satisfies Property 3.4}\} \end{aligned} \tag{3.8}$$

(compare with (2.2); the subscript C stands for chattering). The analog of $(3.1)_0$ is as follows.

$$\begin{aligned} &\text{minimize } \int_a^b c(t,\mu(t))dt \\ &\text{subject to } \mu(t) \in V_C(t) \ . \end{aligned} \qquad (3.1)_{in}$$

The subscript *in* stands for invariant; indeed, the occupational measure reflects an invariance property of the measure. See Artstein [1], Gaitsgory and Leizarowitz [8], Vigodner [13]. Solving $(3.1)_{in}$ leads to a local pointwise optimization requirement analogous to $(3.1)_R$, namely, for each fixed t

$$\begin{aligned} &\text{minimize } c(t,\mu) \\ &\text{subject to } \mu \in V_C(t) \ . \end{aligned} \qquad (3.1)_{inl}$$

The solution $\mu(t)$ is assumed at least measurable. It may not be continuous.

The probability measures description of the limit behaviour of $(3.1)_\varepsilon$, manifested by the pair $(3.1)_{inl}$-$(3.1)_{in}$, is referred to as the chattering variational limit. It follows similar measure valued descriptions, namely the relaxed controls. See Warga [14].

We claim that the local problem $(3.1)_{inl}$ together with the global problem $(3.1)_{in}$ furnish a general variational limit form for $(3.1)_\varepsilon$. The way the solution of the pair $(3.1)_{inl}$–$(3.1)_{in}$ gives rise to an approximate solution for the original problem is very similar to the case of order reduction as was described earlier, and is as follows.

Suppose an optimal solution $\mu(t)$ of $(3.1)_{inl}$–$(3.1)_{in}$ is identified. In the special case where Property 3.4 for $\mu(t)$ can be established with the aid of a feedback control $u(t,z)$, then this control (under general conditions) establishes the properties (i)–(ii) displayed in Section 1 (and in some cases property (iii) can also be verified). In case one needs a general feedback $u(t,z,s)$ for the verification of Property 3.4, there is a need to tie the s-varied control to the small parameter exactly as in the case of order reduction; we leave out the details.

General conditions under which the method is guaranteed to produce a variational limit are as follows. (1) Optimal solutions of $(3.1)_\varepsilon$ are bounded in $R^k \times R^m$, uniformly for all ε near 0, and (2) The system (3.4) is uniformly controllable in the sense that any state z_1 can be steered to any other state z_2 in a finite time, with a bound on this finite time depending only on $|z_1 - z_2|$. We do not display here the proof of this claim, nor the fine details of how a near optimal solution can be constructed. A proof can be worked out using the technique displayed in Artstein and Vigodner [4], Artstein [1], Artstein and Gaitsgory [2, 3], where special cases were treated.

We close this section with a simple example illustrating both the need for the general notion and the way it works.

Example 3.5. Consider the optimal control problem

$$\begin{aligned} &\text{minimize } \int_0^1 ((\rho - t)^2 + \theta^2 + u^2 + v^2)dt \\ &\text{subject to } \varepsilon\tfrac{d\rho}{dt} = u \\ &\qquad \varepsilon\tfrac{d\theta}{dt} = 10 + v \\ &\qquad \rho(0) = \rho_0 \\ &\qquad \theta(0) = \theta_0 \ . \end{aligned} \qquad (3.9)_\varepsilon$$

Here ρ and θ are polar coordinated in the plane. One can use the order reduction method, and furthermore, among the strategies offered by the order reduction, there is an optimal one. Indeed, setting $v = -10$ is a necessary condition for order reduction; then an optimal control pair is easily derived using the analog of $(3.1)_R$ in the case of $(3.9)_\varepsilon$, which yields $(\rho(t), u(t)) = (t, 0)$. The resulting cost, however, is equal to 100, and it is not optimal. Indeed, employing the limit occupational measure approach yields a smaller cost. To this end notice that the trajectory $(\rho(s), \theta(s)) = (t, s)$ is an admissible trajectory, and furthermore, it satisfies Property 3.4. This is established by the local control feedbacks $u(\rho) = \rho - t$ and and $v = 0$. The resulting cost is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \theta^2 d\theta = \frac{(2\pi)^2}{3}$$

and, clearly, it is optimal. Moreover, the global control feedback $u(t, \rho) = \rho - t$ and $v = 0$ establish a near optimal solution. Hence, the correct variational limit of $(3.9)_\varepsilon$ is provided by the chattering limit.

4 Coupled slow and fast motions

The building blocks constructed in the previous section enable us to display the chattering variational limit for the general case of $(2.1)_\varepsilon$.

The first step is to determine the feasible control set $V_C(x, t)$ for the variational limit and the cost associated with each such control. The procedure follows the analogous steps in the previous section; for completeness we reproduce them here with the necessary modifications.

Recall the differential equation of the fast flow in $(2.1)_\varepsilon$, set, however, in the fast time scale as follows.

$$\frac{dz}{ds} = g(x, t, z(s), u(s)) \ . \tag{4.1}$$

Definition 4.1. A pair $(z(\cdot), u(\cdot))$ is admissible for (4.1) if it solves the differential equation.

Definition 4.2. A probability measure μ on $R^k \times R^m$ is a limit occupational measure of (4.1) if there is an admissible pair $(z(\cdot), u(\cdot))$ of (4.1) whose distribution on $[0, T]$ converges, as $T \in \infty$, weakly to μ. Namely, for each measurable $B \subseteq R^k \times R^m$,

$$\begin{aligned} &\lim\nolimits_{T\to\infty} \tfrac{1}{T}\ \lambda\{s \in [0, T] : (z(s), u(s)) \in B\} \\ &\quad \to \mu(B) \end{aligned} \tag{4.2}$$

where λ is the Lebesgue measure.

Note that the collection of occupational measures associated with (4.1) may vary from one slow time t to another and from one slow state x to another. The rational for the following definition is similar to the one concerning Definition 3.3.

Definition 4.3. Let μ be a limit occupational measure of (4.1). The cost associated with μ is defined by

$$c(x, t, \mu) = \int c(x, t, z, u)\mu(dz \times du) \ . \tag{4.3}$$

Property 4.4. The limit occupational measure μ satisfies Property 4.4 if, for an appropriate choice of a control $u = u(z)$ or $u = u(z, s)$ when plugged into (4.1), the distributions of any solution $(z(s), u(s))$ of the resulting differential equation, namely of

$$\frac{dz}{ds} = g(x, t, z(s), u(z, s)) \tag{4.4}$$

converge weakly (see Definition 4.2) to μ.

Observe again that Property 2.1, which characterizes points in the space, reflects a particular case of Property 4.4, namely, when the point in space is identified with the measure supported on that point.

At this stage we are ready to display our generalization to the order reduction special case $(2.1)_0$-$(2.1)_R$, as follows. First, denote by $M(x, t)$ the limit occupational measures of (4.1) and then let

$$\begin{aligned} V_C(x, t) = \{\mu : \mu \ \in M(x, t) \ \ &\text{and} \\ \mu \ \ &\text{satisfies Property 4.4}\} \end{aligned} \tag{4.5}$$

(compare with (2.2) and with (3.8)). Next, we have to determine how the limit occupational measure of the fast flow affects the differential equation of the slow flow. As in the case of identifying the cost (Definitions 3.3, 4.3), the limit effect of the fast flow on the slow flow is via averaging, as follows.

Definition 4.5. Let μ be a probability measure on $R^k \times R^m$. Define

$$f(x, t, \mu) = \int f(x, t, z, u)\mu(dz \times du) \ . \tag{4.6}$$

With the aid of Definition 4.5 we can formulate the analog of $(2.1)_R$, as follows.

$$\begin{aligned} \text{minimize } &\textstyle\int_a^b c(x, t, \mu(x, t))dt \\ \text{subject to } &\tfrac{dx}{dt} = f(x, t, \mu) \\ &x(a) = x_0 \\ &\mu(x, t) \in V_C(x, t) \ . \end{aligned} \qquad (4.7) = (2.1)_{in}$$

Again, the subscript *in* stands for invariant. This time, unlike the x-independent case of the previous section, the problem cannot be reduced to a local one. Still, once the constraint set $V_C(x, t)$ is determined, the problem (4.7) is a standard optimal control problem to which available techniques can be applied. Once an optimal control $\mu(t)$ or $\mu(x, t)$ is identified, it can be made into a near optimal solution in the way described in the preceding section.

To solve (4.7) one should apply the techniques available in the literature, for instance the Lagrange multipliers, the Hamilton-Jacobi-Bellman equation, etc. In this respect the examples worked out in Artstein and Gaitsgory [2, 3] serve as particular cases of the general problem (4.7).

We close the paper with a variant of the example of the previous section, again illustrating both the need for our approach and the way it works.

Example 4.6. Consider the optimal control problem

$$\begin{aligned} &\text{minimize } \int_0^1 (x^2 + (\rho - x)^2 + \theta^2 + u^2 + v^2 + w^2)dt \\ &\text{subject to } \frac{dx}{dt} = x + \theta + w \\ &\qquad \varepsilon \frac{d\rho}{dt} = u \\ &\qquad \varepsilon \frac{d\theta}{dt} = 10 + v \\ &\qquad x(0) = 1 \\ &\qquad \rho(0) = \rho_0 \\ &\qquad \theta(0) = \theta_0 \ . \end{aligned} \tag{4.8$_\varepsilon$}$$

Again, ρ and θ are polar coordinated in the plane. There are many limit occupational measures for each fixed (x, t). A close look, however, reveals that the trajectory $(\rho(s), \theta(s)) = (x, 10s)$ is an admissible trajectory, and furthermore, it satisfies Property 4.4. This is established by the local control feedback $u(\rho) = \rho - x$ and and $v = 0$. The resulting contribution to the cost is equal to $\frac{(2\pi)^2}{3}$. It is easy to see that all other alternatives involve a control v greater or equal to 10, which would result in a larger cost. Hence, when the reduction is made, the relevant control problem is

$$\begin{aligned} &\text{minimize } \int_0^1 (x^2 + \frac{(2\pi)^2}{3} + w^2)dt \\ &\text{subject to } \frac{dx}{dt} = x + w \\ &\qquad x(0) = 1 \end{aligned} \tag{4.8$_R$}$$

and it is the correct variational limit of $(4.8)_\varepsilon$. The number appearing in the cost functional is due to the limit occupational measure. Applying the order reduction would not yield a correct result.

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