Abnormal Extremals for Generic Planar Systems

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1 Introduction

In this paper we deal with the minimum time stabilization problem to the origin for the planar single-input system $\dot{x} = F(x) + uG(x)$, $x \in \mathbb{R}^2$, $u \in [-1,1]$, $F(0) = 0$. In [1] the structure of the set of extremals was studied. Generically all extremals are finite concatenations of regular arcs that are bang or corresponds to a smooth feedback. Moreover the support of extremals is a Whitney stratified set. These information was collected in the definition of *extremal synthesis* and it was proved that a structurally stable extremal synthesis exists. Moreover the projection of the support of extremals was studied. After normalization of the covector, one obtains a two dimensional stratified set in $\mathbb{R}^2 \times S^1$. The projection singularities can be classified in topological sense. Beside the classical folds and cusps, new singularities appears. Some, called vertical, are due to the fact that the target (the origin) is of codimension two, while others are stable and independent of the target properties. These new singularities are called bifold (see figure 2) and ribbon (see figure 3). In particular the ribbon singularity can appear only along abnormal extremals of the problem. The latter are extremal pairs corresponding to the zero level of the Hamiltonian given by the Pontryagin Maximum Principle.

The aim of this paper is to analyze the generic properties of abnormal extremals. We prove that these are finite concatenations of bang arcs (that is corresponding to constant ±1 control). Moreover, the switchings (discontinuity points of the control) happens exactly when the abnormal extremal crosses the set of zeroes of the function $\Delta_A = F \wedge G$. All possible generic singularities of the synthesis on the plane occurring along (projections of) abnormal extremals are classified in [1]. However, one has to singularities indeed appears for some generic system, in particular those corresponding to the new projection singularities, namely bifold and ribbon. The set of possible singularities is formed of 28 (equivalence classes of) singular points, but not all sequences of singularities can be realized. We are able to prove that the generic sequences of singularities along abnormal extremals can be classified with a recognizable set of words. As a by product, we obtain the existence of systems presenting singular points corresponding to projection singularities of ribbon type.

2 Statement of the Main Result

Let Ξ be the set of all couples of $\mathcal{C}^\infty$ vector fields (F,G) such that the origin is an equilibrium point for F that is $F(0) = 0$. For every $(F,G) \in \Xi$ we consider the minimum time stabilization problem to the origin for the control system:

$$\dot{x} = F(x) + uG(x), \qquad x \in \mathbb{R}^2, \quad u \in [-1,1]. \qquad (1)$$

Reversing time we can consider the equivalent problem of reaching every point of the plane in minimum time from the origin. The well known Pontryagin Maximum Principle (see [5]) in this special case states the following. Define for every $(x,p,u) \in \mathbb{R}^2 \times (\mathbb{R}^2)_* \times [-1,1]$, where $(\mathbb{R}^2)_*$ is the set of row vectors $\mathcal{H}(x,p,u) = p \cdot F(x) + u\, p \cdot G(x)$ and $H(x,p) = \max\{p \cdot F(x) + u\, p \cdot G(x) : u \in [-1,1]\}$. If $\gamma : [0,a] \to \mathbb{R}^2$ is a (time) optimal trajectory corresponding to a control $u : [0,a] \to [-1,1]$, then there exist a non trivial *field of covectors along* γ, that is a function $\lambda : [0,a] \to (\mathbb{R}^2)_*$ never vanishing, and a constant $\lambda_0 \leq 0$ such that **i)** $\dot{\lambda}(t) = -\lambda(t) \cdot (\nabla F + u(t)\nabla G)(\gamma(t))$, **ii)** $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$ for almost every $t \in Dom(\gamma)$, **iii)** $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$ for almost every $t \in Dom(\gamma)$. In this case we say that the pair (γ, λ) is extremal. If γ is optimal we say that the pair (γ, λ) is optimal.

Definition 1 *Let (γ, λ) be an extremal pair. If the corresponding Hamiltonian satisfies $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0$ for almost every $t \in Dom(\gamma)$ we say that (γ, λ) is an abnormal extremal.*

The following Theorem, proved in Section 5, describes all the sequences of singularities of abnormal extremals.

Theorem 1 *Under generic assumptions on $(F,G) \in \Xi$, the set of abnormal extremals for the control problem (1) can be classified through a set of words recognizable by an automaton.*

3 Basic Notations

In the following we recall some notation and results of [3, 4]. In [3] it was proved that generically, every extremal trajectory is a concatenation of trajectories corresponding to controls +1, -1 or to a feedback $\varphi(x)$ (called singular) that depends on F, G and their Lie brackets. For later use we define the function: $\Delta_A(x) = F(x) \wedge G(x) = F_1(x)G_2(x) - F_2(x)G_1(x)$. Let $\tau > 0$. We call *reachable set* within time τ the set $\mathcal{R}(\tau) := \{x \in \mathbb{R}^2$: there exists $t \in [0, \tau]$ and a trajectory $\gamma : [0, \tau] \to \mathbb{R}^2$ of (1) such that $\gamma(0) = 0$, $\gamma(t) = x\}$. As in [1], in the following we consider a generic τ. We call $\gamma^\pm : [0, t_f^\pm] \to \mathbb{R}^2$ the extremal trajectories that originate from 0 and correspond to constant control ± 1 where $t_f^\pm$ are the last times in which $\gamma^\pm$ are extremal (if they are less than τ) or τ (otherwise). In [3] it was shown that under generic assumptions (see P1–P9 in [3]) every extremal trajectory exits the origin with constant control $+1$ or -1. Fix $\bar{x} \in \mathcal{R}(\tau)$ and an extremal trajectory $\bar{\gamma} : [0, \bar{a}] \to \mathcal{R}(\tau)$ such that $\bar{\gamma}(0) = 0$ and $\bar{\gamma}(\bar{a}) = \bar{x}$ and define the function: $K_\varepsilon^{\bar{x}, \bar{\gamma}}(x) := \sharp\{$ extremal trajectory $\gamma :\ \gamma(0) = 0,\ \gamma(a) = x,\ |\gamma(t) - \bar{\gamma}(t)| < \varepsilon \forall\ t \in [0, min(a, \bar{a})],\ |a - \bar{a}| < \varepsilon\}$, where $\sharp$ denotes the cardinality of a set. In [1] it is given the following key definition (see figure 1):

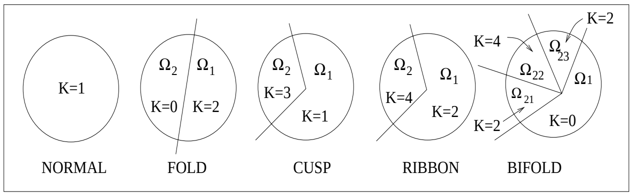


Figure 1

Definition 2 *Fix $\bar{x} \in \mathcal{R}(\tau)$ and an extremal trajectory $\bar{\gamma} : [0, \bar{a}] \to \mathcal{R}(\tau)$ such that $\bar{\gamma}(0) = 0$ and $\bar{\gamma}(\bar{a}) = \bar{x}$. We say that $\bar{x}$ is a normal point along $\bar{\gamma}$ if for ε sufficiently small there exists a neighborhood U of $\bar{x}$ such that $K_\varepsilon^{\bar{x}, \bar{\gamma}}(U) = 1$. We say that $\bar{x}$ is a fold point along $\bar{\gamma}$ if there exists a one dimensional piecewise-$\mathcal{C}^1$ manifold l, with $\bar{x} \in l$, satisfying the following. For ε sufficiently small there exists a neighborhood U of $\bar{x}$ divided by l into two connected components Ω_1, Ω_2 such that $K_\varepsilon^{\bar{x}, \bar{\gamma}}(\Omega_1) = 0$, $K_\varepsilon^{\bar{x}, \bar{\gamma}}(\Omega_2) = 2$, $K_\varepsilon^{\bar{x}, \bar{\gamma}}(l) = 1$. Similarly if the values of $K_\varepsilon^{\bar{x}, \bar{\gamma}}$ on the regions neighboring $\bar{x}$ are 1–3 (respectively 4–2, 0-2-4-2) we we say that $\bar{x}$ is a cusp point (respectively bifold, ribbon) along $\bar{\gamma}$.*

In [1] it is proved that the *ribbon* singularity may occur only along abnormal extremals. In Section 5 we prove that the singularity is in fact realized. In [3, 4] it was proved that the reachable set within time τ is a stratified subset of $\mathbb{R}^2$ and the one and zero dimensional strata are called respectively *Frame Curves* and *Frame Points* (in the following briefly FCs and FPs). Moreover, the authors use the letters X, Y, C, S, K, to indicate respectively the FCs corresponding to subset of $Supp(\gamma^+)$, subsets of $Supp(\gamma^-)$, curve made of switching points, singular trajectories and overlaps, that is curves made of points reached optimally by two distinct trajectories.

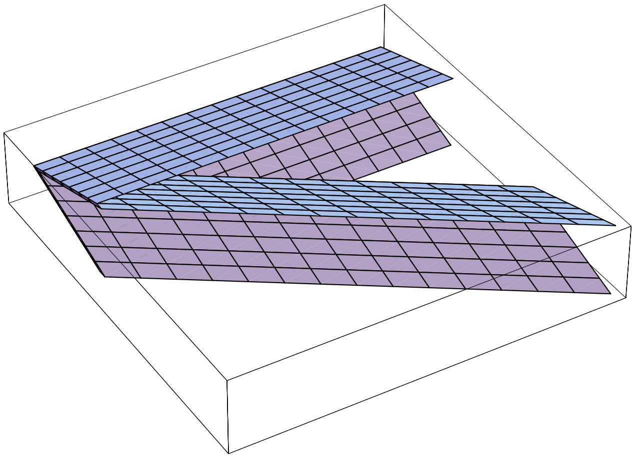


Figure 2: Bifold Singularity

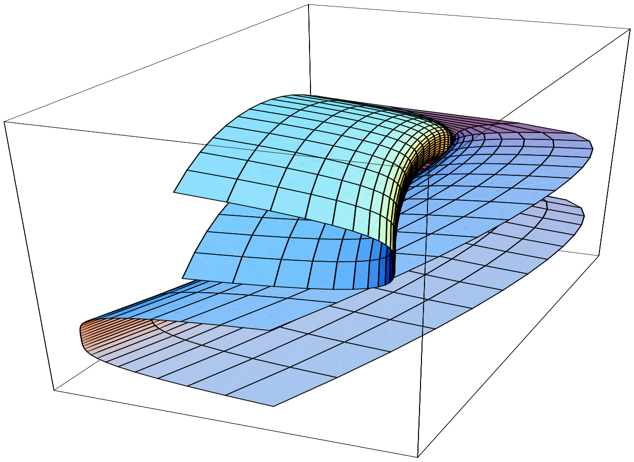


Figure 3: Ribbon Singularity

In [4], it was proved that these are generically all possible FCs. While a FP x that is intersection of two FCs F_1 and F_2 is called a (F_1, F_2) Frame Point. In [1] it was shown that considering the extremal trajectories instead of the optimal ones we do not have any K Frame Curve but we have the following new Frame Curves: a FC called $\bar{C}$ made of switching points on which X and Y points to opposite sides; a FC called W that is an arc of an extremal trajectory characterized by the fact that all its points are fold points; a FC called γ_0 that is an arc of an extremal trajectory that “transports” some special information e.g. it switches every time it meets the locus $\Delta_A^{-1}(0)$ or it evolves into a W FC. More details on the Frame Curves W and γ_0 are given below. Let Γ be the set of all the extremal trajectories for the control problem (1) up to time τ. In [1] it was defined an algorithm that, under generic assumptions, constructs the whole set Γ. First it constructs all the extremal trajectories in a neighborhood of $\gamma^{\pm}$. Then the set of extremal trajectories is subdivided into "strips". Each strip is a one dimensional continuous parametric family of extremal bang–bang trajectories having the same switching strategy. Then the evolution of a strip is studied. In order to do that, the evolution of the boundary of strips are analyzed separately. The more delicate case is that of abnormal extremals that here we treat in more details. The evolution of the interior of a strip may create new strips and cause the subdivision of each strip in smaller strips. In [1] it was proved that the algorithm constructs Γ as a finite union of strips. The set of borders of strips is called $\partial\Gamma$. In [1] it was proved that under generic assumptions, $\partial\Gamma$ is a finite set. The algorithm produces the FCs in the following way: the FCs of kind C and $\bar{C}$ lye in the interior of the strips; the borders of strips contain FCs of kind W, γ_0. In the next Section we see that in the case of abnormal extremals we need a more fine definition for FCs of kind W. Given an extremal trajectory γ, let us define $v^{\gamma}(v_0, t_0; t)$ to be the solution to the Cauchy problem: $\dot{v^{\gamma}}(v_0, t_0; t) = (\nabla F + u(t)\nabla G)(\gamma(t)) \cdot v^{\gamma}(v_0, t_0; t)$, $v^{\gamma}(v_0, t_0; t_0) = v_0$, where $u(t)$ is the control corresponding to γ. In [3], denoting $\bar{v}^{\gamma}(t) := v^{\gamma}(G(\gamma(t)), t; 0)$ it was defined the function: $\theta^{\gamma} : Dom(\gamma) \to [-\pi, \pi]$, $\theta^{\gamma}(t) := arg(\bar{v}^{\gamma}(0), \bar{v}^{\gamma}(t))$, where arg is the angle measured counterclockwise. In the following we use the notation $\theta^{\pm}(t) := \theta^{\gamma^{\pm}}(t)$. In [1] the following times were singled out: $s_1^+ := \min\{t \in]0, t_f^+] : \theta^+(t) = 0, \dot{\theta}^+(0) < 0\}$, $s_1'^+ := \min\{t \in]0, t_f^+] : \theta^+(t) = 0, \dot{\theta}^+(0) > 0\}$. Similarly the times s_1^-, $s_1'^-$ were defined.

4 Properties of Abnormal Extremals

In this section we state the main results about the switching strategies of abnormal extremals.

Definition 3 *Let $\gamma : [0, \tau] \to \mathcal{R}(\tau)$ be (the first component of) an abnormal extremal for the control problem (1) such that it switches at least one time and let t_1 be its first switching time. We refer to the couples (γ, t_1) as Non Trivial Abnormal Extremal (in the following NTAE). By definition a NTAE is maximal if defined on $[0, \tau]$.*

Definition 4 *Let $\gamma : [0, \tau] \to \mathbb{R}^2$ be a NTAE, and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. We set $AA(i) = Supp\left(\gamma|_{[t_i, t_{i+1}]}\right)$ $(i = 1, ..., n(\gamma) - 1)$ and we call it an abnormal arc.*

Theorem 2 *Let $\gamma : [0, \tau] \to \mathbb{R}^2$ be an extremal trajectory for the control problem (1) such that it switches at least one time, $\lambda : [0, \tau] \to \mathbb{R}_2$ the corresponding covector and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Then $\lambda(.)$ is unique (up to the multiplication by a positive constant) and under generic assumptions the following conditions are equivalent:* **(a)** *(γ, t_1) is a NTAE;* **(b)** *$\gamma(t_i) \in \Delta_A^{-1}(0)$ for some $i \in \{1, ..., n(\gamma) - 1\}$;* **(c)** *$\gamma(t_i) \in \Delta_A^{-1}(0)$ for each $i \in \{1, ..., n(\gamma) - 1\}$;* **(d)** *$\gamma(\bar{t}) \in \Delta_A^{-1}(0)$ $(\bar{t} \in Dom(\gamma))$ iff $\bar{t} = t_i$ for some $i \in \{1, ... n(\gamma) - 1\}$. Moreover, under generic assumptions,* **(a)** *(or equivalently* **(b)** *or* **(c)** *or* **(d)***) implies that for every interval $[a, b] \subset [0, \tau]$ $(a < b)$, γ does not correspond to the singular control φ.*

From Theorem 2 and the definition of strip, it follows that if (γ, t_1) is a NTAE then $\gamma \in \partial\Gamma$. Then it is natural to define $\partial\Gamma_A = \{(\gamma, t_1) : \gamma \in \partial\Gamma, t_1 \in \mathbb{R}^+, (\gamma, t_1)$ is a NTAE$\}$.

Definition 5 *Let $x \in \Delta_A^{-1}(0)$, then $(F + G)(x) = \alpha(F - G)(x)$. If $\alpha > 0$ (resp. $\alpha < 0$) we say that at x, $\Delta_A^{-1}(0)$ is direct (resp. inverse).*

Recalling Definitions 2, 4, from Theorem 2 it follows that $AA(i)$ can be only of kind γ_0 or W, but (in the case of abnormal extremals) a more fine definition for the strip borders of kind W is necessary.

Definition 6 *Let $\gamma \in \partial\Gamma_A$, and suppose that it corresponds to a constant control (say +1) in the interval $]b, c[$ $(0 < b < c \leq \tau)$. Let $\mathcal{S}^1$ and $\mathcal{S}^2$ be the two strips such that $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$ and suppose that in the interval $]b, c[$ γ is a strip border of kind W.*

- *We say that in $]b, c[$ γ is a strip border of kind W^C if $\mathcal{S}^1$ and $\mathcal{S}^2$ lie both on the right (resp. on the left) of $\gamma|_{]b,c[}$ and X points to the right (resp. to the left) of $\gamma|_{]b,c[}$ at every point of $supp(\gamma|_{]b,c[})$.*
- *We say that in $]b, c[$ γ is a strip border of kind W^D if $\mathcal{S}^1$ and $\mathcal{S}^2$ lie both on the right (resp. on the left) of $\gamma|_{]b,c[}$ and X points to the left (resp. to the right) of $\gamma|_{]b,c[}$ at every points of $supp(\gamma|_{]b,c[})$.*

If in the interval $]b, c[$ γ is not a strip border of kind W we say that $\gamma|_{]b,c[}$ is a strip border of kind γ_0.

Theorem 3 *Let $\gamma : [0, \tau] \to \mathbb{R}^2$ be an extremal trajectory for the control problem (1) such that it switches at least one time, θ^{γ} the corresponding function (see above) and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_n(\gamma) := \tau$ the sequence of switching times. . Then under generic assumptions the following conditions are equivalent:* **(a)** *(γ, t_1) is a NTAE;* **(b)** *$\theta^{\gamma}(t_i) \in \{0, \pm\pi\}$ for some $i \in \{1, ..., n(\gamma) - 1\}$;* **(c)** *$\theta^{\gamma}(t_i) \in \{0, \pm\pi\}$ for each $i \in$*

$\{1, ..., n(\gamma) - 1\}$; **(d)** *$\theta^\gamma(\bar{t}) \in \{0, \pm\pi\}$, $(\bar{t} \in Dom(\gamma))$ iff $\bar{t} = t_i$ for some $i \in \{1, ...n(\gamma) - 1\}$.*

Theorem 4 *Let (γ, t_1) be a NTAE and $0 =: t_0 < t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Suppose that for some $i \in \{0, 1, ..., n(\gamma) - 1\}$, $\Delta_A^{-1}(0)$ is inverse at the points $\gamma(t_i)$, $\gamma(t_{i+1})$. Then $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i)$.*

From Theorems 2 and 3, using the definitions of the times s_1^+, $s_1'^+$, t_f^+, s_1^-, $s_1'^-$, t_f^-, it follows:

Corollary 1 *Let γ be an extremal trajectory exiting the origin with control $+1$, then its first switching can occur on $\Delta_A^{-1}(0)$ only if $s_1 \neq 0$ (cond. A) or $s_1' \neq 0$ (cond. B) or $|\theta^+(t_f^+)| = \pi$ (cond. C). Moreover at most one of the conditions A, B, C holds and the corresponding time is the first switching time of γ and the first time at which γ^+ intersect $\Delta_A^{-1}(0)$. A similar result holds for γ^- and for the times s_1^-, $s_1'^-$, t_f^-.*

The following two Theorems describe the position of the switching curves of the strips whose borders are abnormal extremals.

Theorem 5 *Let (γ, t_1) be a NTAE, t_i and t_{i+1} two consecutive switching times, $\mathcal{S}$ a strip such that $\gamma \in \partial S$ and U^i, U^{i+1} two small neighborhoods of $\gamma(t_i)$ and $\gamma(t_{i+1})$. Moreover let U_{in}^i and U_{out}^i (resp. U_{in}^{i+1}, U_{out}^{i+1}) be the two connected components of $U^i \setminus \Delta_A^{-1}(0)$ (resp. $U^{i+1} \setminus \Delta_A^{-1}(0)$) chosen in such a way that γ enters U_{in}^i (resp. U_{in}^{i+1}). Under generic conditions we have the following cases:*

(1) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.*
In this case if the switching locus of $\mathcal{S}$ passing trough $\gamma(t_i)$ lies in U_{in}^i (resp. U_{out}^i) then the switching locus of $\mathcal{S}$ passing trough $\gamma(t_{i+1})$ lies in U_{out}^{i+1} (resp. U_{in}^{i+1}).

(2) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.*
In this case if the switching locus of $\mathcal{S}$ passing trough $\gamma(t_i)$ lies in U_{in}^i (resp. U_{out}^i) then the switching locus of $\mathcal{S}$ passing trough $\gamma(t_{i+1})$ lies in U_{in}^{i+1} (resp. U_{out}^{i+1}).

(3) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.*
In this case we have the same conclusion of case **(2)**.

(4) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.*
In this case we have the same conclusion of case **(1)**.

Theorem 6 *Let (γ, t_1) be a NTAE and let $\mathcal{S}^1$ and $\mathcal{S}^2$ be two stripes such that $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$. Let $\bar{t}$ be a switching time for γ and U a small neighborhood of $\gamma(\bar{t})$ such that $U \setminus \Delta_A^{-1}(0)$ has two connected components U_{in} and U_{out}, chosen in such a way γ enter U from U_{in}. Then, under generic conditions, the switching loci of $\mathcal{S}^1$ and $\mathcal{S}^2$ passing trough $\gamma(\bar{t})$ satisfy the following:* **(a)** *they lye both in U_{in} or both in U_{out};* **(b)** *they are tangent to supp(γ) in $\gamma(\bar{t})$.*

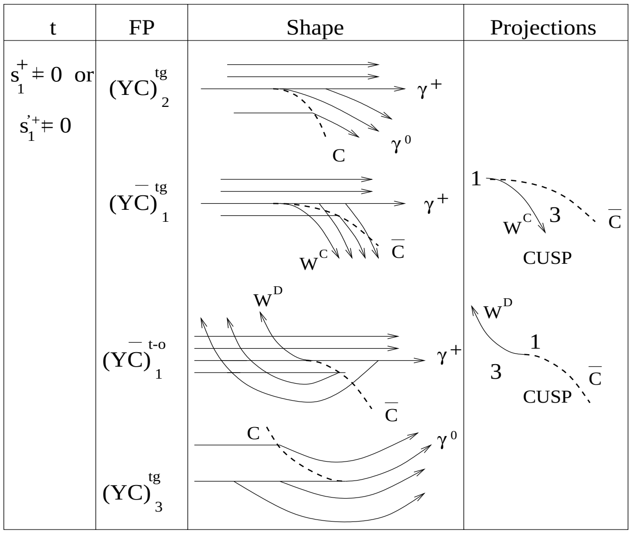


Figure 4

5 Classification of Abnormal Extremals

In [1] a complete classification of the singularities of the extremal synthesis is realized. The singularities involving abnormal extremals along $\gamma^+ \cup \gamma^-$ (respectively γ_0, $W^{C/D}$) are showed in Figure 4 (respectively 5,6). The shape of the singularities involving abnormal are determined by: **i)** $\Delta_A^{-1}(0)$ direct or inverse; **ii)** switching in U_{in} or U_{out} according to Theorem 5; **iii)** all the *essentially different* direction of the exiting abnormal trajectory. Let (γ, t_1) be a NTAE and suppose that its switching times are $t_1 < t_2 < ... < t_n = \tau$. In the following we describe all the possible sequence of singularities of γ by a *set of word* recognizable by an *Automaton*. For details on automata and languages theory we refer to [2].

Definition 7 *Let Σ be a finite set and consider the set Σ^* of ordered n−tuples $s = (\sigma_1, ..., \sigma_k)$, $\sigma_i \in \Sigma$ $(i = 1, ...k)$, $k \geq 0$. We say that $\sigma \in \Sigma$ is a* letter, *Σ is an* alphabet *and $s = (\sigma_1, ..., \sigma_k) \in \Sigma^*$ is a* word of length *k.*

Definition 8 *Let Σ be a finite* alphabet, *an* automaton *$\mathcal{A}$ over Σ consists of a finite set of* states *$\mathbb{S} = \{S^1, ..., S^n\}$, a set of* initial states *$\mathbb{I} \subseteq \mathbb{S}$, a set of* terminal states *$\mathbb{T} \subseteq \mathbb{S}$, a set of* edges *that is a subset*

$\mathbb{E} \subseteq \mathbb{S} \times \Sigma \times \mathbb{S}$. *An* edge *is indicated as* (S_1, σ, S_2) *and we say that it begins at* S_1*, it ends at* S_2 *and it carries the label* σ*.*

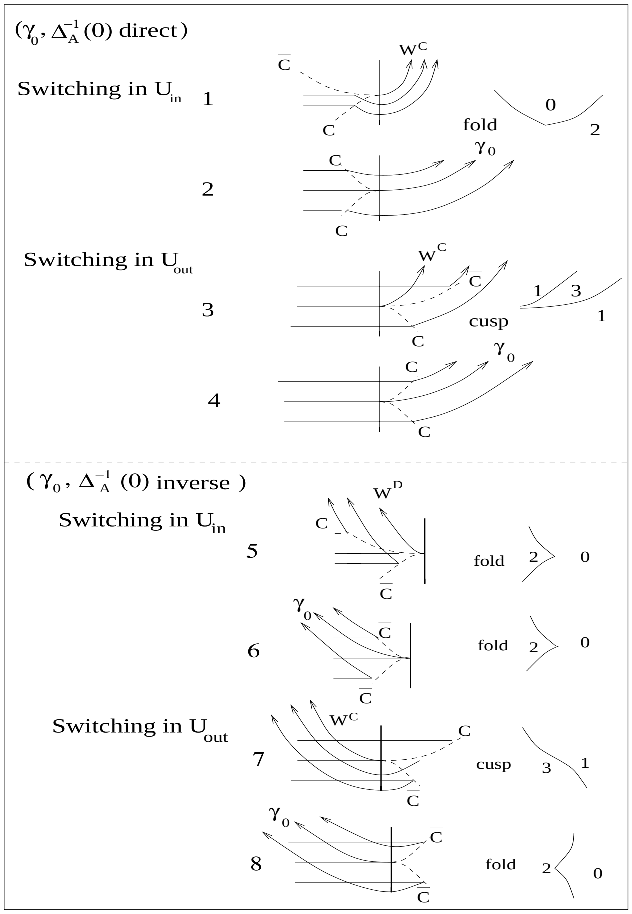


Figure 5

A *path* in $\mathcal{A}$ is a finite sequence of edges of the type $(S_1, \sigma_1, S_2)(S_2, \sigma_2, S_3)...(S_k, \sigma_k, S_{k+1})$. If $S_1 \in \mathbb{I}$ and $S_{k+1} \in \mathbb{T}$ we say that the *path* is successful.

Definition 9 *A set of words* $\Omega \subset \Sigma^*$ *is said to be recognizable by* $\mathcal{A}$ *if for every word* $(\sigma_1, \sigma_2, ..., \sigma_m) \in \Omega$ *of length* m *there exists* $S_1, ..., S_{m+1} \in \mathbb{S}$ *such that:* $(S_i, \sigma_i, S_{i+1}) \in \mathbb{E}$ *for every* $i = 1, ..., m$, $(S_1, \sigma_1, S_2)(S_2, \sigma_2, S_3)...(S_m, \sigma_m, S_{m+1})$ *is a successful path.*

To prove Theorem 1 amounts precisely to construct an *automaton* describing all the possible sequences of singularities along a NTAE. First we build an automaton, naturally associated to a system, with the simplest possible set of edges. By this automaton we can prove that the ribbon singularity is realized, but more than one sequence of singularities may correspond to a recognizable word. Then we describe how to build a more complicated automaton that has the required property that is to every recognizable word it corresponds one and only one sequence of singularities.

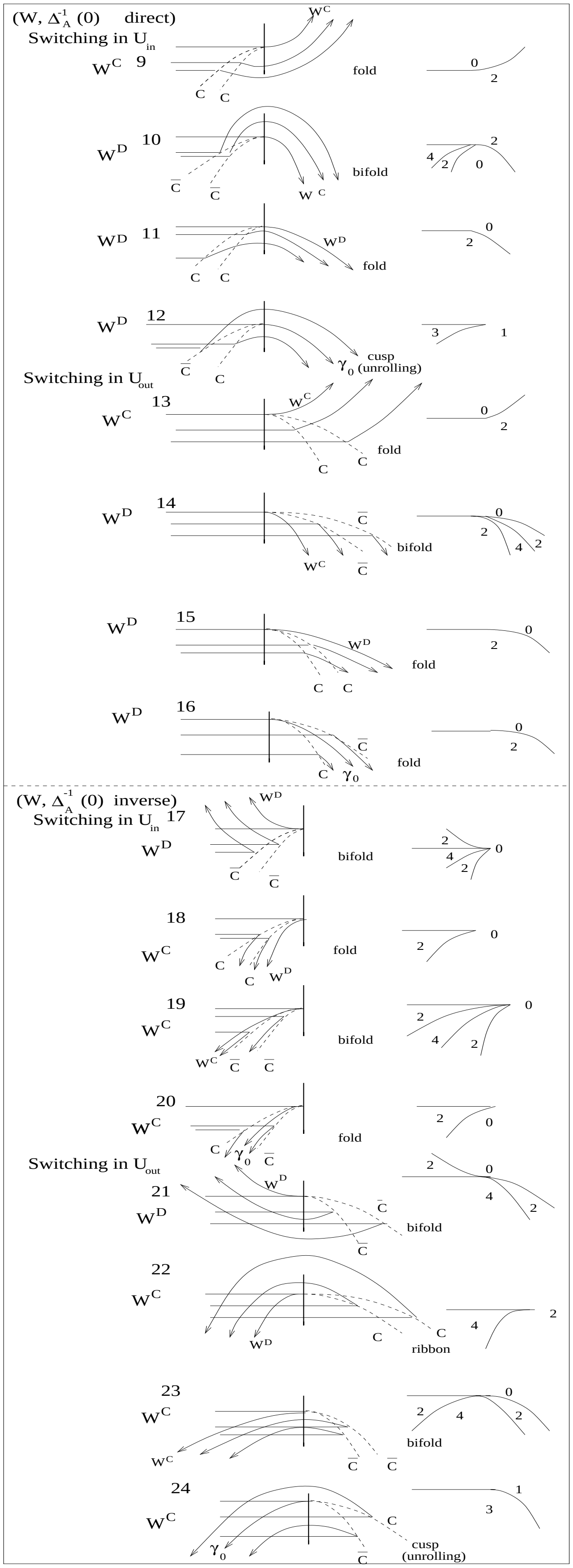


Figure 6

For us the set of *states* is the set of the 28 singularities: $\mathbb{S} := \{(YC)_2^{tg}, (Y\bar{C})_1^{tg}, (Y\bar{C})_1^{t-o}, (YC)_3^{tg}, 1 \div 24\}$ (see [1] for details) and the alphabet is $\mathbb{E} := \{0, \pi\}$ that is (if we are considering two singularities at t_i and t_{i+1}) the set of values assumed by the function $\Delta\theta_i^\gamma := |\theta^\gamma(t_{i+1}) - \theta^\gamma(t_i)|$. The set of *initial states* is constituted by the singularities $(YC)_2^{tg}, (Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}, (YC)_3^{tg}$ and the set of *terminal states* coincides with $\mathbb{S}$. Using Theorems of Section 4 it is easy to write Table A that show how the *edges* connect the *states*. For example from the *state* (singularity) 18, and the *letter* π we may reach the states $14, 15, 16$. This means that the edges of $\mathbb{E}$ that begin at the state 18 are: $(18, \pi, 14), (18, \pi, 15), (18, \pi, 16)$. It is clear that for this automaton every word is recognizable. From Table A we have:

letters→ *states* ↓	0	π
$(YC)_2^{tg}$	1,2,5,6	3,4,7,8
$(Y\bar{C})_1^{tg}$	9,18,19,20	13,22,23,24
$(Y\bar{C})_1^{t-o}$	14,15,16,21	10,11,12
$(YC)_3^{tg}$	1,2,5,6	3,4,7,8
1	13,22,23,24	9,18,19,20
2	3,4,7,8	1,2,5,6
3	9,18,19,20	13,22,23,24
4	1,2,5,6	3,4,7,8
5	10,11,12,17	14,15,16
6	1,2,5,6	3,4
7	13,22,23,24	9
8	3,4,7,8	1,2
9	13,22,23,24	9,18,19,20
10	13,22,23,24	9,18,19,20
11	14,15,16,21	10,11,12,17
12	3,4,7,8	1,2,5,6
13	9,18,19,20	13,22,23,24
14	9,18,19,20	13,22,23,24
15	10,11,12,17	14,15,16,21
16	1,2,5,6	3,4,7,8
17	10,11,12,17	14,15,16
18	10,11,12,17	14,15,16
19	9,18,19,20	13
20	1,2,5,6	3,4
21	14,15,16,21	10,11,12
22	14,15,16,21	10,11,12
23	13,22,23,24	9
24	3,4,7,8	1,2

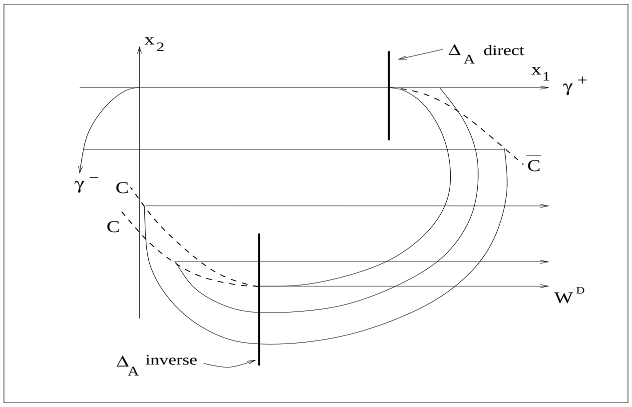
Table A

Theorem 7 *All the states* $1 \div 24$ *can be reached with at most two edges. More precisely only the singularity 17 need in fact two edges. Moreover the singularity number 22 (the ribbon) can be realized with the edge* $((Y\bar{C})_1^{tg}, \pi, 22)$.

To describe in a unique way the set of singularities, with a set of words recognizable by an automaton, we need to include more information in the alphabet. First of all we need to give a name to the entry arrows (e.g. I_1, I_2, I_3, I_4, corresponding respectively to the singularities $(YC)_2^{tg}$, $(Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}$, $(YC)_3^{tg}$). Then we have to introduce more information in the letters (i.e. we need a bigger alphabet). One way is to include the following data relatively to the next singularity: **i)** if $\Delta_A^{-1}(0)$ is direct or inverse (that we indicate with the letters D and I in the following), **ii)** the kind of exiting abnormal arc (i.e. γ_0, W^C, W^D). In this way for the new automaton we have $\mathbb{S}' = \mathbb{S}$ and $\Sigma' = \{0, \pi\} \times \{D, I\} \times \{\gamma_0, W^C, W^D\}$. Every element of Σ' is indicated as a triple (. , . , .). Clearly for this new automaton to every word it corresponds a unique sequence of singularities. Theorem 1 is proved. An explicit example of system that originates a ribbon is the following (choosing properly the functions $F_1(x_2)$ and $F_2(x_2)$):

$$\begin{cases} \dot{x}_1 = \frac{1}{2}\left((1+u) + (1-u)\left(-1 + \frac{3}{2}x_1 + F_1(x_2)\right)\right) \\ \dot{x}_2 = \frac{1}{2}(1-u)\left(\frac{1}{2}x_1(x_1 - 1)(x_1 - \frac{5}{4}) + F_2(x_2)\right. . \end{cases}$$

In particular it is generated at the second switching time from the abnormal extremal exiting the origin with control +1.



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