Sequential Continuous Time Adaptive Control: A Behavioral Approach

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Abstract

We want to design a compensator for a behavior through an appropriate behavioral interconnection. The problem is that the behavior that we want to control is not known. All that is given is a desired interconnected behavior and the prior information that this desired behavior can indeed be achieved by means of regular interconnection. This problem calls for an adaptive flavored strategy. The strategy that we propose is as follows. Measurements are taken during successive time intervals of unit length. Each time a measurement is taken the Most Powerful Unfalsified Model of that measurement and the desired behavior is determined. Since this model contains the desired behavior it is possible to find additional constraints such that the desired behavior is achieved. Moreover these additional constraints can be chosen such that the corresponding interconnection is regular relative to the true unknown behavior. This regularity property makes it possible to invoke the additional constraints in the next time interval by incorporating a transient period. The new measurement therefore satisfies these additional constraints. The procedure is repeated for the new measurement and so on. The main result is that within a finite, though unknown, number of measurements the new measurements are constrained to the desired behavior.

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1 Introduction

The behavioral approach to (linear) systems theory provides elegant tools for controller design and modeling. The present paper deals with the combination of the two, thus leading to an adaptive control system described in behavioral terms. The paper is a direct follow up of [1, 2] where a first attempt was made to approach adaptive control from a behavioral point of view. Here we propose a more realistic setup in that the assumption, made in [1, 2], that successive measurements could be made during the whole times axis $\mathbb{R}$, is now replaced by the situation where a single trajectory is observed. Modeling and controller design is sequentially carried out during successive intervals of time. The main result is that within a finite number of iterations the desired behavior is achieved.

The paper is organized in two sections. In the next section the setup is discussed and a general theorem concerning achievability of controlled behaviors is formulated. The subsequent section deals with the precise iterative scheme and the analysis thereof.

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2 The general setup

The behavior that we want to control is of the form

$$\mathfrak{B} = \{(w, c) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^d) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})c\} \tag{1}$$

The matrices of polynomials $R(\xi)$ and $M(\xi)$ belong to $\mathbb{R}^{\cdot\times q}[\xi]$ and $\mathbb{R}^{\cdot\times d}[\xi]$ respectively. At a later stage they will be assumed to be unknown. The class of behaviors of the form (1) is denoted by $\mathcal{L}$. The variable c is the control variable and w is the to-be-controlled variable. Components of w may be components of c and vice versa. A compensator is a set of laws that restrict the interconnection variable c and therefore also w. It is represented by a polynomial matrix $C(\xi)$ of appropriate dimensions. The compensated behavior is then given by

$$\mathfrak{B}_{\mathrm{c}} = \{(w, c) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^d) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})c \quad C(\frac{d}{dt})c = 0\} \tag{2}$$

Of course in a concrete control problem, the matrix $C(\xi)$ depends on $R(\xi)$ through a *control objective*. The control objective, in some sense, declares certain trajectories acceptable and others unacceptable. Therefore we can view a control objective as a rule that assigns to every behavior a sub-behavior of acceptable trajectories. The definition of control as just the interconnection of two behaviors through dedicated channels may be too general for applications. In [5] two special types of interconnection are defined. The first is *regular* interconnection and the second is *regular feedback* interconnection. Notice the subtle difference in terminology. Roughly speaking, an interconnection is regular if the laws of the controller are algebraically independent of the laws of the system to be controlled. More precisely, if $\mathfrak{B}_i$ is defined by $R_i(\frac{d}{dt})w = 0$, $i = 1, 2$, with $R_i(\xi)$ of full row rank, then the interconnection of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is called regular if $\operatorname{rank}\operatorname{col}(R_1(\xi), R_2(\xi))$ equals $\operatorname{rank} R_1(\xi) + \operatorname{rank} R_2(\xi)$. The interconnection is called a regular feedback interconnection if it is regular and if the McMillan degree of $\operatorname{col}(R_1(\xi), R_2(\xi))$ equals the sum of the McMillan degrees of $R_1(\xi)$ and $R_2(\xi)$. In our context it appears appropriate to restrict the attention to regular interconnections. In [5] it is proven that any sub-behavior of a given behavior defined by $R(\frac{d}{dt})w = 0$ may be achieved by regular interconnection provided that the given behavior is controllable, i.e., $\operatorname{rank} R(\lambda)$ does not depend on $\lambda \in \mathbb{C}$. A similar result holds for the case that the interconnection can be realized through interconnection variables c only.

Whether or not a control objective can be achieved amounts to checking whether, for a given desired sub-behavior, there exists a polynomial matrix $C(\xi)$ such that the equations $R(\dfrac{d}{dt})w = M(\dfrac{d}{dt})c$ and $C(\dfrac{d}{dt})c = 0$ define that sub-behavior. To check this, two extreme sub-behaviors are of interest one for which c is not restricted and one for which c is identically zero.

$$\begin{aligned}\mathfrak{B}_{\text{unc}} &= \{(w,c) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^d) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})c\} \\ \mathfrak{B}_{\max} &= \{(w,c) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^d) \mid R(\frac{d}{dt})w = 0 \quad c = 0\}\end{aligned} \tag{3}$$

The interpretation is that $\mathfrak{B}_{\text{unc}}$ is the uncontrolled behavior and $\mathfrak{B}_{\max}$ the maximally controlled behavior. We call a sub-behavior $\mathfrak{C}$ of $\mathfrak{B}$ *achievable*[1] if there exists a polynomial matrix $C(\xi)$ such that $\mathfrak{C}$ is represented by $R(\dfrac{d}{dt})w = M(\dfrac{d}{dt}c$ and $C(\dfrac{d}{dt})c = 0$. The following theorem, [6, 3, 1, 2], characterizes which sub-behaviors are achievable.

2.1 Theorem
Let $\mathfrak{C} \subset \mathfrak{B}_{\text{unc}}$ be represented by $\tilde{R}(\frac{d}{dt})w = \tilde{M}(\frac{d}{dt})c$ for some polynomial matrices $\tilde{R}(\xi)$ and $\tilde{M}(\xi)$. There exists a polynomial matrix $C(\xi)$ such that $\mathfrak{C}$ is also given by (2) if and only if $\mathfrak{B}_{\max} \subset \mathfrak{C}$.

From Theorem 2.1 it follows that a given control objective is achievable if and only if the desired behavior contains the maximally controlled behavior $\mathfrak{B}_{\max}$. The next result states that the resulting interconnection may be achieved by regular interconnection, provided that $\mathfrak{B}$ is controllable.

2.2 Theorem
Under the conditions of Theorem 2.1, if additionally $\mathfrak{B}$ is controllable, then $C(\xi)$ may be taken such that the resulting interconnection is regular.

[1] In [6, 3] this is called *implementable*.

Proof According to [5, Theorem 6] there exists matrices $R'(\xi)$ and $M'(\xi)$ such that

$$\mathfrak{C} = \{(w,c) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})c \quad R'(\frac{d}{dt})w = M'(\frac{d}{dt})c\} \tag{4}$$

and such that this interconnection is regular. Since $\mathfrak{B}_{\max} \subset \mathfrak{C}$, there holds that $R(\frac{d}{dt})w = 0$ implies $R'(\frac{d}{dt})w = 0$. Therefore $R'(\xi) = F(\xi)R(\xi)$ for a suitably chosen $F(\xi)$. It follows that $\mathfrak{C}$ is also given by

$$\begin{aligned}\mathfrak{C} = \{(w,c) \mid & R(\frac{d}{dt})w = M(\frac{d}{dt})c \\ & (M'(\frac{d}{dt}) - F(\frac{d}{dt})M(\frac{d}{dt}))c = 0\},\end{aligned} \tag{5}$$

which is of course still a regular interconnection. Hence we can take $C(\xi) = M'(\xi) - F(\xi)M(\xi)$. □

Theorem 2.1 also implies that for a given sub-behavior $\mathfrak{B}' \subset \mathfrak{B}$, the control objective is achievable if and only if $\mathfrak{B}_{\max} \subset \mathfrak{B}'$. As already remarked in [1, 2] this is an almost trivial observation. For the convenience of the reader and for future reference we repeat it here,

2.3 Theorem
Let $\mathfrak{B}'$ be defined by $R'(\dfrac{d}{dt})w = M'(\dfrac{d}{dt})c$ and $\mathfrak{C}_{\text{des}}$ by $R(\dfrac{d}{dt})w = 0$, $K_{\text{des}}(\dfrac{d}{dt})c = 0$. Assume furthermore that $\mathfrak{B}_{\max} \subset \mathfrak{C}_{\text{des}} \subset \mathfrak{B}' \subset \mathfrak{B}$. Then there exists a polynomial matrix $K'_{\text{des}}(\xi)$ such that $\mathfrak{C}_{\text{des}}$ is also defined by $R'(\dfrac{d}{dt})w = M'(\dfrac{d}{dt})c$, $K'(\dfrac{d}{dt})c = 0$.

Theorem 2.3 classifies all sub-behaviors of $\mathfrak{B}$ for which the control objective is achievable. Following [1, 2] we call these behaviors $\mathfrak{C}$-controllable. It should be remarked that if $\mathfrak{B}'$ is not controllable, then the step from $\mathfrak{B}'$ to $\mathfrak{C}_{\text{des}}$ need not be achievable by regular interconnection.

2.4 Definition
A behavior $\mathfrak{B}' \subset \mathfrak{B}$ defined by $R'(\dfrac{d}{dt})w = M'(\dfrac{d}{dt})c$ is called $\mathfrak{C}$-controllable if there exists $K'(\xi)$ such that $\mathfrak{C} = \{w \mid R'(\dfrac{d}{dt})w = M'(\dfrac{d}{dt})c \quad K'(\dfrac{d}{dt})c = 0\}$.

3 Adaptive control

We now turn to the adaptive part. As announced in Section 2, the entries of the matrices $R(\xi)$ and $M(\xi)$ are unknown. All that is given is their number of columns. What is assumed to be known, however, is the desired sub-behavior that we want to achieve through appropriate controller design. The desired sub-behavior is denoted by $\mathfrak{C}_{\text{des}}$. To be achievable at all, we assume moreover that $\mathfrak{B}_{\max} \subset \mathfrak{C}_{\text{des}}$. In [1, 2] it was assumed that we could observe elements of the behavior on the whole time axis. An iterative design was proposed based on consecutive measurements. Of course, this idea is of no practical relevance as it is totally unrealistic to assume that observations are available on the whole time axis $\mathbb{R}$. In this paper we consider a more realistic setup. Measurements are taken during time intervals of unit length. Each time a measurement has been completed a model of that measurement is derived and a controller interconnection is determined that achieves the control objective for that particular model. We do not make any assumptions as to how the observed trajectories are generated other than that they belong to the unknown behavior. The way we propose to model an observation trajectory is by means of the *Most Powerful Unfalsified Model* (MPUM), see [4]. In [4] this is defined as the smallest behavior in $\mathcal{L}$, the class of behaviors that admit a kernel representation, that explains (or does not falsify) the observation. However, due to the prior knowledge that we assume, we use a modified version of MPUM. The prior knowledge that we assume is that we know $\mathfrak{C}_{\text{des}}$ and moreover that $\mathfrak{C}_{\text{des}}$ is achievable, in other words $\mathfrak{B}$ is $\mathfrak{C}_{\text{des}}$ controllable. Since $\mathfrak{C}_{\text{des}}$ is contained in $\mathfrak{B}$, it appears natural to look for the smallest behavior that contains $\mathfrak{C}_{\text{des}}$ and does not falsify the observed trajectory. Below we prove that this behavior indeed exists. To distinguish this model from the MPUM as defined in [4], we call it the Most Powerful Unfalsified $\mathfrak{C}_{\text{des}}$ Controllable Model, MPUCM for short.

The following theorem, which generalizes Proposition 12 in [4], implies the existence of the smallest behavior admitting a kernel representation containing $\mathfrak{C}_{\text{des}}$ and a given trajectory $\bar{w} \in \mathfrak{B}|_I$. Here I is an open interval in $\mathbb{R}$.

3.1 Theorem
Let $I \subset \mathbb{R}$ be an open interval. Let $R_i(\xi) \in \mathbb{R}^{g\times q}[\xi]$, $i = 0, 2$ and let the behaviors $\mathfrak{B}_i$ be defined by $R_i(\frac{d}{dt})w = 0$, $i = 0, 2$. Assume that $\mathfrak{B}_0 \subset \mathfrak{B}_2$ and $\bar{w}|_I \in \mathfrak{B}_2|_I$. Then there exists an $R_1(\xi)$ of appropriate size and a unique behavior $\mathfrak{B}_1$, defined by $R_1(\frac{d}{dt})w = 0$, such that $\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \mathfrak{B}_2$ and $\bar{w}|_I \in \mathfrak{B}_1|_I$. Moreover $\mathfrak{B}_1$ is the smallest behavior with these properties.

Proof The proof is the same as that of Theorem 3.1 in [1, 2] and is therefore omitted. □

In view of Theorem 2.3 we can now introduce the following notation.

3.2 Definition
Let $\bar{w} \in \mathfrak{B}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ defined by $R_0(\frac{d}{dt})w = 0$. Furthermore let $I \subset \mathbb{R}$ be an open interval. The smallest behavior in $\mathcal{L}$ containing $\mathfrak{B}_0$ that does *not* falsify $\bar{w}|_I$ is called the MPUM of $\bar{w}|_I$ and $\mathfrak{B}_0$ and is denoted by MPUM$(\mathfrak{B}_0, \bar{w}|_I)$. If $\mathfrak{C} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R'(\frac{d}{dt})w = 0,\ \ K'(\frac{d}{dt})c = 0\}$ for suitable matrices $R'(\xi)$, $K'(\xi)$, then the smallest behavior in $\mathcal{L}$ containing $\mathfrak{C}$ and $\bar{w}$ is called the Most Powerful Unfalsified $\mathfrak{C}$ Controllable model of $\bar{w}$ and $\mathfrak{B}_0$ and is denoted by MPUCM$(\mathfrak{C}, \bar{w}|_I)$.

Once the MPUCM is obtained, we specify a sub-behavior that corresponds to our control specifications. In view of Theorem 2.1 the control specifications can be met if and only if the sub-behavior contains the maximally controlled behavior.

The resulting scheme is as follows.

3.3 Iterative scheme
1. Initialization:
   (a) Observe $(w, c) \in \mathfrak{B}$ on the time interval $I_0 = [0, \frac{1}{2})$.
   (b) Determine $\mathfrak{B}_0 = \text{MPUCM}(\mathfrak{C}_{\text{des}}, (w, c)|_{I_0})$.
   (c) Determine $R_0(\xi)$, $M_0(\xi)$, and $C_0(\xi)$ such that $\mathfrak{B}_0 = \{(w, c) \mid R_0(\frac{d}{dt})w = M_0(\frac{d}{dt})c\}$ and $\mathfrak{C}_{\text{des}} = \{(w, c) \mid R_0(\frac{d}{dt})w = M_0(\frac{d}{dt})c, C_0(\frac{d}{dt})c = 0\}$. Define $\mathfrak{C}_0 = \{c \mid C_0(\frac{d}{dt})c = 0\}$.
   (d) Implement $C_0(\xi)$ during the time interval $(0.5, 1.5)$.
2. At the k-th iteration:
   (a) Observe $(w, c) \in \mathfrak{B} \cap \mathfrak{C}_k$ on the time interval $I_k = [k - \frac{1}{2}, k + \frac{1}{2})$.
   (b) Determine $\mathfrak{B}_k = \text{MPUCM}(\mathfrak{B}_{k-1}, (w, c)|_{I_k})$.
   (c) Determine $R_k(\xi)$, $M_k(\xi)$, and $C_k(\xi)$ such that $\mathfrak{B}_k = \{(w, c) \mid R_k(\frac{d}{dt})w = M_k(\frac{d}{dt})c\}$ and $\mathfrak{C}_{\text{des}} = \{(w, c) \mid R_k(\frac{d}{dt})w = M_k(\frac{d}{dt})c, C_k(\frac{d}{dt})c = 0\}$, $\mathfrak{C}_k = \{c \mid C_k(\frac{d}{dt})c = 0\} \subset \mathfrak{C}_{k-1}$.
   (d) Implement $C_k(\xi)$ during the time interval $(k + 0.5, k + 1.5)$.

The following theorem states that within a finite number of iteration we have achieved the control objective.

3.4 Theorem
Consider the iterative scheme 3.3. For all $w \in \mathfrak{B}$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$:

(i) $w|_{[k,\infty)} \in \mathfrak{C}_{\text{des}[k,\infty)}$.

(ii) $\mathfrak{B}_k = \mathfrak{B}_{k_0}$.

Proof (i). By construction there holds that for all k $\mathfrak{C}_{k+1} \subset \mathfrak{C}_k$. Just like in [1, 2] this implies that there exists k_0 such that for all $k \geq k_0$, there holds $\mathfrak{C}_k = \mathfrak{C}_{k_0}$.

Moreover, by construction, for all k

$$(R_k(\frac{d}{dt})w)(t) = (M_k(\frac{d}{dt})c)(t) \qquad t \in I_0 \cup \ldots \cup I_k \tag{6}$$

$$(C_{k-1}(\frac{d}{dt})c)(t) = 0 \qquad t \in I_k$$

The first line follows from the fact that $\mathfrak{B}_0 \subset \ldots \subset \mathfrak{B}_k$. For $k \geq k_0 + 1$, since $\mathfrak{C}_{k-1} = \mathfrak{C}_k$, this implies:

$$(R_k(\frac{d}{dt})w)(t) = (M_k(\frac{d}{dt})c)(t)t \in I_k \tag{7}$$
$$(C_k(\frac{d}{dt})c)(t) = 0,$$

which in turn implies that for $k \geq k_0 + 1$ and $t \in I_k$:

$$(R(\frac{d}{dt})w)(t) = (M(\frac{d}{dt})c)(t) \tag{8}$$
$$(C_{\text{des}}(\frac{d}{dt})c)(t) = 0$$

(ii). Let k_0 be as in Part (i) and let $k \geq k_0$. Since $(w, c)|_{I_{k+1}} \in \mathfrak{C}_{\text{des}}|_{I_{k+1}} \subset \mathfrak{B}_k$, it follows that $\mathfrak{B}_{k+1}$ =MPUCM$(\mathfrak{B}_k, (w, c)|_{I_{k+1}}) = \mathfrak{B}_k$. □

3.5 Remark
The implementation of the controller $C_k(\xi)$ is not trivial. The interconnection can be chosen to be regular. However, to be able to implement a set of additional constraint instantaneously the interconnection should be of the type regular feedback interconnection. Invoking an interconnection that is just regular may require preparation of the states. See [5] for an extensive discussion. Here we just assume that the implementation can be done. We hope to report on this issue in a more satisfactory manner in the near future.

4 Conclusions

We have made a second step towards a behavioral theory of adaptive control. Issues that should be addressed further include in particular the role of regular and regular feedback interconnections. For other other points of further research we refer to [1, 2]

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