On the relationship between two-dimensional behaviors decompositions and the factor skew-primeness property

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Abstract

In this paper, the possibility of obtaining certain decompositions with finite-dimensional intersection, for a given two-dimensional behavior, are investigated and related to the factor skew-primeness property of suitable matrix pairs, involved in the behavior description. The analysis carried on here represents a first step toward a complete generalization of the preliminary results about direct sum decompositions presented in [2].

1 Introduction

In the last decade, the behavioral approach to dynamic systems [8, 14] has been fruitfully applied to the multidimensional, in particular two-dimensional (2D), context [9, 16]. Within the behavioral setting, several classic results that hold true for 2D state space models have found a natural generalization. Among them, we recall the autonomous/controllable decomposition [5, 9, 13, 16], that holds true for every (linear, shift-invariant and complete) 2D behavior. This decomposition constitutes the straightforward, even though nontrivial, extension of the well-known "free evolution"/"forced evolution" decomposition, that holds true for every trajectory of a (linear and shift-invariant) 2D state space model.

The relevance of this decomposition, which has been intensively investigated also in the context of 1D behaviors, is immediately apparent. In fact, it both simplifies the analysis of the system properties and allows to afford relevant issues like stability, stabilizability and control design [12]. However, while in the 1D case it is always possible to express a behavior $\mathfrak{B}$ as a direct sum of its (uniquely determined) controllable part and of some autonomous behavior, in the two-dimensional case this is not always feasible. Indeed, the controllable part is always uniquely determined, but the case often occurs that all autonomous parts involved in the behavior decomposition have a nontrivial intersection with the controllable one [13, 16]. Nevertheless, it is always possible to obtain a decomposition in which the autonomous part has a finite-dimensional intersection with the controllable part [13].

In this paper we aim at extending these results, and, in particular, those presented in a recent paper [2] and concerned with the direct sum decomposition, to a more general setting. In fact, given a 2D behavior $\mathfrak{B}$ and one of its *sub-behaviors* $\mathfrak{B}_1$, i.e. another behavior included in $\mathfrak{B}$, we aim to investigate under what conditions a further behavior $\mathfrak{B}_2$ can be found, such that (s.t.) $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, and $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is finite-dimensional autonomous. This constitutes a complete generalization of the decomposition theorem, as it represents a decomposition with a somehow "minimal" intersection, in which one of the two terms is *a priori* fixed.

Significantly enough, the possibility of obtaining such a decomposition is related to the algebraic properties of certain matrix pairs (A, B), where B is a Laurent polynomial (*L-polynomial*) matrix involved in the kernel description of $\mathfrak{B}_1$, and A is a special matrix such that $\mathfrak{B} = \ker(AB)$. So, in the 2D case the situation is fairly different from the one arising with 1D behaviors, where a decomposition $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, with $\mathfrak{B}_1 \cap \mathfrak{B}_2$ autonomous and hence finite-dimensional, always exists [1]. Throughout the paper, we will steadily consider linear, shift-invariant and complete behaviors, which are the sets of solutions of a family of linear and shift-invariant difference equations [9], and refer to them simply as *behaviors*.

Before proceeding, we introduce some notation. Given any nonzero L-polynomial $p \in \mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]$, the *Laurent variety* of p is $\mathcal{V}_L(p) := \{(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}, \alpha \cdot \beta \neq 0 : p(\alpha, \beta) = 0\}$. If $\mathcal{I}$ is a set of L-polynomials, its Laurent variety $\mathcal{V}_L(\mathcal{I})$ is, by definition, the intersection of the Laurent varieties of all its elements. If H is a matrix of full (row or column) rank, we denote by $\mathcal{V}_L(H)$ the Laurent variety of its maximal order minors. Also, if H is a matrix of full (row or column) rank, we say that H is *L-primitive w.r.t. the variable* z_1 (z_2, respectively) if the g.c.d. of its maximal order minors is devoid of factors belonging to $\mathbb{R}[z_1, z_1^{-1}]$ ($\mathbb{R}[z_2, z_2^{-1}]$, respectively), except for units.

In the paper, primeness properties of a matrix, i.e. factor primeness and zero primeness [6, 7], refer to the ring of L-polynomials. A full column (row) rank L-polynomial matrix H is right (left) zero prime (rZP/lZP) if and only if $\mathcal{V}_L(H)$ is empty, and it is right (left) factor prime (rFP/lFP) if and only if $\mathcal{V}_L(H)$ consists of a finite number of points [6].

A pair (A, B) of L-polynomial matrices is said to be *(internally) zero skew-prime* [11, 15, 17], if there exist L-polynomial matrices X and Y, of suitable dimensions, that solve the two-sided Bézout equation $XA+BY = I$. Similarly, (A, B) is said to be *(internally) factor skew-prime* if there exist L-polynomial matrices X_i and Y_i, of suitable dimensions, that solve the two two-sided diophantine equations $X_iA+BY_i = \psi_i I$, with ψ_i a suitable L-polynomial in $\mathbb{R}[z_i, z_i^{-1}], i = 1, 2$. It is easy to realize that (A, B) is (internally) factor skew-prime if and only if there exist two factor coprime L-polynomials, say p_1 and p_2, s.t. $\tilde{X}_iA + B\tilde{Y}_i = p_iI$, holds for suitable L-polynomial matrices $\tilde{X}_i$ and $\tilde{Y}_i$, $i = 1, 2$. Equivalently, the pair (A, B) is (internally) factor skew-prime if and only if it is skew-prime both in $\mathcal{R}_1 := \mathbb{R}(z_1)[z_2, z_2^{-1}]$ and in $\mathcal{R}_2 := \mathbb{R}(z_2)[z_1, z_1^{-1}]$. Of course, if either A is rZP/rFP or B is lZP/lFP, then the above equations are trivially solvable and hence (A, B) is an (internally) zero/factor skew-prime pair. Throughout the paper, we will omit the specification "internally".

2 Factor skew-primeness

Given a matrix pair (A, B), with A a $p\times r$ L-polynomial matrix and B an $r \times q$ L-polynomial matrix, and $p+q \geq r$, we may want to know whether the given pair is factor skew-prime or not. In the 1D case, the most significant result that provides an answer to this question is the well-known theorem, due to Roth [10], stating that a pair (A, B) is skew-prime if and only if the matrices $\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} B & I_r \\ 0 & A \end{bmatrix}$ are equivalent, by this meaning that they can be obtained one from the other by means of elementary transformations, and hence share the same Smith form. Since the factor skew-primeness property is, in fact, equivalent to the skew-primeness in both rings $\mathcal{R}_1$ and $\mathcal{R}_2$, we can use Roth's theorem in these two rings (by evaluating the Smith forms of both previous matrices), and hence get an explicit answer to our question. For this reason, factor skew-primeness is easier to test than zero skew-primeness, which requires the use of algorithms based on the Gröbner bases [4].

Proposition 2.1, below, gives a simple sufficient condition for factor skew-primeness that will prove to be useful for the following analysis.

Proposition 2.1 *Let A be a $p \times r$ full column rank L-polynomial matrix and let B be an $r \times q$ full row rank L-polynomial matrix. If $\mathcal{V}_L(A) \cap \mathcal{V}_L(B)$ consists of a finite number of points, the pair (A, B) is factor skew-prime. Moreover, if $r = 1$, also the converse holds true.*

Proof If $\mathcal{V}_L(A) \cap \mathcal{V}_L(B)$ consists of a finite number of points, then the ideal generated by the maximal order minors of A and B, $m_i(A)$ and $m_i(B)$ respectively, includes both an L-polynomial in the variable z_1 alone, say $\psi_1(z_1)$, and an L-polynomial in the variable z_2 alone, say $\psi_2(z_2)$ [6]. This amounts to saying that there exist L-polynomials $x_1^{(j)}, \dots, x_{n_p}^{(j)}$ and $y_1^{(j)}, \dots, y_{n_q}^{(j)}$ ($n_p := \binom{p}{r}$ and $n_q := \binom{q}{r}$), s.t.

$$\psi_j(z_j) = \sum_{i=1}^{n_p} x_i^{(j)} m_i(A) + \sum_{i=1}^{n_q} m_i(B) y_i^{(j)}, \qquad j = 1, 2.$$

Set $\psi_A^{(j)} := \sum_{i=1}^{n_p} x_i^{(j)} m_i(A)$ and $\psi_B^{(j)} := \sum_{i=1}^{n_q} m_i(B) y_i^{(j)}$. Clearly, $\psi_A^{(j)} + \psi_B^{(j)} = \psi_j(z_j)$. Moreover, as $m_i(A)I_r = \text{adj}(S_iA)(S_iA)$, where S_i is the $r \times p$ selection matrix that singles out the $r \times r$ submatrix of A corresponding to $m_i(A)$, and, similarly, $m_i(B)I_r = (B\tilde{S}_i)\text{adj}(B\tilde{S}_i)$, where $\tilde{S}_i$ is the $q \times r$ selection matrix s.t. $\det(B\tilde{S}_i) = m_i(B)$, then

$$\begin{aligned} \psi_A^{(j)} I_r &= \sum_{i=1}^{n_p} x_i^{(j)} \text{adj}(S_iA)(S_iA) = X_jA, \\ \psi_B^{(j)} I_r &= \sum_{i=1}^{n_q} (B\tilde{S}_i)\text{adj}(B\tilde{S}_i) y_i^{(j)} = BY_j, \end{aligned}$$

where $X_j := \sum_{i=1}^{n_p} x_i^{(j)} \text{adj}(S_iA)S_i$ and $Y_j := \sum_{i=1}^{n_q} \tilde{S}_i \text{adj}(B\tilde{S}_i) y_i^{(j)}$. Therefore,

$$\psi_j(z_j)I_r = \psi_A^{(j)} I_r + \psi_B^{(j)} I_r = X_jA + BY_j, \quad j = 1, 2.$$

This proves that the pair (A, B) is factor skew-prime.

Conversely, assume $r = 1$. If $\mathcal{V}_L(A) \cap \mathcal{V}_L(B)$ would consist of an infinite number of points, the entries of both A and B would be multiples of some nontrivial L-polynomial m. But then, for every pair of L-polynomial vectors X and Y, the L-polynomial $XA + BY$ would be a multiple of m, too, thus contradicting the factor skew-primeness assumption. ■

3 Decompositions with finite-dimensional intersection

As we know [13], a direct sum controllable/autonomous decomposition is a rare occurrence. A less ambitious task is that of obtaining a controllable/autonomous decomposition with finite-dimensional intersection, and indeed it has been shown that such a decomposition always exists [13]. Aiming to extend this result to a broader context, we now look for conditions ensuring that a given behavior $\mathfrak{B}$ admits a decomposition in which one of the two terms is a priori chosen (the subbehavior $\mathfrak{B}_1$) and the other one is constrained to have a finite-dimensional intersection with the first one.

Theorem 3.1 *Set* $\mathfrak{B} := \ker(AB)$, *with* A *a* $p \times r$ *L-polynomial matrix and* B *an* $r \times q$ *L-polynomial matrix. Let* F *be a minimal left annihilator (MLA, for short) [9] of* B, *and set* $\hat{A} := \begin{bmatrix} A \\ F \end{bmatrix}$ *and* $\mathfrak{B}_1 := \ker B$. *Also, let* G *be a minimal right annihilator (MRA) of* $\hat{A}$. *The following facts hold true:*

i) If there exist polynomial matrices $\bar{A}$ *and* $\bar{B}$ *of suitable sizes s.t.* $[\, A \quad \bar{A} \,] \begin{bmatrix} B \\ -\bar{B} \end{bmatrix} = 0$, *and each of these block matrices is a minimal annihilator (left and right, respectively) of the other, then there exists a behavior* $\mathfrak{B}_2$ *s.t.* $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, *with* $\mathfrak{B}_1 \cap \mathfrak{B}_2$ *finite-dimensional.*

ii) If A *is of full column rank,* B *is of full row rank, and* $\mathcal{V}_L(A) \cap \mathcal{V}_L(B)$ *consists of a finite number of points, then* $\mathfrak{B}$ *can be expressed as*

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2, \tag{1}$$

for some $\mathfrak{B}_2$, *with* $\mathfrak{B}_1 \cap \mathfrak{B}_2$ *finite-dimensional.*

iii) If there exists a behavior $\mathfrak{B}_2$ *s.t.* $\mathfrak{B}$ *can be expressed as in (1), with* $\mathfrak{B}_1 \cap \mathfrak{B}_2$ *a finite-dimensional autonomous behavior, then the pair* $(\hat{A}, B)$ *is factor skew-prime, and* $G = BT$ *for some L-polynomial matrix* T.

PROOF i) Set $\mathfrak{B}_2 := \ker\bar{B}$. Of course, as $\begin{bmatrix} B \\ -\bar{B} \end{bmatrix}$ is rFP, $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is a finite-dimensional autonomous behavior. This ensures (see Lemma A.1 in [13]) that $\mathfrak{B}_1 + \mathfrak{B}_2 = \ker(AB) = \mathfrak{B}$.

ii) As a first step, we show that it is sufficient to prove the above decomposition result in the case when A is a diagonal matrix, namely $p = r$ and $A = \psi I_r$, for some L-polynomial ψ, with $\mathcal{V}_L(\psi) \cap \mathcal{V}_L(B)$ a finite set. In fact, if $\mathcal{V}_L(A) \cap \mathcal{V}_L(B)$ consists of a finite number of points, an L-polynomial matrix X can be found s.t. $XA = \psi I_r$ and $\mathcal{V}_L(\psi) \cap \mathcal{V}_L(B)$ is finite [12]. So, upon setting $\tilde{\mathfrak{B}} := \ker(\psi I_r)B = \ker(XAB) \supseteq \ker(AB) = \mathfrak{B}$, if we prove that a behavior $\tilde{\mathfrak{B}}_2 = \ker\tilde{H}$ exists s.t. $\tilde{\mathfrak{B}} = \mathfrak{B}_1 + \tilde{\mathfrak{B}}_2$, with $\mathfrak{B}_1 \cap \tilde{\mathfrak{B}}_2$ finite-dimensional (i.e., $\begin{bmatrix} \tilde{H} \\ B \end{bmatrix}$ rFP), then $\mathfrak{B} = \mathfrak{B} \cap \tilde{\mathfrak{B}} = \mathfrak{B} \cap (\ker\tilde{H} + \ker B) = \ker \begin{bmatrix} \tilde{H} \\ AB \end{bmatrix} + \ker B = \ker \begin{bmatrix} \tilde{H} \\ AB \end{bmatrix} + \mathfrak{B}_1$. Of course, $\begin{bmatrix} \tilde{H} \\ AB \\ B \end{bmatrix}$ is rFP, too, and hence (1) holds true for $\mathfrak{B}_2 := \ker \begin{bmatrix} \tilde{H} \\ AB \end{bmatrix}$.

We aim to show, now, that the result holds for any behavior $\tilde{\mathfrak{B}} = \ker((\psi I_r)B)$, with ψ an L-polynomial s.t. $\mathcal{V}_L(\psi) \cap \mathcal{V}_L(B)$ is a finite set. To this end, factorize ψ into the product of its distinct irreducible factors ($p_i \neq p_j$ if $i \neq j$), each of them appearing with its corresponding multiplicity $\nu_i > 0$: $\psi = p_1^{\nu_1} p_2^{\nu_2} \dots p_t^{\nu_t}$. We aim to obtain q distinct factorizations of ψ, i.e., $\psi = a_i b_i$, $i = 1, \dots, q$, in such a way that, by defining $\tilde{H} := \operatorname{diag}\{a_1, a_2, \dots, a_q\}$ and $L := \operatorname{diag}\{b_1, b_2, \dots, b_q\}$, the $(r+q) \times q$ matrix $R := \begin{bmatrix} B \\ \tilde{H} \end{bmatrix}$ and the $r \times (q+r)$ matrix $T := [\, \psi I_r \quad | \quad -BL \,]$ are each a minimal annihilator (right and left, respectively) of the other. These facts ensure that $\ker\tilde{H} \cap \mathfrak{B}_1 = \ker \begin{bmatrix} B \\ \tilde{H} \end{bmatrix}$ is a finite-dimensional behavior, and [2] that $\ker\tilde{H} + \mathfrak{B}_1 = \tilde{\mathfrak{B}}$.

Notice that since matrices T and R have "complementary" sizes (by this meaning that the number of rows of the first matrix plus the number of columns of the second one coincides with the common dimension), and $TR = 0$, to prove that each of them is a minimal annihilator of the other, by Lemma A.7 in [12], it will be sufficient to show that T is lFP and that $\det \tilde{H} = \det \psi I_r = \psi^r$.

We, now, provide an algorithm for obtaining the aforementioned q factorizations $\psi = a_i b_i$, $i = 1, 2, \dots, q$, by distributing certain L-polynomials $p_j^{\nu_j}$ in b_i, and leaving the remaining ones in a_i. Notice that since $\mathcal{V}_L(\psi) \cap \mathcal{V}_L(B)$ is finite, every irreducible factor p_i, appearing in the decomposition of ψ, cannot be a common divisor of all maximal (i.e. rth) order minors of B. Set $b_i^{(0)} := 1$, for every index i, and consider, first, p_1. As p_1 cannot be a common factor of all minors of B, there exists some $r \times r$ submatrix of B, say the one corresponding to the columns $j_1 < j_2 < \dots < j_r$, whose determinant is factor coprime with p_1. So, we can set $b_i^{(1)} := b_i^{(0)}$, for $i \in \{j_1, j_2, \dots, j_r\}$, and $b_i^{(1)} := p_1^{\nu_1} b_i^{(0)}$ for the remaining indices. By assuming the same reasoning, at the second step, we attribute $p_2^{\nu_2}$ as a common factor of $q-r$ of the $b_i^{(2)}$'s, while for the remaining r indices, corresponding to an $r \times r$ submatrix of B whose determinant is coprime with p_2, we set $b_i^{(2)} := b_i^{(1)}$. By proceeding in this way, we distribute all factors of ψ

among the b_i's. Finally, we set $a_i := \psi/b_i$. It is easily seen that, due to condition $|\mathcal{V}_L(\psi)\cap\mathcal{V}_L(B)| < +\infty$ and the way all factors of ψ have been distributed among the b_i's, T is lFP. As every $p_j^{\nu_j}$, $j = 1,\dots,t$, has been attributed to $q-r$ of the b_i's, then, of course, there are $q-r$ among the a_i's which are factor coprime with p_j and r of them which have $p_j^{\nu_j}$ as a common factor. So, $\det\tilde{H} = \prod_{i=1}^q a_i = \prod_{j=1}^t (p_j^{\nu_j})^r = \psi^r = \det(\psi I_r)$. This proves that R is rFP.

iii) Assume that there exists some $\ell\times q$ L-polynomial matrix H s.t. $\mathfrak{B}_2 := \ker H$ satisfies (1), and hence, in particular, $\begin{bmatrix} B \\ H \end{bmatrix}$ is rFP. Therefore, L-polynomial matrices E_i and C_i, $i = 1,2$, can be found s.t.

$$V_i := \begin{bmatrix} E_i & B \\ -C_i & H \end{bmatrix}$$

satisfies $\psi_i := \det V_i \in \mathbb{R}[z_i, z_i^{-1}]$. Set $U_i := \mathrm{adj}V_i$, so that $U_iV_i = V_iU_i = \psi_i I_{r+\ell}$, and partition U_i, accordingly to V_i, as follows:

$$U_i = \begin{bmatrix} U_{11}^{(i)} & U_{12}^{(i)} \\ U_{21}^{(i)} & U_{22}^{(i)} \end{bmatrix},$$

with $U_{11}^{(i)}$ of size $(\ell+r-q)\times r$. As $\begin{bmatrix} U_{11}^{(i)} & U_{12}^{(i)} \end{bmatrix}$ is a full row rank left annihilator of $\begin{bmatrix} B \\ H \end{bmatrix}$, if we denote by $\begin{bmatrix} X & Y \end{bmatrix}$ an MLA of $\begin{bmatrix} B \\ H \end{bmatrix}$, we get the identity

$$\begin{bmatrix} U_{11}^{(i)} & U_{12}^{(i)} \end{bmatrix} = P_i \begin{bmatrix} X & Y \end{bmatrix},$$

where P_i is a suitable nonsingular square L-polynomial matrix. Moreover, as $\begin{bmatrix} X & Y \end{bmatrix}$ is an MLA of $\begin{bmatrix} B \\ H \end{bmatrix}$, then, by the assumption $\ker B + \ker H = \mathfrak{B} = \ker(AB)$, it follows (see Lemma A.1 in [13]) that $\ker(XB) = \ker(AB)$, and therefore there exists an L-polynomial matrix P s.t. $XB = PAB$, which amounts to saying that there exists L-polynomial matrices P and Q for which

$$X = \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} A \\ F \end{bmatrix} = \begin{bmatrix} P & Q \end{bmatrix}\hat{A}. \tag{2}$$

This implies $U_{11}^{(i)} = P_i \begin{bmatrix} P & Q \end{bmatrix}\hat{A}$, $i = 1,2$. But since

$$\begin{bmatrix} E_i & B \end{bmatrix}\begin{bmatrix} U_{11}^{(i)} \\ U_{21}^{(i)} \end{bmatrix} = \psi_i I_r,$$

it follows that $(E_iP_i\begin{bmatrix} P & Q \end{bmatrix})\hat{A} + BU_{21}^{(i)} = \psi_i I_r$, $i = 1,2$, and therefore the pair $(\hat{A}, B)$ is factor skew-prime.

Finally, from (2) we get $XG = \begin{bmatrix} P & Q \end{bmatrix}\hat{A}G = 0$, which proves that $\begin{bmatrix} G \\ 0 \end{bmatrix}$ is a right annihilator of $\begin{bmatrix} X & Y \end{bmatrix}$. As a consequence, it is related to the MRA $\begin{bmatrix} B \\ H \end{bmatrix}$ of $\begin{bmatrix} X & Y \end{bmatrix}$ by the identity: $\begin{bmatrix} G \\ 0 \end{bmatrix} = \begin{bmatrix} B \\ H \end{bmatrix} T$, where T is an L-polynomial matrix. Consequently, $G = BT$. ■

As for zero skew-primeness [2], for a given behavior $\mathfrak{B}$ and a given sub-behavior $\mathfrak{B}_1 := \ker B$, there can be choices of A for which $\ker(AB)$ coincides with $\mathfrak{B}$ and the matrix pair (A, B) is factor skew-prime, and choices for which not both requirements are simultaneously fulfilled. If the problem is solvable, namely at least one matrix A can be found that meets both conditions, we may look for a complete characterization of the set of matrices that exhibit these two properties, i.e.

$$\mathcal{A}_{\rm FSP} := \{A : \mathfrak{B} = \ker(AB) \text{ and} (A,B) \text{is factor skew-prime}\}.$$

Proposition 3.2 [3] *Let $\mathfrak{B}$ be a behavior and $\mathfrak{B}_1 :=$ $\ker B$ one of its sub-behaviors. Let F be an MLA of B. If $\mathcal{A}_{\rm FSP} \neq \emptyset$, and hence there exists an L-polynomial matrix A^* s.t. $\mathfrak{B} = \ker(A^*B)$ and the pair (A^*, B) is factor skew-prime, then $\mathcal{A}_{\rm FSP}$ is the set of matrices A satisfying the following conditions:*

1. *there exist matrices D_i, with entries in $\mathcal{R}_i$, s.t. $A^* = D_iA$, $i = 1,2$;*

2. *L-polynomial matrices P, P^*, Q and Q^* exist s.t. $A = \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} A^* \\ F \end{bmatrix}$ and $A^* = \begin{bmatrix} P^* & Q^* \end{bmatrix}\begin{bmatrix} A \\ F \end{bmatrix}$.*

Matrices belonging to $\mathcal{A}_{\rm FSP}$ do not necessarily provide kernel descriptions of the same behavior. However, if A and A^* are both in $\mathcal{A}_{\rm FSP}$, then the controllable parts of $\ker A$ and of $\ker A^*$ coincide [3]. This can be equivalently stated by saying that all matrices in $\mathcal{A}_{\rm FSP}$ share the same minimal right annihilators, which, in the sequel, will be denoted by G (see Theorem 3.1). The set $\mathcal{A}_{\rm FSP}$ hence exhibits weaker properties with respect to the analogous one, $\mathcal{A}_{\rm ZSP}$, for the zero skew-prime case [2]). In order to obtain a set that behaves as $\mathcal{A}_{\rm ZSP}$, we have to introduce the following subset of $\mathcal{A}_{\rm FSP}$:

$$\begin{array}{rcl} \mathcal{A}^*_{\rm FSP} & := & \{A : \mathfrak{B} = \ker(AB), (A,B) \text{ is factor skew-prime}, \\ \ker A & = & \ker\begin{bmatrix} A \\ F \end{bmatrix}\} = \left\{ A \in \mathcal{A}_{\rm FSP} : \ker A = \ker\begin{bmatrix} A \\ F \end{bmatrix} \right\} \\ & = & \{A \in \mathcal{A}_{\rm FSP} : F = PA, \ \exists\, P \text{ L-polynomial matrix}\}. \end{array}$$

The set $\mathcal{A}^*_{\rm FSP}$ behaves exactly as we would expect. Indeed, it is not hard to prove that matrices in $\mathcal{A}^*_{\rm FSP}$ provide kernel representations of the same behavior. Also, it can be easily seen that if A is a matrix s.t. $\ker(AB) = \mathfrak{B}$ and L-polynomial matrices $\bar{A}$ and $\bar{B}$, of suitable sizes, can be found, s.t. $\begin{bmatrix} A & \bar{A} \end{bmatrix}\begin{bmatrix} B \\ -\bar{B} \end{bmatrix} = 0$, where each of the above block matrices is a minimal annihilator of the other, then A is in $\mathcal{A}^*_{\rm FSP}$.

4 Special cases for which the above conditions are equivalent

As we have seen in Theorem 3.1, if we assume that $\mathfrak{B} := \ker(AB)$, $\mathfrak{B}_1 := \ker B$, F is an MLA of B, $\hat{A} := \begin{bmatrix} A \\ F \end{bmatrix}$ and G is an MRA of $\hat{A}$, we get a set of sufficient conditions that, in general, are not necessary. As a matter of fact, examples can be given proving that none of the implications of Theorem 3.1 can be reversed. Situations exist, however, where all the above statements turn out to be equivalent ones. This happens, for instance, when A is of full column rank and B is of full row rank. In order to analyze this situation we need the following lemma.

Lemma 4.1 [3] *Let $\mathfrak{B} = \ker(AB)$, with A a $p \times r$ L-polynomial matrix and B an $r \times q$ L-polynomial matrix, be a behavior. If the pair (A, B) is factor skew-prime, and either A or B is nonsingular square, then there exist matrices $\bar{A} \in \mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]^{p\times(p+q-r)}$ and $\bar{B} \in \mathbb{R}[z_1, z_2, z_1^{-1},\ z_2^{-1}]^{(p+q-r)\times q}$ s.t. $[\,A \quad \bar{A}\,] \begin{bmatrix} B \\ -\bar{B} \end{bmatrix} = 0$, where each of these block matrices is a minimal annihilator (left and right, respectively) of the other.*

Theorem 4.2 [A is of full column rank and B is of full row rank] *Let $\mathfrak{B} = \ker(AB)$, with A a $p\times r$ full column rank L-polynomial matrix and B an $r \times q$ full row rank L-polynomial matrix, be a behavior. Set $\mathfrak{B}_1 := \ker B$. The following facts are equivalent:*

- *i) there exist matrices $\bar{A}$ and $\bar{B}$ of suitable sizes s.t. $[\,A \quad \bar{A}\,] \begin{bmatrix} B \\ -\bar{B} \end{bmatrix} = 0$, where each block matrix is a minimal annihilator of the other;*
- *ii) $\mathfrak{B}$ can be expressed as $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, for some $\mathfrak{B}_2$, with $\mathfrak{B}_1 \cap \mathfrak{B}_2$ a finite-dimensional autonomous behavior;*
- *iii) the pair (A, B) is factor skew-prime.*

Proof i) ⇒ ii) as well as ii) ⇒ iii) have been proved in Theorem 3.1. So, we are remained to prove that iii) implies i). Without loss of generality, we can assume [3] that A is L-primitive w.r.t. the variable z_1, and that L-polynomial matrices L and H, can be found s.t.

$$[\,A \quad L\,] \begin{bmatrix} B \\ H \end{bmatrix} = 0, \tag{3}$$

where each of the above block matrices is a minimal annihilator, over $\mathcal{R}_1$, of the other, and L is of full row rank. Apply to L the primitive factorization algorithm [7], and express it as $L = \Delta_L \bar{L}$, where Δ_L is nonsingular square, with $\det \Delta_L \in \mathbb{R}[z_1, z_1^{-1}]$, and $\bar{L}$ is of full row rank and L-primitive w.r.t. the variable z_1. Of course, Δ_L is, in particular, of full column rank, while $\bar{L}$ is of full row rank, and $\mathcal{V}_L(\Delta_L) \cap \mathcal{V}_L(\bar{L})$ consists of a finite number of points. This ensures, by Proposition 2.1, that the pair $(\Delta_L, \bar{L})$ is factor skew-prime. But then, as Δ_L is nonsingular square, by Lemma 4.1 we can find L-polynomial matrices $\tilde{L}$ and S, of suitable sizes, s.t.

$$[\,\Delta_L \quad \tilde{L}\,] \begin{bmatrix} \bar{L} \\ -S \end{bmatrix} = 0,$$

where each of the above block matrices is a minimal annihilator of the other. Therefore, $L = \Delta_L \bar{L} = \tilde{L} S$. Moreover, S is nonsingular square and, by Lemma A.7 in [12], its determinant coincides (up to units) with $\det \Delta_L \in \mathbb{R}[z_1, z_1^{-1}]$. Consider, now, the identity

$$[\,A \quad \tilde{L}\,] \begin{bmatrix} B \\ \tilde{H} \end{bmatrix} = 0, \tag{4}$$

where $\tilde{H} := SH$. Again each block matrix in (4) is L-polynomial and a minimal annihilator (left and right, respectively) over $\mathcal{R}_1$ of the other. We, first, show that $[\,A \quad \tilde{L}\,]$ is lFP. Of course, $[\,A \quad \tilde{L}\,]$ is prime over $\mathcal{R}_1$. If Q is a $p \times p$ L-polynomial matrix, with $\det Q \in \mathbb{R}[z_1, z_1^{-1}]$, which is a common left divisor of A and $\tilde{L}$, then Q must be a left divisor of $L = \tilde{L} S = \Delta_L \bar{L}$ and hence, by the essential unicity (up to unimodular matrices) of the left divisor Δ_L in any primitive factorization [7], a left divisor of Δ_L. This means that Q is a left divisor of the lFP matrix $[\,\Delta_L \quad \tilde{L}\,]$, and hence it must be a unit.

Consider, now, the L-polynomial matrix $\begin{bmatrix} B \\ \tilde{H} \end{bmatrix}$. In general, it is not right factor prime, but only right prime over $\mathcal{R}_1$, and hence the g.c.d. of its maximal order minors, say ϕ, is an L-polynomial in the variable z_1 alone. Assume that ϕ factorizes as $\phi = p_1^{\nu_1} p_2^{\nu_2} \dots p_t^{\nu_t}$, where the p_i's are distinct irreducible L-polynomials in $\mathbb{R}[z_1, z_1^{-1}]$ and the ν_i's are positive integers. We aim to provide an algorithm to iteratively modify matrices $\tilde{L}$ and $\tilde{H}$ in such a way that their product is still $-AB$, the matrix $[\,A \quad \tilde{L}\,]$ is still lFP, but the number of distinct irreducible factors appearing in ϕ is strictly decreasing.

Set $L^{(0)} := \tilde{L}$, $H^{(0)} := \tilde{H}$ and $\phi^{(0)} := \phi$. Consider the first irreducible factor in ϕ, namely p_1. Then [3], there exists at least one maximal order minor of $[\,A \quad L^{(0)}\,]$ (let us call it $\tilde{m}$), corresponding to a $p \times p$ submatrix which includes all r columns of A, which is not a multiple of p_1. Let $i_1, \dots, i_{p-r} \in \{1, 2, \dots, p + q - r\}$ be the indices of the columns of $L^{(0)}$ involved in such a submatrix, and let $j_1, \dots, j_q \in \{1, 2, \dots, p + q - r\}$ be the indices of the remaining columns of $L^{(0)}$. The maximal order minor of the submatrix of $\begin{bmatrix} B \\ H^{(0)} \end{bmatrix}$ obtained by selecting the rows of $H^{(0)}$ of indices $j_1, \dots, j_q$ is, of course, a maximal order minor of $H^{(0)}$, and it can be expressed (see Lemma A.7 in [12]), up to units, as $\tilde{m}\phi^{(0)}$. Let S_j be the $q \times (p+q-r)$ selection matrix that singles out the rows of $H^{(0)}$ of indices $j_1, \dots, j_q$, and hence s.t.

$\det(S_j H^{(0)}) = \tilde{m}\phi^{(0)}$. By applying Lemma 4.1 in [7], we can factorize $S_j H^{(0)}$ as $\Delta_1 M_j$, with $\det \Delta_1 = p_1^{\nu_1}$ and $\det M_j = \tilde{m}\phi^{(0)}/p_1^{\nu_1}$.

Denote by V_1 the $(p+q-r)\times(p+q-r)$ L-polynomial matrix whose columns of indices $i_1, \dots, i_{p-r}$ are the canonical vectors $\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_{p-r}}$, while the columns of indices $j_1, \dots, j_q$ are those of $S_j^T \Delta_1$. Set, now, $L^{(1)} := L^{(0)} V_1$, $H^{(1)} := V_1^{-1} H^{(0)}$, and $\phi^{(1)} := \phi^{(0)}/p_1^{\nu_1}$. Of course, $AB = -L^{(1)} H^{(1)}$. In order to show that $[\,A \quad L^{(1)}\,]$ is still lFP, we first observe that, from the identity

$$[\,A \quad L^{(1)}\,] = [\,A \quad L^{(0)}\,] \begin{bmatrix} I_r & 0 \\ 0 & V_1 \end{bmatrix}$$

one gets $\mathcal{V}_L([\,A \quad L^{(0)}\,]) \subseteq \mathcal{V}_L([\,A \quad L^{(1)}\,]) \subseteq \mathcal{V}_L([\,A \quad L^{(0)}\,]) \cup \mathcal{V}_L(\det V_1)$. On the other hand, as $\mathcal{V}_L([\,A \quad L^{(1)}\,])$ is trivially included in itself, we get $\mathcal{V}_L([\,A \quad L^{(1)}\,]) \subseteq [\mathcal{V}_L([\,A \quad L^{(0)}\,]) \cup \mathcal{V}_L(\det V_1)] \cap \mathcal{V}_L([\,A \quad L^{(1)}\,]) \subseteq \mathcal{V}_L([\,A \quad L^{(1)}\,])$, and hence $\mathcal{V}_L([\,A \quad L^{(1)}\,]) = [\mathcal{V}_L([\,A \quad L^{(0)}\,]) \cup \mathcal{V}_L(\det V_1)] \cap \mathcal{V}_L([\,A \quad L^{(1)}\,])$. As the columns of indices $i_1, \dots, i_{p-r}$ are the same in $L^{(0)}$ and in $L^{(1)}$, $\tilde{m}$ is a maximal order minor also of $[\,A \quad L^{(1)}\,]$, and hence $\mathcal{V}_L(\tilde{m}) \supseteq \mathcal{V}_L([\,A \quad L^{(1)}\,])$. Therefore, by keeping in mind also that $\mathcal{V}_L(\det V_1) = \mathcal{V}_L(p_1)$, we get $[\mathcal{V}_L([\,A \quad L^{(0)}\,]) \cup \mathcal{V}_L(\det V_1)] \cap \mathcal{V}_L([\,A \quad L^{(1)}\,]) \subseteq \mathcal{V}_L([\,A \quad L^{(0)}\,]) \cup [\mathcal{V}_L(p_1) \cap \mathcal{V}_L(\tilde{m})] = \mathcal{V}_L([\,A \quad L^{(0)}\,]) \cup \mathcal{V}_L(p_1, \tilde{m})$.

As p_1 and $\tilde{m}$ are devoid of common factors, $\mathcal{V}_L([\,A \quad L^{(1)}\,])$ differs from $\mathcal{V}_L([\,A \quad L^{(0)}\,])$ in a finite number of points (belonging to $\mathcal{V}_L(p_1, \tilde{m})$), and hence $[\,A \quad L^{(1)}\,]$ is lFP, too. Finally, by noticing that the maximal order minor $\tilde{m}\phi^{(0)}$ of $\begin{bmatrix} B \\ H^{(0)} \end{bmatrix}$ is now replaced by $\tilde{m}\phi^{(1)}$ in $\begin{bmatrix} B \\ H^{(1)} \end{bmatrix}$, the g.c.d. of the maximal order minors of $\begin{bmatrix} B \\ H^{(1)} \end{bmatrix}$ is now $\phi^{(1)}$.

By considering, now, all remaining irreducible factors, $p_2, \dots, p_t$, we can perform the same algorithm now presented, thus obtaining, after t steps (t being the number of distinct irreducible factors in ϕ), matrices $L^{(t)}$ and $H^{(t)}$ s.t. $[\,A \quad L^{(t)}\,]$ is lFP and $AB = -L^{(t)} H^{(t)}$, while $\begin{bmatrix} B \\ H^{(t)} \end{bmatrix}$ is now rFP. ■

Remarks Notice that if A is of full column rank then it has no nontrivial right divisors, and hence G is the empty matrix. Also, as B is of full row rank, even F is the empty matrix and hence $\mathcal{A}^*_{\mathrm{FSP}} = \mathcal{A}_{\mathrm{FSP}}$ and $\hat{A} = A$.

Finally, the result [13] stating that every 2D complete behavior $\mathfrak{B}$ admits an autonomous/controllable decomposition $\mathfrak{B} = \mathfrak{B}_a + \mathfrak{B}_c$, with finite-dimensional intersection $\mathfrak{B}_a \cap \mathfrak{B}_c$, can be viewed as a simple corollary of Theorem 4.2.

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