

# A two-variable approach to solve the polynomial Lyapunov equation

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## Abstract

A two-variable polynomial approach to solve the one-variable polynomial Lyapunov equation is put forward. Such approach yields an iterative solution method based on the method of Faddeev for the computation of matrix resolvents. The resulting algorithm is especially suitable for applications requiring symbolic or exact computation.

## 1 Introduction

Let  $R$  be a real nonsingular  $q \times q$  polynomial matrix in the indeterminate  $\xi$ , i.e.,

$$R(\xi) = R_0 + R_1\xi + \dots + R_L\xi^L$$

where  $R_i$  is a real  $q \times q$  constant matrix,  $i = 1, \dots, L$ , and  $\det(R)$  is not the zero polynomial. For some positive integer  $p$ , let  $Q$  be a real  $p \times q$  polynomial matrix in the indeterminate  $\xi$  such that  $QR^{-1}$  is a matrix of strictly proper rational functions. Finally, let  $\Sigma$  be a  $p \times p$  signature matrix. The equation

$$R(-\xi)^T X(\xi) + X(-\xi)^T R(\xi) = Q(-\xi)^T \Sigma Q(\xi) \quad (1)$$

in the unknown  $q \times q$  polynomial matrix  $X$  is called the *polynomial Lyapunov equation* (in the following simply PLE).

The PLE arises in various areas of mathematics, for example in the computation of integrals of quadratic functionals of the variables of a system and of their derivatives [10, 11]. It also arises in stability theory for higher order differential equations as a generalization of the usual Lyapunov stability test for first order systems [7, 14]. The solution of PLEs is also required in iterative algorithms for spectral factorization based on Newton's method [5]; in such case the right-hand side of the PLE is a general *para-Hermitian*  $q \times q$  polynomial matrix  $Z(\xi)$ , i.e.  $Z$  satisfies  $Z(\xi) = Z(-\xi)^T$ , and it is not given in the factored form  $Q(-\xi)^T \Sigma Q(\xi)$ . However, the algorithm presented in this communication can be

used to solve also such more general form of PLE; we refer to [8] for details.

In this communication we present a new algorithm to solve the PLE. The procedure we propose is based on lifting the problem from the one-variable polynomial context in which it is originally formulated to a two-variable setting, thus associating a two-variable polynomial equation called the *lifted polynomial Lyapunov equation* (LPLE in the following) to the PLE (see [12] for another application of such lifting technique). Such an approach allows to cast the problem of solving the PLE as that of finding the inverse image of an element of a finite-dimensional vector space under the action of a linear operator, a problem that can be solved in an iterative way using the Faddeev method for finding the inverse of a matrix (see [4] for another application of the Faddeev method).

Iterative approaches for the solution of the scalar PLE have already appeared in the literature, but the algorithm presented in this communication significantly differs from them: it is based on a two-variable polynomial approach and it uses a Faddeev sequence method. Its advantages are manifold: it can be applied both to scalar and to multivariable problems, it requires no preprocessing of the matrices involved in the PLE, and it is suitable especially for problems requiring symbolic or exact computations. Such generality arises from the possibility of using the calculus of quadratic differential forms (QDF's) developed in [13]. QDF's are quadratic functionals of the variables of a system and their derivatives, which are intimately related to two-variable polynomial matrices.

The paper is organized as follows: in Section 2 we review some notions regarding polynomial matrices and shifts in a single variable. Such notions are extended to the case of symmetric two-variable polynomials in Section 3, where we define the Lyapunov operator, a two-variable shift operator acting on a finite-dimensional vector space associated with the PLE; such operator is of fundamental importance in our two-variable framework for the study of the PLE. In Section 4 we associate

with the PLE a two-variable polynomial equation called the LPLE, we discuss the relationship of the PLE with the LPLE, and we show how the Lyapunov operator relates to the LPLE. In Section 5 we state our iterative algorithm to solve the LPLE and we show how to use a solution to the LPLE in order to compute a solution to the PLE. We also apply the algorithm to a PLE involving  $2 \times 2$  matrices in a worked example. No proofs are provided for the result presented in this communication; they can be found in the forthcoming paper [8].

## 2 $R$ -equivalence, $R$ -canonicity, and the shift operator in one variable

In this section we review some notions regarding polynomial matrices in a single variable. The material of this section is not new, although the terminology may differ from that used elsewhere; see [1], [2] and [13].

Let  $R$  be an element of  $\mathbb{R}^{q \times q}[\xi]$ , the set of  $q \times q$  real polynomial matrices in the indeterminate  $\xi$ . We call  $R$  *nonsingular* if the polynomial  $\det(R)$  does not vanish identically.  $R$  induces an equivalence relation on the set  $\mathbb{R}^{1 \times q}[\xi]$  as follows.

**Definition 2.1** *Two  $1 \times q$  polynomial matrices  $D_1, D_2 \in \mathbb{R}^{1 \times q}[\xi]$  are called  $R$ -equivalent if there exists a polynomial matrix  $P \in \mathbb{R}^{1 \times q}[\xi]$  such that  $D_1 - D_2 = PR$ . A polynomial matrix  $D \in \mathbb{R}^{1 \times q}[\xi]$  is called  $R$ -canonical if the rational matrix  $DR^{-1}$  is strictly proper.*

It is easily shown that every  $1 \times q$  polynomial matrix  $D$  admits a unique  $R$ -canonical polynomial matrix  $D'$  which is  $R$ -equivalent to  $D$ . Such matrix  $D'$  can be computed as  $D' = SR = D - P'R$ , where  $P'$  denotes the polynomial part and  $S$  the strictly proper rational part of  $DR^{-1} = P' + S$ . We denote the  $R$ -canonical representative  $D'$  of the  $R$ -equivalence class of  $D$  with  $D \bmod R$ . Uniqueness follows from the observation that if  $D'' = D - P''R$  is another  $R$ -canonical matrix  $R$ -equivalent to  $D$  with  $P''$  polynomial, then  $(D' - D'')R^{-1} = P' - P''$  is both strictly proper and polynomial, and consequently zero.

We denote the subset of  $\mathbb{R}^{1 \times q}[\xi]$  consisting of all  $R$ -canonical polynomial matrices with  $\mathcal{C}_R^{1 \times q}[\xi]$ . It is easily verified that  $\mathcal{C}_R^{1 \times q}[\xi]$  is a vector space identified in a natural way with the vector space of  $R$ -equivalence classes in  $\mathbb{R}^{1 \times q}[\xi]$ .  $\mathcal{C}_R^{1 \times q}[\xi]$  is finite-dimensional; the relevant result is the following.

**Proposition 2.2** *The space  $\mathcal{C}_R^{1 \times q}[\xi]$  is a finite dimensional vector space over  $\mathbb{R}$  of dimension  $n = \deg(\det(R))$ .*

We proceed to define the *one-variable polynomial shift operator*  $S$  acting on  $\mathcal{C}_R^{1 \times q}[\xi]$ .

**Definition 2.3** *The one-variable polynomial shift operator  $S : \mathcal{C}_R^{1 \times q}[\xi] \rightarrow \mathcal{C}_R^{1 \times q}[\xi]$  is defined as*

$$S(D(\xi)) := \xi D(\xi) \bmod R(\xi)$$

It is immediate to see that  $S$  is a linear operator. Its characteristic polynomial is uniquely determined by  $\det(R)$ , as stated in the following proposition.

**Proposition 2.4** *The characteristic polynomial  $\chi_S(\xi)$  of  $S$  is*

$$\chi_S(\xi) = \prod_{i=1}^n (\xi - \lambda_i)$$

where  $n = \deg(\det(R(\xi)))$  and  $\lambda_i, i = 1, \dots, n$  are the roots of  $\det(R)$ .

The notions of  $R$ -equivalence and  $R$ -canonicity, and the definition of  $S$  are extended to matrices in  $\mathbb{R}^{k \times q}[\xi]$  in a row-wise manner. In the following we denote the subspace of  $R$ -canonical elements of  $\mathbb{R}^{k \times q}[\xi]$  with  $\mathcal{C}_R^{k \times q}[\xi]$ .

## 3 $R$ -equivalence, $R$ -canonicity, and the shift operator in two variables

In this section we first introduce the notions of  $R$ -equivalence and  $R$ -canonicity; such concepts were originally developed in the context of quadratic differential forms, see [13]. We then proceed to define the shift operator in a two-variable polynomial context. Such shift operator is of fundamental importance for the development of the algorithm that constitutes the main result of this communication, and for reasons that will be apparent in Section 4 it is called the *Lyapunov operator*.

We denote the set of  $q \times q$  real polynomial matrices in the two indeterminates  $\zeta$  and  $\eta$  with  $\mathbb{R}^{q \times q}[\zeta, \eta]$ , and we call an element  $Y$  of such space *symmetric* if  $Y(\zeta, \eta) = Y(\eta, \zeta)^T$ . It is a matter of straightforward verification to see that  $\mathbb{R}^{q \times q}[\zeta, \eta]$  is a vector space over  $\mathbb{R}$ , and that the subset of  $\mathbb{R}^{q \times q}[\zeta, \eta]$  consisting of all symmetric polynomial matrices is a subspace. We denote such subspace with  $\mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$ .

Now consider a nonsingular one-variable polynomial matrix  $R \in \mathbb{R}^{q \times q}[\xi]$ . In Definition 2.1 we considered an equivalence relation on the space of one-variable polynomial matrices induced by  $R$ ; we now define an equivalence relation induced by  $R$  on the space of symmetric two-variable polynomial matrices  $\mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$ .

**Definition 3.1** *Two symmetric  $q \times q$  polynomial matrices  $Y_1, Y_2 \in \mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$  are called  $R$ -equivalent if there*

exists a polynomial matrix  $P \in \mathbb{R}^{q \times q}[\zeta, \eta]$  such that  $Y_1(\zeta, \eta) - Y_2(\zeta, \eta) = R(\zeta)^T P(\zeta, \eta) + P(\eta, \zeta)^T R(\eta)$ . A polynomial matrix  $Y \in \mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$  is called  $R$ -canonical if  $R(\zeta)^{-T} Y(\zeta, \eta) R(\eta)^{-1}$  is a matrix of strictly proper two-variable rational functions.

Every  $Y \in \mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$  admits a unique  $R$ -canonical two-variable symmetric polynomial matrix  $Y'$  which is  $R$ -equivalent to  $Y$ . Such  $R$ -canonical representative  $Y'$  of the  $R$ -equivalence class of  $Y$  can be obtained as follows. Following Section 3 of [13], compute a factorization of  $Y$  as  $Y(\zeta, \eta) = M(\zeta)^T N(\eta)$ . Compute the (one-variable)  $R$ -canonical representatives  $M' = M \bmod R$  of  $M$  and  $N' = N \bmod R$  of  $N$ . Then  $Y'(\zeta, \eta) = M'(\zeta)^T N'(\eta)$ . Uniqueness of  $Y'$  can be proved as in Proposition 4.9 of [13]. In the following we will denote the  $R$ -canonical representative of  $Y(\zeta, \eta)$  with  $Y \bmod R$ .

The subset of  $\mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$  consisting of all  $R$ -canonical two-variable symmetric polynomial matrices is denoted with  $\mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$ , and it can be identified in a natural way with the vector space of  $R$ -equivalence classes in  $\mathbb{R}_{\text{sym}}^{q \times q}[\zeta, \eta]$ .  $\mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  is a finite-dimensional subspace, as stated in the following proposition.

**Proposition 3.2**  $\mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  is a finite-dimensional vector space over  $\mathbb{R}$  of dimension  $n(n+1)/2$ , where  $n = \deg(\det(R))$ .

We proceed to define the two-variable shift operator  $\mathcal{L}_R$  acting on  $\mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  (compare Definition 2.3 of the one-variable polynomial shift).

**Definition 3.3** The two-variable shift operator  $\mathcal{L}_R : \mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta] \rightarrow \mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  is defined as

$$\mathcal{L}_R(Y(\zeta, \eta)) := (\zeta + \eta)Y(\zeta, \eta) \bmod R$$

For reasons that will become evident in the next section, in the following we will refer to the operator  $\mathcal{L}_R$  as the *Lyapunov operator* associated with  $R$ .

Observe that  $\mathcal{L}_R$  is linear. Its characteristic polynomial is completely determined by  $\det(R)$  (compare Proposition 2.4).

**Proposition 3.4** The characteristic polynomial  $\chi_{\mathcal{L}_R}(\xi)$  of the Lyapunov operator  $\mathcal{L}_R$  is

$$\chi_{\mathcal{L}_R}(\xi) := \prod_{1 \leq i \leq j \leq n} (\xi - (\lambda_i + \lambda_j)) \quad (2)$$

where  $n = \deg(\det(R))$  and  $\lambda_1, \dots, \lambda_n$  are the zeros of  $\det(R)$ .

## 4 The lifted polynomial Lyapunov equation

In this section we define our two-variable polynomial framework for the PLE. We first lift the problem of solving the PLE from the one-variable polynomial context in which it is originally formulated to a two-variable polynomial context, introducing a two-variable polynomial equation associated with the matrices  $R$ ,  $Q$  and  $\Sigma$  of the PLE.

**Definition 4.1** The equation

$$(\zeta + \eta)Y(\zeta, \eta) \bmod R = Q(\zeta)^T \Sigma Q(\eta) \quad (3)$$

in the unknown  $R$ -canonical symmetric two-variable polynomial matrix  $Y \in \mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  is called the lifted polynomial Lyapunov equation (LPLE).

Solvability of the PLE is equivalent to solvability of the LPLE, as stated in the following proposition.

**Proposition 4.2** Let  $R \in \mathbb{R}^{q \times q}[\xi]$  be nonsingular, let  $Q \in \mathbb{R}^{p \times q}[\xi]$  be  $R$ -canonical and let  $\Sigma$  be a  $p \times p$  signature matrix. Then the following two statements are equivalent:

- (a) There exists a solution  $X \in \mathbb{R}^{q \times q}[\xi]$  to the PLE (1);
- (b) There exists a solution  $Y \in \mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  to the LPLE (3).

Besides being interesting in its own right, the result of Proposition 4.2 provides a method for constructing a solution  $X$  to the PLE from a given solution  $Y$  to the LPLE. Indeed, observe that if  $Y$  is a solution to the LPLE then by definition of  $R$ -equivalence there exists a polynomial matrix  $P \in \mathbb{R}^{q \times q}[\zeta, \eta]$  such that

$$\begin{aligned} (\zeta + \eta)Y(\zeta, \eta) + R(\zeta)^T P(\zeta, \eta) + P(\eta, \zeta)^T R(\eta) \\ = Q(\zeta)^T \Sigma Q(\eta). \end{aligned}$$

A solution to the PLE is then obtained from  $P(\zeta, \eta)$  by substituting  $\zeta = -\xi$  and  $\eta = \xi$ , yielding  $X(\xi) := P(-\xi, \xi)$ . Actually, it can be shown that an  $R$ -canonical solution  $X$  to the PLE can be expressed directly in terms of a solution  $Y$  to the LPLE. The relevant result is the following.

**Proposition 4.3** Let  $Y \in \mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$  be a solution to the LPLE. Then an  $R$ -canonical solution  $X \in \mathcal{C}_R^{q \times q}[\xi]$  to the PLE is given by

$$X(\xi) := - \lim_{|\mu| \rightarrow \infty} \mu R(\mu)^{-T} Y(\mu, \xi). \quad (4)$$

Proposition 4.2 and equation (4) show that *in order to solve the PLE one may proceed by solving the LPLE first and subsequently constructing a R-canonical solution to the PLE directly from the solution to the LPLE.*

If we denote the right-hand side of the LPLE by  $\Phi(\zeta, \eta) := Q(\zeta)^T \Sigma Q(\eta)$ , then the LPLE can be written in a compact way as  $\mathcal{L}_R(Y) = \Phi$ , with  $\mathcal{L}_R$  the Lyapunov operator defined in Definition 3.3. From Proposition 3.4 a necessary and sufficient condition for the existence of a unique solution *to the LPLE* is immediate. It is remarkable that the same condition also characterizes the existence of a unique *R-canonical solution to the PLE.*

**Proposition 4.4** *Let  $R \in \mathbb{R}^{q \times q}[\xi]$  be nonsingular, let  $Q \in \mathbb{R}^{p \times q}[\xi]$  be R-canonical and let  $\Sigma$  be a  $p \times p$  signature matrix. Let  $n = \deg(\det(R))$  and let  $\lambda_1, \dots, \lambda_n$  be the zeros of  $\det(R)$ . Then the following three statements are equivalent.*

(a) *The following condition is satisfied:*

$$\lambda_i + \lambda_j \neq 0 \quad \text{for all } i, j = 1, 2, \dots, n. \quad (5)$$

(b) *The LPLE has a unique (R-canonical) solution.*

(c) *The PLE has a unique R-canonical solution.*

For obvious reasons we call condition (5) the *invertibility condition* for the operator  $\mathcal{L}_R$ . Observe that such condition is satisfied when  $R$  is Hurwitz, i.e. when  $\det(R)$  is an Hurwitz polynomial; in this sense the result of Proposition 4.4 constitutes a generalization of Theorem 4.8 of [13].

## 5 An iterative algorithm to solve the PLE

In this section we state an iterative procedure for solving the PLE under the assumption that the invertibility condition (5) is satisfied. Our algorithm is inspired by the Faddeev method for computing the inverse  $A^{-1}$  and the resolvent  $(zI_n - A)^{-1}$  of a square matrix  $A$ , which we now briefly review.

Assume that the  $n \times n$  matrix  $A$  is invertible, and let  $\chi_A(z) = \det(zI_n - A) := z^n + \chi_1 z^{n-1} + \dots + \chi_{n-1} z + \chi_n$  be the characteristic polynomial of  $A$ . Observe that  $\chi_n = (-1)^n \det(A) \neq 0$  and that from Cayley-Hamilton's theorem it follows that  $\chi_A(A) = 0$ . Consequently,  $A(A^{n-1} + \chi_1 A^{n-2} + \dots + \chi_{n-1} I_n) = -\chi_n I_n$ , and therefore the inverse of  $A$  can be computed as

$$A^{-1} = -\frac{1}{\chi_n} (A^{n-1} + \chi_1 A^{n-2} + \dots + \chi_{n-1} I_n)$$

Of course such formula involve the knowledge of the coefficients of the characteristic polynomial  $\chi_A(z)$  of

$A$ , but it turns out that the  $\chi_i$ 's can also be computed in an iterative way, knowing only the traces of certain matrices related to  $A$  (see Section IV.5 of [3] for more details).

An analogous approach can be used in order to compute the unique solution  $\hat{x} = A^{-1}b$  to the linear equation  $Ax = b$ . Indeed, the following iterations produce  $\hat{x}$ :

$$x_0 := b, \quad (6)$$

$$x_k := Ax_{k-1} + \chi_k b, \quad (k = 1, 2, \dots, n-1) \quad (7)$$

$$\hat{x} := -\frac{1}{\chi_n} x_{n-1}. \quad (8)$$

We now proceed to illustrate how such techniques can be applied in order to compute a solution to the LPLE. Recall from the discussion after Proposition 4.3 that the LPLE can be interpreted as the linear equation  $\mathcal{L}_R(Y) = \Phi$  in the finite-dimensional vector space  $\mathcal{C}_{R, \text{sym}}^{q \times q}[\zeta, \eta]$ , where  $\mathcal{L}_R$  is the Lyapunov operator and  $\Phi(\zeta, \eta) := Q(\zeta) \Sigma Q(\eta)$ . Observe also that the characteristic polynomial of  $\mathcal{L}_R$  is known once  $\det(R)$  has been computed, compare (2). Consequently, in order to come up with a procedure for solving the LPLE (and therefore, as it follows from Proposition 4.3, with a procedure for solving the PLE) we only need to adapt the iterations (6)-(8) to the case at hand. Such simple observation allows us to state the main result of this paper.

**Proposition 5.1** *Let  $R \in \mathbb{R}^{q \times q}[\xi]$  be nonsingular, let  $Q \in \mathbb{R}^{p \times q}[\xi]$  be R-canonical and let  $\Sigma$  be a  $p \times p$  signature matrix. Assume that the invertibility condition (5) holds. Let  $\chi_{\mathcal{L}_R}(z) = z^d + \chi_1 z^{d-1} + \dots + \chi_{d-1} z + \chi_d$  be the characteristic polynomial of the Lyapunov operator  $\mathcal{L}_R$  as given by Eqn. (2), with  $d = n(n+1)/2$  and  $n = \deg(\det(R))$ . Denote the right-hand side of the LPLE by  $\Phi(\zeta, \eta) = Q(\zeta)^T \Sigma Q(\eta)$ . Consider the iterations:*

$$Y_0 := \Phi, \quad (9)$$

$$Y_k := \mathcal{L}_R(Y_{k-1}) + \chi_k \Phi, \quad (10)$$

for  $k = 1, 2, \dots, d-1$ . Then the two-variable polynomial matrix

$$Y := -\frac{1}{\chi_d} Y_{d-1} \quad (11)$$

is the unique solution to the LPLE. The unique R-canonical solution  $X$  to the PLE is computed as

$$X(\xi) := -\lim_{|\mu| \rightarrow \infty} \mu R(\mu)^{-T} Y(\mu, \xi). \quad (12)$$

We proceed to illustrate the application of the iterative procedure (9)-(12) with an example.

**Example 1** In this example we illustrate the algorithm step by step for a PLE with parameters

$$R(\xi) = \begin{pmatrix} -3 + \xi & -6 - 3\xi + 3\xi^2 \\ 0 & -2 - \xi + \xi^2 \end{pmatrix},$$

$$Q(\xi) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix  $R$  is column reduced (see Section 6.3 of [6]) with column degrees 1 and 2, and consequently the set of  $R$ -canonical (one-variable) polynomial matrices consists of all matrices with constant entries in the first column and entries of polynomial degree at most 1 in the second column (see Section 6.3 of [6]). Observe that the matrix  $Q$  is  $R$ -canonical.

The characteristic polynomial of  $\mathcal{L}_R$  is computed from Eqn. (2); the zeros of  $\det(R)$  are  $-1$ ,  $2$  and  $3$ , and consequently

$$\chi_{\mathcal{L}_R}(z) = z^6 - 16z^5 + 85z^4 - 130z^3 - 236z^2 + 776z - 480.$$

Since the constant term is nonzero, the invertibility condition holds and the PLE is solvable with a unique  $R$ -canonical solution  $X$  (see Proposition 4.4).

The iterations (9)-(10) generate the following sequence of two-variable  $R$ -canonical matrices  $Y_i(\zeta, \eta)$ ,  $i = 1, \dots, 5$ :

$$Y_1 = \begin{pmatrix} -10 & 2(-13 + \eta) \\ 2(-13 + \zeta) & 5(-16 + \eta + \zeta) \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} 25 & 6(16 - 3\eta) \\ 6(16 - 3\zeta) & 5(89 - 15\eta - 15\zeta + 2\zeta\eta) \end{pmatrix}$$

$$Y_3 = \begin{pmatrix} 20 & 8(-1 + 3\eta) \\ 8(-1 + 3\zeta) & 10(-95 + 39\zeta + 39\eta - 13\zeta\eta) \end{pmatrix}$$

$$Y_4 = \begin{pmatrix} -116 & 8(-56 + 11\eta) \\ 8(-56 + 11\zeta) & 20(19 - 41\eta - 41\zeta + 26\zeta\eta) \end{pmatrix}$$

$$Y_5 = \begin{pmatrix} 80 & 96(4 - \eta) \\ 96(4 - \zeta) & 600(1 + \zeta + \eta - \zeta\eta) \end{pmatrix}.$$

It can be verified that

$$Y(\zeta, \eta) := \frac{1}{480} Y_5(\zeta, \eta) =$$

$$= \begin{pmatrix} \frac{1}{6} & \frac{4}{5} - \frac{1}{5}\eta \\ \frac{4}{5} - \frac{1}{5}\zeta & \frac{5}{4} + \frac{5}{4}\eta + \frac{5}{4}\zeta - \frac{5}{4}\zeta\eta \end{pmatrix}$$

is a solution to the LPLE  $\mathcal{L}_R(Y(\zeta, \eta)) = Q(\zeta)^T \Sigma Q(\eta)$ . The unique  $R$ -canonical solution  $X$  to the PLE is computed from  $Y$  as in (12):

$$X(\xi) = \begin{pmatrix} -\frac{1}{6} & -\frac{4}{5} + \frac{1}{5}\xi \\ \frac{7}{10} & \frac{23}{20} + \frac{13}{20}\xi \end{pmatrix} \square$$

We conclude this section with some remarks regarding symbolic and exact computational issues and some possible modifications of the iterations (9)-(12) in order to speed up computations when specially structured matrices  $R$  are present.

**Remark 1** Knowledge of the characteristic polynomial of  $\mathcal{L}_R$  is fundamental for being able to apply the procedure spelled out in Proposition 5.1. Observe that in the context of symbolic or exact computation it is not advisable to compute the characteristic polynomial of  $\mathcal{L}_R$  from the zeros  $\lambda_i$  of  $\det(R)$  as might be suggested by Eqn. (2). An efficient rational algorithm to compute the coefficients of  $\chi_{\mathcal{L}_R}$  directly from the coefficients of  $\det(R)$  can be designed using Faddeev-type recursions. The reader is referred to [8] for more details.

**Remark 2** The algorithm (9)-(12) involves the computation of the  $R$ -canonical representatives of  $(\zeta + \eta)Y_{k-1}(\zeta, \eta)$  for  $k = 1, 2, \dots, d - 1$ . It is easy to see that if one defines the matrix  $Y'_k(\xi) := -\lim_{|\mu| \rightarrow \infty} \mu R(\mu)^{-T} Y_k(\mu, \xi)$  it holds that  $(\zeta + \eta)Y_{k-1}(\zeta, \eta) \bmod R = (\zeta + \eta)Y_{k-1}(\zeta, \eta) + R(\zeta)^T Y'_{k-1}(\eta) + Y'_{k-1}(\zeta)^T R(\eta)$ . There are good reasons why an implementation directly based on this formula for  $Y'_{k-1}$  should be avoided. First of all, the definition of  $Y'_{k-1}$  requires the knowledge of the rational matrix  $R(\xi)^{-1}$ , which in an exact computation context is a delicate issue. Second, a limit operation is required, and this may require a large processing time even for matrices of small dimensions. The authors have devised a Faddeev-type recursion that enables the computation of  $Y'_{k-1}$  with polynomial operations only and which only requires division between the highest-power coefficients of certain univariate polynomials. Such recursion is described in detail in [9].  $\square$

**Remark 3** In many cases the matrix  $R(\xi) = R_0 + R_1\xi + \dots + R_L\xi^L$  has the property that its leading coefficient matrix  $R_L$  is nonsingular; for example, the scalar PLE  $r(-\xi)x(\xi) + x(-\xi)r(\xi) = q(-\xi)q(\xi)$ , where  $r, q$  and  $x \in \mathbb{R}[\xi]$  is such a case. The authors have developed an algorithm that takes advantage of the assumption  $\det(R_L) \neq 0$ ; see [9].  $\square$

## 6 Conclusions

The algorithm for solving the PLE presented in this communication works directly with the polynomial matrices that constitute the PLE; no preprocessing or transformations to canonical forms are required. The amount of bookkeeping necessary to perform the computations is minimal and the procedure is straightforward to implement. Moreover, the methods employed make the algorithm especially suitable for exact and symbolic computation purposes.

The application of the two-variable polynomial framework proposed in this paper to the solution of other polynomial equations relevant for systems and control applications is currently under investigation. Other di-

rections of research currently pursued involve also the case of singular Lyapunov operators.

### References

- [1] Fuhrmann, P.A., *Linear systems and operators in Hilbert space*, McGraw-Hill, New York, 1981.
- [2] Fuhrmann, P.A., *A polynomial approach to linear algebra*, Springer-Verlag, Berlin, 1996.
- [3] Gantmacher, F.R., *The theory of matrices I*, Chelsea Publ. Co., New York, N.Y., 1959.
- [4] Hanzon, B. and Peeters, R.L.M., “A Faddeev sequence method for solving Lyapunov and Sylvester equations”, *Linear Algebra and Its Applications*, vol. 241–243, pp. 401–430, 1996.
- [5] Ježek, J. and Kučera, V., “Efficient Algorithm for Matrix Spectral Factorization”, *Automatica*, Vol. 21, no. 6, pp. 663–669, 1985.
- [6] Kailath, T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.
- [7] Kalman, R.E., “Algebraic characterization of polynomials whose zeros lie in certain algebraic domains”, *Proc. Nat. Acad. Sciences*, vol. 64, pp. 818–823, 1969.
- [8] Peeters, R.L.M. and Rapisarda, P., “A two-variable approach to solve the polynomial Lyapunov equation”, will appear in *Systems and Control Letters*.
- [9] Peeters, R.L.M. and Rapisarda, P., “A new approach to solve the polynomial Lyapunov equation”, Report M 99-003, Dept. Mathematics, University of Maastricht, 1999.
- [10] Talbot, A., “The evaluation of integrals of products of linear system responses- Part I”, *Quart. Journ. Mech. and Applied Math.*, Vol. 12, no. 4, pp. 488–503, 1959.
- [11] Talbot, A., “The evaluation of integrals of products of linear system responses - Part II. Continued-fraction expansion”, *Quart. Journ. Mech. and Applied Math.*, Vol. 12, no. 4, pp. 504–520, 1959.
- [12] Trentelman, H.L. and Rapisarda, P., “New algorithms for polynomial  $J$ -spectral factorization”, *Mathematics of Control, Signals and Systems*, vol. 12, pp. 24–61, 1999.
- [13] Trentelman, H.L. and Willems, J.C., “On quadratic differential forms”, *SIAM J. Control Opt.*, vol. 36, no. 5, pp. 1703–1749, 1998.
- [14] Willems, J.C. and Fuhrmann, P.A., “Stability theory for high order equations”, *Linear Algebra and Its Applications*, vol. 167, pp. 131–149, 1992.