

All unmixed solutions of the algebraic Riccati equation using Pick matrices

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Abstract

In this short paper we study the existence of positive and negative semidefinite solutions of the algebraic Riccati equation corresponding to linear quadratic problems with an indefinite cost functional. An important role is played by certain two-variable polynomial matrices associated with the algebraic Riccati equation. We characterize all unmixed solutions in terms of the Pick matrices associated with these two-variable polynomial matrices. As a corollary it turns out that the signatures of the extremal solutions are determined by the signatures of particular Pick matrices.

1 Introduction

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be such that (A, B) is a controllable pair. Let $Q \in \mathbb{R}^{n \times n}$ be symmetric and let $R \in \mathbb{R}^{m \times m}$ be nonsingular. Finally, let $S \in \mathbb{R}^{m \times n}$. The quadratic equation

$$A^T K + KA + Q - (KB + S^T)R^{-1}(B^T K + S) = 0 \quad (1)$$

in the unknown $n \times n$ matrix K is called the (continuous-time) algebraic Riccati equation, in the following called simply the ARE. Since its introduction in control theory at the beginning of the sixties, the ARE has been studied extensively because of its prominent role in linear quadratic optimal control and filtering, H_∞ -optimal control, differential games, and stochastic filtering and control (see [?] for a discussion on the ARE

and its applications and an overview of the existing literature).

In this communication we formulate reasonably simple necessary and sufficient conditions for the existence of at least one real positive semidefinite solution, or of at least one real negative semidefinite solution to the ARE. Some partial results regarding the solution to such problem were obtained in [?, ?, ?, ?, ?]; for an overview of such results and of their relation with the classical problem of the existence of nonnegative storage functions for dissipative systems, we refer to [?].

The necessary and sufficient condition presented in this communication is based on the signs of certain constant $n \times n$ matrices which in the following will be called the *Pick matrices* associated with the ARE; such matrices are constructed in a straightforward way from the parameters appearing in the ARE. It turns out that a real symmetric positive semidefinite solution of the ARE (1) exists if and only if (i) the ARE has at least one real symmetric solution; and (ii) a suitable Pick matrix is negative semidefinite (likewise, the existence of at least one negative semidefinite solution depends on the positive semidefiniteness of a suitable Pick matrix).

In the process of establishing such conditions we obtain a number of intermediate results, among which a new characterization of all unmixed real symmetric solutions of the ARE, and a new characterization of its supremal and infimal real symmetric solutions. Such characterizations are all given in terms of the Pick matrices associated with the ARE.

A few words on notation: in this communication we

adopt the usual symbols \mathbb{R} and \mathbb{C} in order to denote the set of real and complex numbers, respectively. The open and closed right half-planes of \mathbb{C} are denoted respectively by \mathbb{C}_+^0 and \mathbb{C}_+ . Given $\lambda \in \mathbb{C}$, its complex conjugate is denoted by $\bar{\lambda}$. The space of n dimensional real, respectively complex, vectors is denoted by \mathbb{R}^n , respectively \mathbb{C}^n , and the space of $m \times n$ real, respectively complex, matrices, by $\mathbb{R}^{m \times n}$, respectively $\mathbb{C}^{m \times n}$. If $A \in \mathbb{R}^{m \times n}$, then $A^T \in \mathbb{R}^{n \times m}$ denotes its transpose, and if $A \in \mathbb{C}^{m \times n}$, then $A^* \in \mathbb{C}^{n \times m}$ denotes its conjugate transpose \bar{A}^T . The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; analogously, the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $n \times m$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{n \times m}[\xi]$, and that consisting of all $n \times m$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{n \times m}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{n \times m}[\xi]$, $R^\sim(\xi)$ is defined as $R^\sim(\xi) := R(-\xi)^T$. For a given finite-dimensional Euclidean space X , we denote by $\mathcal{C}^\infty(\mathbb{R}, X)$ the set of all infinitely often differentiable functions from \mathbb{R} to X , and with $\mathfrak{D}(\mathbb{R}, X)$ the subset of $\mathcal{C}^\infty(\mathbb{R}, X)$ consisting of those functions having compact support. Finally, if K is a symmetric $n \times n$ matrix, the quadratic form on \mathbb{R}^n defined by $x \mapsto x^T K x$ is denoted by $|x|_K^2$.

2 Quadratic differential forms

In this communication we use extensively the concepts and tools developed in the behavioral context; the reader is referred to the textbook [?] or the paper [?] for a thorough exposition. Among such concepts the most widely used in this communication is that of quadratic differential form introduced in [?], which we now review.

In many modeling and control problems for linear systems it is necessary to study quadratic functionals of the system variables and their derivatives. An efficient representation for such quadratic functionals is by means of two-variable polynomial matrices as follows. Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$; then Φ can be written in the form $\Phi(\zeta, \eta) = \sum_{h,k=0}^N \Phi_{h,k} \zeta^h \eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{q_1 \times q_2}$ and N is an integer. The two-variable polynomial matrix Φ induces a bilinear functional from $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ defined as $L_\Phi(w_1, w_2) = \sum_{h,k=0}^N \left(\frac{d^h w_1}{dt^h}\right)^T \Phi_{h,k} \frac{d^k w_2}{dt^k}$. If Φ is a *symmetric two-variable polynomial matrix*, i.e. if $q_1 = q_2$ and $\Phi_{h,k} = \Phi_{k,h}^T$ for all h, k , then it induces also a quadratic functional Q_Φ from $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by $Q_\Phi(w) := L_\Phi(w, w)$. We will call Q_Φ the *quadratic differential form* (in the following abbreviated with QDF) associated with Φ . We denote the set of all symmetric $q \times q$ two-variable polynomial matrices by $\mathbb{R}_s^{q \times q}[\zeta, \eta]$.

The association of two-variable polynomial matrices with QDF's allows to develop a calculus that has applications in stability theory, optimal control, and H_∞ -control (see [?], [?] and [?]). We restrict attention to a couple of concepts that are used extensively in this communication. The first one is the map $\partial : \mathbb{R}_s^{q \times q}[\zeta, \eta] \rightarrow \mathbb{R}^{q \times q}[\xi]$ defined by

$$\partial\Phi(\xi) := \Phi(-\xi, \xi).$$

Observe that for every $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, $\partial\Phi$ is para-hermitian, i.e. $\partial\Phi = (\partial\Phi)^\sim$. Another notion used in this paper is that of derivative of a QDF. Given a QDF Q_Φ we define its derivative as the QDF $\frac{d}{dt}Q_\Phi$ defined by $(\frac{d}{dt}Q_\Phi)(w) := \frac{d}{dt}(Q_\Phi(w))$. Q_Φ is called the derivative of Q_Ψ if $\frac{d}{dt}Q_\Psi = Q_\Phi$. In terms of the two-variable polynomial matrices associated with the QDF's, this relationship is expressed equivalently as $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta)$.

In this communication we also use integrals of QDF's. In order to make sure that the integrals exist, we assume that the trajectories on which the QDF acts are of compact support, that is, they belong to $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Given a QDF Q_Φ , we define its integral as the functional from $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ to \mathbb{R} acting as $\int Q_\Phi(w) = \int_{-\infty}^{+\infty} Q_\Phi(w) dt$. We call a QDF Q_Φ *average nonnegative*, if $\int Q_\Phi \geq 0$, i.e., $\int_{-\infty}^{+\infty} Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. A QDF can be tested for average nonnegativity by analyzing the behavior of the para-Hermitian matrix $\partial\Phi$ on the imaginary axis. Indeed, it is proven in [?] that $\int Q_\Phi \geq 0 \iff \partial\Phi(i\omega) \geq 0 \quad \forall \omega \in \mathbb{R}$.

3 Storage functions, polynomial spectral factorization, and Pick matrices

In the context of dissipative systems, the concepts of storage function and of dissipation function emerge in the framework of QDF's as follows. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$; the QDF Q_Ψ is said to be a *storage function* for Q_Φ (or: Ψ is a storage function for Φ) if the *dissipation inequality* $\frac{d}{dt}Q_\Psi \leq Q_\Phi$ holds. A QDF Q_Δ is a *dissipation function* for Q_Φ (or: Δ is a dissipation function for Φ) if $Q_\Delta \geq 0$ and $\int Q_\Phi = \int Q_\Delta$. There is a close relationship between storage functions, average nonnegativity, and dissipation functions (for a proof of the following result see [?], Proposition 5.4.).

Proposition 1 *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The following conditions are equivalent:*

1. $\int Q_\Phi \geq 0$,
2. Φ admits a storage function,
3. Φ admits a dissipation function.

Moreover, there exists a one-one relation between storage functions Ψ and dissipation functions Δ for Φ , defined by $\frac{d}{dt}Q_\Psi = Q_\Phi - Q_\Delta$ or, equivalently, $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta)$.

Since storage functions measure the energy stored inside a system, it is to be expected that they are related to the memory, to the state, of the system. Using the concept of state map introduced in [?], one can formalize such intuition as follows (for a proof see [?]).

Proposition 2 *Let \mathcal{B} be represented by $w = M(\frac{d}{dt})l$, and let $X \in \mathbb{R}^{n \times d}[\xi]$ induce a state map for \mathcal{B} . Let P be a symmetric $q \times q$ matrix, and define the two-variable polynomial matrix $\Phi(\zeta, \eta) = M^T(\zeta)PM(\eta)$. Let Q_Ψ be a storage function for Q_Φ . Then Q_Ψ is a quadratic function of the state, i.e. there exists a symmetric $n \times n$ matrix K such that $Q_\Psi(l) = |X(\frac{d}{dt})l|_K^2$ for all $l \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d)$, equivalently, $\Psi(\zeta, \eta) = X^T(\zeta)KX(\eta)$.*

Given an average nonnegative QDF, in general there exist an infinite number of storage functions; however, they all lie between two extremal storage functions (see [?], Theorem 5.7):

Proposition 3 *Let $\int Q_\Phi \geq 0$. Then there exist storage functions Ψ_- and Ψ_+ such that any other storage function Ψ for Φ satisfies $Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$.*

In the following we call Q_{Ψ_-} the *smallest* and Q_{Ψ_+} the *largest storage function* of Q_Φ . Under additional assumptions on Φ , Q_{Ψ_-} and Q_{Ψ_+} can be computed as follows:

Proposition 4 *Let $\Phi(\zeta, \eta) \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Assume $\det(\partial\Phi) \neq 0$ and $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then the smallest and the largest storage functions Ψ_- and Ψ_+ of Φ can be constructed as follows: let H and A be semi-Hurwitz, respectively semi-anti-Hurwitz, polynomial spectral factors of $\partial\Phi$ (i.e. $\partial\Phi = H\tilde{H} = A\tilde{A}$ with all the roots of $\det(H)$ in \mathbb{C}_-^0 and all roots of $\det(A)$ in \mathbb{C}_+^0). Then*

$$\Psi_+(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - A^T(\zeta)A(\eta)}{\zeta + \eta},$$

$$\Psi_-(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - H^T(\zeta)H(\eta)}{\zeta + \eta}.$$

It also turns out that if P is para-hermitian, and if $P(i\omega) > 0$ for all $\omega \in \mathbb{R}$, then for every factorization of the scalar polynomial $\det(P)$ as $\det(P) = f\tilde{f}$, where f and \tilde{f} have no common roots, there exists a polynomial spectral factorization of P as $P = F\tilde{F}$, with $\det(F) = f$ (see [?]):

Proposition 5 *Let $P \in \mathbb{R}^{m \times m}[\xi]$ be para-hermitian. Assume that $P(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Then for every*

factorization $\det(P) = f\tilde{f}$, with $f \in \mathbb{R}[\xi]$ such that f and \tilde{f} are coprime, there exists $F \in \mathbb{R}^{m \times m}[\xi]$ such that $P = F\tilde{F}$ and $\det(F) = f$.

We finally introduce the Pick matrices associated with average nonnegative quadratic differential forms. We consider a two-variable symmetric polynomial matrix $\Phi(\zeta, \eta) \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, and we assume that $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. We introduce Pick matrices in the special case that the singularities of $\partial\Phi$ are semi-simple, i.e. every singularity λ of $\partial\Phi$ has the property that its multiplicity as a root of $\det(\partial\Phi)$ is equal to the rank deficiency of $\partial\Phi(\lambda)$ at λ . (The definition of Pick matrix in the general case is notationally more involved and is not considered in this communication. However, the results presented here hold also in the general case.)

Definition 6 *Let $f \in \mathbb{R}[\xi]$ be such that $\det(\partial\Phi) = f\tilde{f}$ and (f, \tilde{f}) are coprime. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of f . We adopt the convention that if the multiplicity of λ_i as a root of $\det(\partial\Phi)$ is m_i , then λ_i appears in this list m_i times. (It is easy to see that, $\partial\Phi$ being para-hermitian, all the other singularities of $\partial\Phi$ are $-\lambda_1, -\lambda_2, \dots, -\lambda_n$, i.e. the roots of \tilde{f} .) Now for $i = 1, 2, \dots, n$, let $v_i \in \mathbb{C}^q$ be such that $\partial\Phi(\lambda_i)v_i = 0$, and such that the v_i 's associated to the same roots of $\det(\partial\Phi)$ are linearly independent. The Pick matrix associated with f is the matrix T_f whose (i, j) -th element is $\frac{v_i^* \Phi(\lambda_i, \lambda_j) v_j}{\lambda_i + \lambda_j}$.*

Note that $T_f = T_f^* \in \mathbb{C}^{n \times n}$, where $2n$ is the degree of $\det(\partial\Phi)$.

4 The ARE and storage functions

In this section we study the connection between the existence of real symmetric solutions of the ARE, and average nonnegativity of a given QDF associated with the ARE.

We associate with the ARE (1) the system with manifest variable $w = \text{col}(x, u)$ represented by $\frac{d}{dt}x = Ax + Bu$ (here $\text{col}(x, u) = \begin{pmatrix} x^T & u^T \end{pmatrix}$). Such equation constitutes a kernel representation of the behavior $\mathfrak{B} = \{\text{col}(x, u) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m) \mid \frac{d}{dt}x = Ax + Bu \text{ is satisfied}\}$. Since by assumption the pair (A, B) is controllable, \mathfrak{B} can also be represented in image form; one such representation can be computed as follows. Let $X \in \mathbb{R}^{n \times m}[\xi]$ and $U \in \mathbb{R}^{m \times m}[\xi]$ induce a right coprime factorization of the rational matrix $(\xi I_n - A)^{-1}B$. Then \mathfrak{B} is represented in observable image form as $\mathfrak{B} = \text{Im}(\begin{pmatrix} X(\frac{d}{dt})^T & U(\frac{d}{dt})^T \end{pmatrix})$. Observe that any such X yields a minimal state map $X(\frac{d}{dt})$ for \mathfrak{B} .

Given the matrices $Q = Q^T \in \mathbb{R}^{n \times n}$, $R = R^T \in \mathbb{R}^{m \times m}$ and $S \in \mathbb{R}^{m \times n}$ appearing in the ARE, and the

polynomial matrices X and U , we define the symmetric $m \times m$ two-variable polynomial matrix Φ by

$$\Phi(\zeta, \eta) = \begin{pmatrix} X(\zeta)^T & U(\zeta)^T \end{pmatrix} \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \begin{pmatrix} X(\eta) \\ U(\eta) \end{pmatrix}. \quad (2)$$

Now assume $R > 0$. The next result connects the average nonnegativity of the QDF associated with (??) with the existence of real symmetric solutions to the ARE (1) and with the existence of storage functions for Q_Φ (see also Ch. 8 of [?], [?] and Ch. 5. of [?]).

Theorem 7 *Let $\Phi(\zeta, \eta)$ be defined by (??), where X and U are such that $X(\xi)U(\xi)^{-1}$ is a right coprime factorization of $(\xi I_n - A)^{-1}B$. Assume $R > 0$. Then the following statements are equivalent:*

1. $\int Q_\Phi \geq 0$,
2. *There exists a real symmetric solution to the ARE.*

In fact, for every $K = K^T \in \mathbb{R}^{n \times n}$ the following conditions are equivalent:

- (i) $-K$ satisfies the ARE,
- (ii) $|X(\frac{d}{dt})l|_K^2$ is a storage function for Q_Φ with associated dissipation function $\Delta(\zeta, \eta) = F(\zeta)^T F(\eta)$ where $F(\xi) := R^{-\frac{1}{2}}(-B^T K + S)X(\xi) + R^{\frac{1}{2}}U(\xi)$.

5 Pick matrices and the ARE

In this section we present the main result of this communication, namely necessary and sufficient conditions for the existence of sign-definite solutions of the ARE; such conditions are given in terms of the Pick matrices associated with the Hurwitz and anti-Hurwitz factorizations of $\det(\partial\Phi)$, with Φ given by (??).

Since $\partial\Phi$ is para-hermitian, $\det(\partial\Phi)$ has even degree. Actually, it turns out that $\deg(\det(\partial\Phi)) = 2n$, with n the dimension of \mathfrak{B} defined in section 4 (for a proof, see [?], Prop. 5.4.1). Assume now that $\int Q_\Phi \geq 0$, equivalently, $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. According to Theorem ?? this is equivalent to the existence of a real symmetric solution of the ARE. Observe that every polynomial spectral factorization of $\partial\Phi$ as $\partial\Phi = F \sim F$ with $F \in \mathbb{R}^{m \times m}[\xi]$ yields a factorization of $\det(\partial\Phi)$ as $\det(\partial\Phi) = f \sim f$, with $f = \det(F)$ and $\deg(f) = n$. Let \mathcal{F} be the set of all polynomials of degree n , with positive highest degree coefficient, that can occur as the determinant of a polynomial spectral factor of $\partial\Phi$:

$$\mathcal{F} := \{f \in \mathbb{R}[\xi] \mid f(\xi) = f_0 + f_1\xi + \dots + f_n\xi^n, f_n > 0, \text{ and there exists } F \in \mathbb{R}^{m \times m}[\xi] \text{ such that } \partial\Phi = F \sim F \text{ and } \det(F) = f\}. \quad (3)$$

Also, let \mathcal{S} be the set of all real symmetric solutions of the ARE:

$$\mathcal{S} := \{K \in \mathbb{R}^{n \times n} \mid K = K^T \text{ and } K \text{ satisfies the ARE}\}.$$

For any $K \in \mathcal{S}$, denote $A_K := A - BR^{-1}(B^T K + S)$ and let χ_{A_K} be the characteristic polynomial of A_K . The following result states that there is a one-one correspondence between \mathcal{F} and \mathcal{S} .

Theorem 8 *$\mathcal{S} \neq \emptyset$ if and only if $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. In that case there exists a bijection between \mathcal{F} and \mathcal{S} . Such bijection $\text{Ric} : \mathcal{F} \rightarrow \mathcal{S}$ is defined as follows. For any $f \in \mathcal{F}$, let $F \in \mathbb{R}^{m \times m}[\xi]$ be such that $f = \det(F)$ and $\partial\Phi = F \sim F$. Then define $\text{Ric}(f) = K$, where $K = K^T \in \mathbb{R}^{n \times n}$ is the unique solution of*

$$\frac{\Phi(\zeta, \eta) - F^T(\zeta)F(\eta)}{\zeta + \eta} = X^T(\zeta)(-K)X(\eta). \quad (4)$$

For any $K \in \mathcal{S}$ we have $\partial\Phi = (F_K) \sim F_K$, where $F_K(\xi) := R^{-1/2}(B^T K + S)X(\xi) + R^{1/2}U(\xi)$. Furthermore, for any $K \in \mathcal{S}$ we have $\det(F_K) = \sqrt{\det(R)} \chi_{A_K}$, whence $\det(\partial\Phi) = \det(R) (\chi_{A_K}) \sim \chi_{A_K}$, and $K = \text{Ric}(\sqrt{\det(R)} \chi_{A_K})$.

If we strengthen the assumptions of Theorem ?? to include $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$, then the one-to-one correspondence between polynomials and the set of real symmetric solutions of the ARE can be made even more explicit. In order to illustrate such result we introduce \mathcal{F}_{cop} , the set of all real polynomials f such that the determinant of $\partial\Phi$ admits a factorization $f \sim f$ with f and $f \sim$ coprime:

$$\mathcal{F}_{\text{cop}} = \{f \in \mathbb{R}[\xi] \mid f(\xi) = f_0 + f_1\xi + \dots + f_n\xi^n, f_n > 0, (f, f \sim) \text{ coprime and } \det(\partial\Phi) = f \sim f\}.$$

It is easily seen that if $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ then $\mathcal{F}_{\text{cop}} \neq \emptyset$ if and only if $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Hence it follows from Proposition 5, that $\mathcal{F}_{\text{cop}} \subset \mathcal{F}$. In the remainder of this section we assume that $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$.

Note that if $f \in \mathcal{F}_{\text{cop}}$ and $K = \text{Ric}(f)$ then according to Theorem ??, $f = \sqrt{\det(R)} \chi_{A_K}$, so χ_{A_K} and $(\chi_{A_K}) \sim$ are coprime, equivalently, $\sigma(A_K) \cap \sigma(-A_K) = \emptyset$. If a solution K of the ARE satisfies this property, we call it *unmixed*. The set of all unmixed solutions of the ARE is denoted by \mathcal{S}_{unm} . It follows immediately from Theorem ?? that Ric defines a bijection between \mathcal{F}_{cop} and \mathcal{S}_{unm} , and we now show how to use the Pick matrix T_f in order to compute, given an $f \in \mathcal{F}_{\text{cop}}$, the unmixed solution $K = \text{Ric}(f)$.

Theorem 9 Assume $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$ if and only if $\mathcal{F}_{\text{cop}} \neq \emptyset$. Assume that this holds. Then $\text{Ric} : \mathcal{F}_{\text{cop}} \rightarrow \mathcal{S}_{\text{unm}}$ is a bijection. Now define the $n \times n$ zero state matrix S_f associated with f as $S_f := (X(\lambda_1)v_1 \ \dots \ X(\lambda_n)v_n)$, where the λ_i 's are the roots of $\det(\partial\Phi)$ and the v_i 's are the associated directions, see section 3. Then for all $f \in \mathcal{F}_{\text{cop}}$ the zero state matrix S_f is non-singular. Furthermore, for any $f \in \mathcal{F}_{\text{cop}}$, the corresponding solution $\text{Ric}(f) \in \mathcal{S}_{\text{unm}}$ is given by $\text{Ric}(f) = -(S_f^*)^{-1}T_f S_f^{-1}$.

We now turn to the problem of establishing necessary and sufficient conditions for the existence of sign-definite solutions to the ARE. Our main result in this direction is an immediate consequence of Theorem ??, and is based on the result of Theorem 3, namely that the largest (smallest) storage function for Φ is associated with an anti-Hurwitz (Hurwitz) factorization of $\partial\Phi$. Let K_- and K_+ be the smallest, respectively largest real symmetric solution of the ARE.

Corollary 10 (Main result) Let $\Phi(\zeta, \eta)$ be defined as in (??). Assume that $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Factor $\det(\partial\Phi) = (f_A)^\sim f_A = (f_H)^\sim f_H$, where f_A and f_H have their roots in \mathbb{C}_+ and \mathbb{C}_- , respectively. Then $K_- = -(S_{f_A}^*)^{-1}T_{f_A} S_{f_A}^{-1}$ and $K_+ = -(S_{f_H}^*)^{-1}T_{f_H} S_{f_H}^{-1}$. Consequently, the ARE (1) has a negative semidefinite (negative definite) solution if and only if the Pick matrix T_{f_A} is positive semidefinite (respectively positive definite). It has a positive semidefinite (positive definite) solution if and only if the Pick matrix T_{f_H} is negative semidefinite (respectively negative definite).

6 Conclusions

In this communication we applied concepts and tools coming from the calculus of QDF's to the problem of formulating necessary and sufficient conditions for the existence of (semi-) definite solutions. The authors believe that such conditions are more easily verified than those already (see [?]). Indeed, the known conditions require to check an infinite number of matrices for positive semidefiniteness; moreover, the dimensions of the matrices involved in such check are not upper bounded. The condition illustrated in this communication requires instead to check the positive semidefiniteness of *only one* $n \times n$ Hermitian matrix which is easily constructed from the parameters of the ARE.

References

[1] B.D.O. Anderson, Corrections to: Algebraic properties of minimal degree spectral factor. *Automatica*, Vol. 11, pp. 321 - 322, 1975.
 [2] Bittanti, S., Laub, A.J. and J.C. Willems (Eds.), *The Riccati Equation*, Springer-Verlag, Berlin, 1991.

[3] Coppel, W.A., *Linear Systems*, Notes on Pure Mathematics, vol. 6, Australian National University, Canberra, 1972.
 [4] Molinari, B.P., Conditions for non-positive solutions of the linear matrix inequality, *IEEE Trans. Aut. Contr.*, vol. AC-29, pp. 804-806, 1975.
 [5] Moylan, P.J., On a frequency domain condition in linear optimal control theory, *IEEE Trans. Aut. Contr.*, vol. AC-29, p. 806, 1975.
 [6] Polderman, J.W. and J.C. Willems, *Introduction to Mathematical System Theory: a Behavioral Approach*, Springer-Verlag, Berlin, 1997.
 [7] Rapisarda, P., *Linear Differential Systems*, Ph.D. Thesis, University of Groningen, The Netherlands, 1998.
 [8] Rapisarda, P. and J.C. Willems, State maps for linear systems, *SIAM J. Control Opt.*, vol. 35, nr. 3, pp. 1053-1091, 1997.
 [9] Trentelman, H.L., When does the algebraic Riccati equation have a negative semi-definite solution?, in: V.D. Blondel, E.D. Sontag, M. Vidyasagar and J.C. Willems, eds., *Open problems in Mathematical Systems and Control Theory*, pp. 229 - 237, Springer, 1998.
 [10] Trentelman, H.L. and R. Van der Geest, The Kalman-Yakubovic-Popov Lemma in a behavioural framework, *Systems and Control Letters*, vol. 32, nr. 5,, pp. 283-290, December 1997.
 [11] Trentelman, H.L. and J.C. Willems, Every storage function is a state function, *Systems and Control Letters*, vol. 32, nr. 5, pp. 249-260, December 1997.
 [12] Willems, J.C and H.L. Trentelman, On quadratic differential forms, *SIAM J. Control Opt.*, vol. 36, no. 5, pp. 1703-1749, September 1998.
 [13] Trentelman, H.L. and J.C. Willems, H_∞ -control in a behavioural context: the full information case, *IEEE Transactions on Automatic Control*, Vol. 44, No. 3, pp. 521 - 536, 1999.
 [14] Willems, J.C., Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Aut. Contr.*, vol. AC-16, pp. 621-634, 1971.
 [15] Willems, J.C., On the existence of a nonpositive solution to the Riccati equation, *IEEE Trans. Aut. Contr.*, vol. AC-19, pp. 592-593, 1974.
 [16] Willems, J.C., Paradigms and puzzles in the theory of dynamical systems, *IEEE Trans. Aut. Control*, vol. 36, nr. 3, pp. 259-294, April 1991.
 [17] Willems, J.C. and H.L. Trentelman, Synthesis of dissipative systems using quadratic differential forms, Part 1: The coupling condition, Part 2: Controller implementation, Part 3: Special cases", Manuscripts, January 2000, Submitted to *IEEE Transactions on Automatic Control*.