Stabilizability, Uncertainty and the Choice of Sampling Rate [1]

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Abstract

We shall in this contribution study a class of first order sampled-data control systems with unknown nonlinear structure and with sampling rate not necessary fast enough, aiming at understanding how stabilizability depends quantitatively upon the choice of the sampling rate and the "size" of the uncertainty. We shall show that if the unknown nonlinear function has a linear growth rate with its "slope" (denoted by L) being a measure of the "size" of uncertainty, then the sampling rate should not exceed 1/L multiplied by a constant (≈ 7.53) for the system to be globally stabilizable. If, however, the unknown nonlinear function has a growth rate faster than linear, and if the system is disturbed by noises modeled as the standard Brownian motion, then an example is given, showing that the corresponding sampled-data system is not stabilizable in general, no matter how fast the sampling rate is.

1 Introduction

Sampled-data control systems are prevalent in practice due to the rapid developments and applications of digital computers. Over the past several decades, extensive research efforts have been devoted to sampled-data linear control systems (see e.g. [1] [2]). However, there are only a few papers devoted to the theoretical investigation of sampled-data nonlinear control systems (see e.g. [3]-[5], [7]-[9]), and most of which are only concerned with the case where the sampling rate is fast enough.

In comparison with the linear case, the investigation of sampled-data nonlinear control systems is more complicated not only due to the impossibility of obtaining explicit solutions of nonlinear equations, but also due to the structure complexity of the closed-loop system—-a hybrid system consisting of both continuous- and discrete-time signals. Moreover, just as there is a fundamental difference between adaptive stabilizability of continuous- and discrete-time nonlinear models (see, [6]), a sampled-data nonlinear controller derived from a standard continuous-time stabilizing controller may indeed lose its stability [10], unless the sampling rate is sufficiently fast [3]. However, due to physical constrains, sufficiently fast sampling rate is usually not feasible in practice. Thus, a central issue in sampled-data control is how to properly chose the sampling rate, and the most difficult part is to understand what the sampled-data feedback cannot do if the sampling period is not small enough.

Unlike most of the previous research, we will in this paper discuss sampled-data (robust/adaptive) control of a typical class of first order nonlinear systems with uncertain structure and/or noises, aiming at understanding how stabilizability of the system depends quantitatively upon the choice of the sampling rate and the "size" of the uncertainty. We will treat deterministic and stochastic systems separately, and will prove the following main results:

(a) The proper choice of the sampling rate h is closely related to the " slope" L of the unknown nonlinear function in consideration. To be precise, if h is larger than L^{-1} multiplied by a constant ($\approx$7.53), then there exists no sampled-data control which can globally stabilize the prescribed class of uncertain nonlinear systems; if, however, h is less than L^{-1} multiplied by $log 4$, then a stabilizing sampled-data controller for the whole class of uncertain systems can be constructed.

(b) In the stochastic case where the random noise is described by the standard Brownian motion, the unknown system is globally stabilizable whenever the nonlinear function has a linear growth rate and the sampling rate satisfies $h < 0.15L^{-1}$. If, however,

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the unknown nonlinear function has a growth rate faster than linear, then we give an example showing that even though the continuous-time stabilizing controller exists, there is no stabilizing sampled-data control, no matter how fast the sampling rate is.

In section **2**, we will formulate the problem considered in the paper and present the main results. Section 3 will present some auxiliary lemmas which play a key role in the proof of the main results. Some concluding remarks will be given in Section **4**.

2 The Main Results

2.1 Deterministic Systems

Consider the following basic control system:

$$\dot{x}_t = f(x_t) + u_t, \quad t \geq 0, x_0 \in R^1. \tag{1}$$

The system signals are assumed to be sampled at a constant rate $h > 0$, and the input is assumed to be implemented via the familiar zero-order hold device (piecewise constant function):

$$u_t = u_{kh}, \quad kh \leq t < (k+1)h \tag{2}$$

where u_{kh} depends on $\{x_0, x_h, ..., x_{kh}\}$.

Definition 2.1. $\{u_t, t \geq 0\}$ is called *a sampled-data feedback control* if at each step k, u_{kh} is a causal function of the past and present sampled data $\{x_0, x_h, \cdots, x_{kh}\}$, i.e., there exists a function $g_k(\cdot) : R^{k+1} \to R^1$ such that $u_{kh} = g_k(x_0, x_h, \cdots, x_{kh})$.

The nonlinear function f in (1) is assumed to be unknown but belongs to the following class of functions:

$$G_c^L = \{f | f \text{ is locally Lipschiz and satifies } |f(x)| \leq L|x| + c, \forall x \in R^1\} \tag{3}$$

where $c > 0$ and $L > 0$ are constants. A function f is called locally Lipschiz if, for any $R > 0$, there exists a constant L such that $|f(x) - f(y)| \leq L|x-y|, \forall (x,y) : |x| \leq R, |y| \leq R$.

In the above definition, L is (the upper bound of) the "slope" of the function $f \in G_c^L$, which may be regard as a measure of the size of the uncertainty and plays a crucial role in the determination of the sampling rate h as will be shown by the following theorems.

Theorem 2.1. *Let $b > 0$ be the unique positive solution of the following equation*

$$\frac{1}{b}\{\frac{10}{3} + log(\frac{b+2}{b-2}) + log(3 + 2b^{-1})\} = \frac{2}{3} \tag{4}$$

If $Lh > b$, then for any $c > 0$ and any sampled-data control $\{u_{kh}, k \geq 0\}$ there always exists a function $f^* \in G_c^L$, such that the state signal of (1)-(2) corresponding to f^* with initial point $x_0 = 0$ satisfies $(k \geq 1)$

$$|x_{kh}| \geq (\frac{Lh}{2})^{k-1} \cdot ch \underset{k\to\infty}{\longrightarrow} \infty.$$

We remark that the value of b determined by (4) can be shown to be approximately 7.53. The following theorem shows that once Lh is suitably small, a stabilizing sampled-data feedback control can be constructed.

Theorem 2.2. *Let $Lh < log4$, $L > 0$ and $c > 0$. Then the following sampled-data control ($a \stackrel{\triangle}{=} e^{Lh/2}$)*

$$u_{kh} = -(c + L|x_{kh}|)sgn(x_{kh}) - \frac{(2-a)L}{2a} \cdot x_{kh} \tag{5}$$

is globally stabilizing for the system (1) with any $f \in G_c^L$. Moreover,

$$\lim_{t\to\infty} |x_t| \leq \frac{(3a-2)(a^2-1)}{(2-a)a} \cdot \frac{2c}{L}, \quad \forall f \in G_c^L.$$

2.2 Stochastic Systems.

We now consider the following stochastic control system

$$dx_t = f(x_t)dt + u_t dt + \sigma dw_t, \tag{6}$$

where the unknown nonlinear function f belongs to G_c^L defined by (3), and where $\{w_t\}$ is the standard Brownian motion, and $\sigma > 0$.

Theorem 2.3. *For any $f \in G_c^L$, let the sampled-data control (2) be defined by*

$$u_{kh} = -(1+\lambda)Lx_{kh}, \tag{7}$$

where $\lambda > 0$ is a constant. If the sampling rate h satisfies

$$Lh < \frac{\lambda}{(1+\lambda)(\sqrt{2}+1+\lambda)}, \tag{8}$$

then the closed-loop system (6)-(7) is globally stable, i.e. for any $f \in G_c^L$,

$$\overline{\lim_{t\to\infty}}\, Ex_t^2 < \infty, \quad \forall x_0 \in R^1.$$

Remark 2.1. The right-hand-side(RHS) of (8) takes its maximum value (≈ 0.15) at $\lambda = \sqrt{\sqrt{2}+1}$.

In the above, we have constrained ourselves to the case where the nonlinear function has a linear growth rate. A natural question is: Can we find a stabilizing sampled-data control for systems where the unknown nonlinear function has a nonlinear growth rate? The following theorem gives us a negative answer for a class of nonlinear stochastic systems, even in the case where the nonlinear function is known *a priori*.

Theorem 2.4. *Consider the stochastic control system (6). Assume that*

(i) u_t is continuous on $[kh,(k+1)h), \forall k \geq 0$, and

$$|u_t - u_{kh}| \leq M, \quad \forall t \in [kh,(k+1)h), \quad \forall k \geq 0,$$

for some constant $M > 0$. Also,

$$\sigma\{u_t, t \in [kh,(k+1)h)\} \subseteq \sigma\{w_t, t \leq kh\}.$$

where $\sigma\{x\}$ denotes the σ-algebra generated by x.

(ii) The function $f(x)$ is locally Lipschitz and there exist two positive constants R_0 and δ such that

$$xf(x) \geq |x|^{2+\delta}, \quad \forall x : |x| \geq R_0.$$

Then for **any** *$h > 0$ and* **any** *feedback control satisfying (i), the closed-loop system is unstable in the sense that:*

$$Ex_T^2 = \infty, \quad \forall T > 0.$$

Remark 2.2. a) The class of sampled-data control defined by condition (i) includes the standard zero-order hold device (2) as a special case. It also includes other familiar hold devices such as the first order hold device, etc.;

b) The condition (ii) in Theorem 2.4 is obviously satisfied if we take $f(x) = |x|^{1+\delta} sgn(x)$. In this case, it is easy to show that the simple state feedback control $u_t = -|x_t|^{1+\delta} sgn(x_t) - x_t$ will globally stabilize the system in the sense that $\sup_{T>0} Ex_T^2 < \infty, \forall x_0$. This example in conjunction with Theorem 2.4 demonstrates the fundamental differences between continuous-time control and sampled-data control.

3 Some Auxiliary Lemmas

The proofs of our main theorems are rather involved, we shall therefore omit the details here and only present several auxiliary lemmas that play a key role in the proof. First, we introduce a definition.

Definition 3.1. *Consider the following two sampled-data control systems:*

$$\Sigma_f : \quad \begin{cases} \dot{x} = f(x) + u_t, & t \geq 0, \quad x(t_0) = a, \\ u_t = u_{kh}, & kh \leq t < (k+1)h; \end{cases} \tag{9}$$

$$\Sigma_g : \quad \begin{cases} \dot{z} = g(z) + u_t, & t \geq 0, \quad z(t_0) = a, \\ u_t = u_{kh}, & kh \leq t < (k+1)h. \end{cases} \tag{10}$$

Under the same sampled-data control sequence $\{u_t\}$, the above two systems Σ_f and Σ_g are called **N-step equivalent** *starting from the same initial point $a \in R^1$, if the sampled signals or observations of the two systems are equal, i.e., $x_{t_0+kh} = z_{t_0+kh}$, $k = 0, 1, \cdots N$.*

Such an **equivalent** *relationship will be denoted by*

$$\Sigma_f \overset{a}{\underset{N}{\Longleftrightarrow}} \Sigma_g, \quad s.t.\{u_t\}.$$

If $N = 1$, we will simply use the notation $\Sigma_f \overset{a}{\Longleftrightarrow} \Sigma_g$, s.t.$u$ to denote that $x(t_0 + h) = z(t_0 + h)$ when $x(t_0) = z(t_0) = a$ and $u_t \equiv u, t_0 \leq t < t_0 + h$.

The following lemmas are instrumental in the proof of Theorem 2.1.

Lemma 3.1. *Consider the one dimensional autonomous system:*

$$\begin{cases} \dot{x} = \phi(x), & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{11}$$

where $\phi(\cdot)$ is local Lipschiz. Then

(i) The trajectory x(t) is a monotonous function of t;

(ii) For any $T > 0$, and $x_T \neq x_0$, the necessary and sufficient condition for $x(T) = x_T$ is $\int_{x_0}^{x_T} \frac{dx}{\phi(x)} = T$ together with $\phi(x) \neq 0$ on $[min(x_T, x_0), max(x_T, x_0)]$.

Remark 3.1. The state signal $x(t)$ defined by (1)-(2) is monotonous in any fixed sampling interval. With the help of (ii), we can calculate how much time it will take for the state $x(t)$ to travel from one point to another or how far $x(t)$ will travel in a certain time period. The first part (i) of Lemma 3.1 can be found in [11], but for the sake of easy reference, we still give a simple proof here in Appendix A.

Lemma 3.2. *Let the function $\hat{g} \in G_c^L$ satisfy $\hat{g}(z) \equiv L|z_0| + c$, for $z \geq |z_0|$, such that the state signal of the system*

$$\Sigma_{\hat{g}} : \begin{cases} \dot{z} = \hat{g}(z) + u_0, & t \geq 0 \\ z(0) = z_0 \end{cases} \tag{12}$$

satisfies $z(1) = z_1 > |z_0| > 0$. Then there exists a function $g_1 \in G_c^L$ satisfying $g_1(z_1) = Lz_1 + c$, and $g_1[z_0, |z_0|] = \hat{g}$, $g_1[|z_0|, z_1] \geq 0$, such that the state signal of the following system:

$$\Sigma_{g_1} : \begin{cases} \dot{x} = g_1(x) + u_0, & t \geq 0 \\ x(0) = z_0 \end{cases} \tag{13}$$

satisfies $x(1) = z_1$, *where by definition* $f_1[\alpha, \beta] = f_2$ *means* $f_1(x) = f_2(x)$, $\forall x \in [min(\alpha, \beta), max(\alpha, \beta)]$.

Remark 3.2. Lemma 3.2 shows that although $\sum_{\hat{g}}$ and $\sum_{g_1}$ are one-step equivalent by Definition 3.1, the terminal values $\hat{g}(z_1)$ and $g_1(z_1)$ can be quite different. This key fact makes it possible for us to construct the nonstabilizable system in the proof of Theorem 2.1 later on.

Similar to Lemma 3.2, we have the following "adjoint" lemma.

Lemma 3.2'. *Let the function* $\hat{g} \in G_c^L$ *satisfy* $\hat{g}(z) \equiv -L|z_0| - c, z \leq -|z_0|$ *such that the state signal of the system:*

$$\begin{cases} \dot{z} = \hat{g}(z) + u_0, & t \geq 0 \\ z(0) = z_0 \end{cases} \tag{14}$$

satisfies: $z(1) = z_1 < -|z_0| < 0$, *then there exists a function* $g_1 \in G_c^L$ *satisfying* $g_1(z_1) = Lz_1 - c$, *and* $g_1[-|z_0|, z_0] = \hat{g}$, $g_1[z_1, -|z_0|] \leq 0$, *such that the state signal of the following system:*

$$\begin{cases} \dot{x} = g_1(x) + u_0, & t \geq 0 \\ x(0) = z_0 \end{cases} \tag{15}$$

satisfies $x(1) = z_1$.

Lemma 3.3. *If we explicitly denote the system (1)-(2) as* $Sys(f, x_0, h, \{u_{kh}\})$, *then for any positive constant* λ, *there is a "linear time-transforming" relationship between the state signal* $x(t)$ *of the system (1)-(2) and the state signal* $z(t)$ *of the system* $Sys(\lambda f, x_0, \frac{1}{\lambda}h, \{\lambda u_{kh}\})$, i.e.,

$$z(t) = x(\lambda t), \forall t \geq 0.$$

This lemma will make it possible to transfer the general sampling rate case $h > 0$ to the special case $h = 1$ in the proofs of the main theorem to be given in the sequel.

The key idea behind the proof of Theorem 2.1 is as follows: given any sampled-data feedback $\{u_{kh}\}$, we try to find a "worst case" function $f^* \in G_c^L$ such that the corresponding system is not stabilizable.

The following lemma is useful in the proof of Theorem 2.2.

Lemma 3.4. *Let the system* $\dot{x} = g(x) + u_0, x(0) = x_0$ *satisfy*

(i) $g \in G_c^L, c \geq 0$ *and* $0 < L < log4$;

(ii) $|x_0| > 2 \cdot \dfrac{2c}{L} \cdot \dfrac{a-1}{2-a}$, *where* $a \triangleq e^{\frac{L}{2}}$.

If $u_0 = -(c + L|x_0|)sgn(x_0) - \dfrac{2-a}{2a}Lx_0$, *then we have* $|x(1)| \leq \mu \cdot |x_0|$, *where* $\mu \in (0,1)$ *is a constant.*

The proof of Theorem 2.3 is not difficult, however, the proof of Theorem 2.4 is not straightforward. To prove Theorem 2.4, we need the following property on the standard Brownian motion (see [7], P.33):

Property 1. *Let* w_t *be the standard Brownian motion, and let* η *be a Markov time defined by*

$$\eta \triangleq inf\{t \geq 0 : w_t = -a + bt\},$$

where $a > 0$, $0 \leq b < \infty$. *Then the probability density of* η *is* $p_\eta(t) = \frac{a}{\sqrt{2\pi}t^{3/2}} exp\{-(bt-a)^2/2t\}$.

By Property 1, it is clear that the following lemma is true.

Lemma 3.5. *For any* T *and* $c_1 > 0$, *we have*

$$P\{\sigma w_t > c_1 t - 1, \quad \forall t \in [0, T]\} > 0,$$

where w_t *is the standard Brownian motion.*

Lemma 3.6. *Let a function* $f(x) \in C^1(R^1)$ *satisfy* $f'(x) > 0, \forall x > a - 1$, *and* $f(a-1) + b > 0$, *where* $a, b \in R^1$ *are two constants. Also, let* $x(t)$ *and* $y(t)$ *be two continuous functions of* t, *which satisfy*

$$\begin{cases} x(t) \geq a + \int_0^t f(x_s)ds + bt - 1, & t \geq 0; \\ x(0) = a; \end{cases}$$

and

$$y(t) = a + \int_0^t f(y_s)ds + bt - 1.$$

Then

$$x(t) > y(t) \geq a - 1, \quad \forall t \geq 0.$$

Lemma 3.7. *For any positive constans* T, ν, g_0 *and any* $x_0 \in R^1$, *there exists a constant* $b > 0$ *such that the solution of the following integral equation:*

$$z_t = x_0 + \int_0^t (|z_s - x_0 + 1|^{1+\nu} - g_0)ds + bt - 1 \tag{16}$$

is blow up at time T, *i.e.,* $z(T) = \infty$.

4 Concluding Remarks

In this paper, we have tried to understand quantitatively the following questions: (i) How the stabilizability of sampled-data control systems with uncertain nonlinear structure depend upon the (not necessarily small) value of the sampling period and upon the "size" of the uncertainty ? And (ii) what the sampled-data feedback control cannot do ? As a starting point towards the study of these fundamental issues, we have considered a typical class of first-order sampled-data control systems with unknown nonlinearity, and have obtained several concrete theoretical results. These results show that, among other things, the choice of the sampling rate h should be of the magnitude $O(\frac{1}{L})$ with a suitable "O" constant, where L is the "slope" of the unknown nonlinear function. For further investigation, it is desirable to bridge the gap between the bounds for Lh in Theorems 2.1 and 2.2 as has been done for the pure discrete-time case in [11] , and to study more general systems. Finally, we remark that, since our main result—Theorem 2.1 is a "negative" type result, it is also valid for more general class of uncertain systems which include (1) as a special case.

References

[1] K.J.Åström and B.Wittenmark, *Computer Controlled Systems–Theory and Design*. Englewood Cliffs, N.J., Prentice-Hall, 1984.

[2] T.W.Chen and B.Francis, *Optimal Sampled-Data Control Systems*. Springer-Verlag: London, 1995.

[3] A.R.Teel, D. Nešić and P.V. Kokotović, *A note on input-to-state stability of sampled-data nonlinear systems*, Proceedings of the 37th IEEE Conference on Decision & Control·Tampa, Florida USA·December 1998, pp.2473-2478

[4] B.Castillo, S.Di Gennaro, S.Monaco and D.Nomand-Cyrot, *On regulation under sampling*, IEEE Trans. Automat. Contr., 42 (1997), pp. 864-868.

[5] L. Hou, A.N.Michel and H.Ye, *Some qualitative properties of sampled-data control systems*, IEEE Trans. Automat. Contr., 42 (1997), pp.1721-1725.

[6] L. Guo, *On critical stability of discrete-time adaptive nonlinear control*, IEEE Trans. Automat. Contr., 42 (1997), pp. 1488-1499.

[7] A. Brockwell, K. Borovkov and R. Evans, *Stability of an adaptive regulation for partially Known Nonlinear Stochastic Systems*, SIAM J. Control Optim. 37 (1999), pp. 1553-1567.

[8] E. Skafidas, A. Fradkov, R. J. Evans and I.M.Y. Mareels, *Trajectory-approxim ation-based adaptive control for nonlinear systems under matching conditions*, Automatica, 34 (1998), pp. 287-299.

[9] I.M.Y. Mareels, H. B. Penfold and R. J. Evans, *Controlling nonlinear time-varying systems via Euler approximations*, Automatica, 28 (1992), pp. 681-696.

[10] L.L. Xie, *Adaptive Control of Discrete-Time Nonlinear Systems with Parametric or Structure Uncertainty*, Ph.D. Dissertation, Institute of Systems Science, Chinese Academy of Sciences. May, 1999.

[11] L.L. Xie and L. Guo, *On the capability of feedback for controlling uncertain nonlinear systems*, Proc. of the 38th IEEE-CDC, Phoenix, AZ, December 7-10, 1999, pp. 2833-2838.