# A Sliding Mode Output Feedback Controller for an Aircraft System with Flexible Modes

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**Abstract**

This paper considers a framework for sliding mode control design. LMI based techniques are used to design the output dependent switching surface and to determine a control law to effect a sliding motion. The reachability condition is not required to be satisfied globally. Instead sliding is only expected to take place within a subset of the state-space referred to as the sliding patch. The LMI optimisation seeks to maximise the sliding patch subject to certain constraints. The efficacy of the proposed control design method is demonstrated on a non-trivial design example – an aircraft system representation which includes the flexible modes of the airframe.

## 1 Introduction

Most of the early developments in the area of sliding mode control for nominal linear systems with bounded uncertainty assumed that all the internal plant states were accessible. Unfortunately this assumption is very limiting from a practical viewpoint. In practice either an observer must be designed to provide an estimate of the internal unmeasured states or the design methods must be modified to make allowance for the fact that effectively only a subset of the states are available for use in the control law. For linear systems with no uncertainty the problem of hyperplane design using output information has been investigated by El-Khazali & DeCarlo [7, 8]. For uncertain systems, approaches have been reported by Hui & Żak [12] who propose an algorithm for output dependent hyperplane design based upon eigenvector methods. Edwards & Spurgeon [5] demonstrate that the hyperplane design problem is equivalent to a static output feedback problem and provide necessary and sufficient conditions in terms of the system structure for a stable reduced order motion to exist. Having designed the surface, it is necessary to develop a controller to induce and sustain a sliding motion. A common design methodology which appears in the work of Heck *et al.* and DeCarlo *et al.* [1, 7, 10, 11] is based on synthesising a static output feedback gain numerically to ensure the so-called reachability condition is satisfied. In [6] it is demonstrated that in order for these numerical schemes to terminate satisfactorily, at best, a certain structural constraint must be satisfied and a particular choice of sliding surface must be made. The approach adopted by Kwan [14] is to estimate *a-priori* the magnitude of certain unmeasured states and to use this value in a simple dynamic system which is guaranteed to be larger in norm than the states as they evolve. This circumvents some of the restrictions of Żak *et al.* , however the controller is now a dynamical one.

In this paper a new output feedback based sliding mode controller synthesis procedure is proposed. Both the existence and reachability problems are solved using Linear Matrix Inequality (LMI) optimisation. The linear component of the sliding mode controller is designed to maximise the so-called *sliding patch* [15]. The proposed design algorithms are applied to a realistic aircraft system which includes flexible modes.

## 2 System Description & Problem Statement

Consider the uncertain dynamic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t,x,u) \\ y(t) &= Cx(t) \end{aligned} \qquad (1)$$

where the state vector $x \in \mathbb{R}^n$, the control signal $u \in \mathbb{R}^m$, the output signal $y \in \mathbb{R}^p$ and the condition $m \leq p < n$ is fulfilled. The system triple $(A, B, C)$ is assumed to be known and appropriately dimensioned, with the constraint of controllability imposed on the matrix pair $(A, B)$. Further it is assumed that any invariant zeros of the triple $(A, B, C)$ lie in the left half of the complex plane, the matrix product $CB$ has full rank and that all the plant inputs and outputs are independent. The function $f(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ represents the lumped sum of the non-linearities and/or uncertainties which are assumed to lie in the range space of the input distribution matrix:

$$f(t,x,u) = B\xi(t,x,u) \qquad (2)$$

In the last equation $\xi(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is assumed to satisfy

$$\|\xi(t,x,u)\| < k_1\|u\| + \alpha(t,y) \qquad (3)$$

for some known scalar valued function $\alpha(t,y)$ and a positive constant $k_1 < 1$.

The first phase of the variable structure control method concerns designing a switching hyperplane so that when the system trajectories are confined to this surface, the overall dynamic behaviour of the system is stable and has some desired characteristics. In the static output feedback case this entails constructing a matrix $F \in \mathbb{R}^{m\times p}$ so that the motion on the hyperplane

$$\mathcal{S} = \{x \in \mathbb{R}^n \, : \, FCx = 0\} \tag{4}$$

satisfies the design specifications. It will be seen that the matrix $F$ may be determined as the solution to an output feedback stabilization problem for a particular subsystem triple.

The second phase of the design scheme will focus on the formulation of a control law that will drive the states onto the sliding surface in finite time. The intention is to synthesise a gain matrix $G \in \mathbb{R}^{m\times p}$ so that the control law

$$u(t) = -Gy(t) + u_n(t) \tag{5}$$

guarantees sliding takes place in finite time. The non-linear switching component $u_n(t)$ will be described in the sequel.

In [11] the gains are synthesised numerically using an LMI approach so that the reaching condition

$$s^{\mathrm{T}}\dot{s} < 0 \tag{6}$$

is satisfied *globally* where $s(t) := Fy(t)$ is the so-called *switching function*. It is shown in [6] that the requirement that (6) is satisfied globally imposes an additional and unnecessary structural constraint on the system and at best restricts the choice of sliding surface. In [5] the global requirement is dropped and a sliding motion is shown to exist only in a region of the state space sometimes referred to as the *sliding patch*; it is shown that the linear control component can be designed so that the sliding patch is globally attractive in finite time. In practice the controller in [5], although explicitly identified, tends to be high gain in nature which is a disadvantage when trying to design for real systems with a view to implementation. The work which will be described here seeks to overcome these deficiencies.

The following lemma provides a convenient canonical form for the triple $(A, B, C)$.

**Lemma 1** *Let $(A,B,C)$ be a linear system with $p > m$ and rank$(CB) = m$. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:*

*a) the system matrix can be written as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{7}$$

*where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and when partitioned has the structure*

$$A_{11} = \left[\begin{array}{cc|c} A_{11}^o & A_{12}^o & A_{121}^m \\ 0 & A_{22}^o & A_{122}^m \\ \hline 0 & A_{21}^o & A_{22}^m \end{array}\right] \tag{8}$$

*where $A_{11}^o \in \mathbb{R}^{r\times r}$, $A_{22}^o \in \mathbb{R}^{(n-p-r)\times(n-p-r)}$ and $A_{21}^o \in \mathbb{R}^{(p-m)\times(n-p-r)}$ for some $r \geq 0$ and the pair $(A_{22}^o, A_{21}^o)$ is completely observable.*

*Furthermore the invariant zeros of the system are the $r$ eigenvalues of $A_{11}^o$.*

*b) the input distribution matrix has the form*

$$B = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \tag{9}$$

*c) the output distribution matrix has the form*

$$C = \begin{bmatrix} 0 & T \end{bmatrix} \tag{10}$$

*where $T \in \mathbb{R}^{p\times p}$ and is nonsingular.*

**Proof**:
This is based on the canonical form from [5]. An additional coordinate transformation has been introduced which scales the last $m$ coordinates to effect the special structure of the input distribution matrix.

■

## 3 A framework for sliding mode control design

### 3.1 The existence problem

Without loss of generality the sliding surface matrix can be parameterised as

$$F := F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^{-1} \tag{11}$$

where $K \in \mathbb{R}^{m\times(p-m)}$ and $F_2 \in \mathbb{R}^{m\times m}$ is a nonsingular matrix [5]. Here for simplicity assume that $F_2 = I_m$ which implies that $FCB = I_m$. If $\begin{bmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ represents a partition of the states commensurate with the system matrix in equation (7) of Lemma 1 then it can easily be shown that during the sliding motion

$$\dot{x}_1(t) = (A_{11} - A_{12}KC_1)\,x_1(t) \tag{12}$$

where

$$C_1 := \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{p-m} \end{bmatrix} \tag{13}$$

This reveals the static output feedback nature of the hyperplane design problem. In this paper an LMI approach will be considered for the synthesis of $K$ based on the work of Benton & Smith [2]. The problem they consider (in the notation consistent with this section) is one of synthesizing a symmetric positive definite matrix $P_1$ such that the matrix inequalities

$$P_1(A_{11}+A_{12}K_{sf})+(A_{11}+A_{12}K_{sf})^{\mathrm{T}}P_1+2dP_1 < 0 \quad (14)$$
$$P_1A_{11}+A_{11}^{\mathrm{T}}P_1+2dP_1-\sigma C_1^{\mathrm{T}}C_1 < 0 \quad (15)$$

hold for some $\sigma > 0$ where $d > 0$ is a given design scalar. In inequality (14) the gain $K_{sf} := -A_{12}^{\mathrm{T}}P_{1,are}$ where $P_{1,are}$ is the stabilising solution to the Algebraic Ricatti Equation

$$P_1A_{11}+A_{11}^{\mathrm{T}}P_1+2dP_1-P_1A_{12}A_{12}^{\mathrm{T}}P_1=-\epsilon I \quad (16)$$

where $\epsilon > 0$ is a design scalar. The problem of synthesizing a $P_1$ and $\sigma$ satisfying (14)-(15) is a convex feasibility problem and LMI methods can be employed for its solution [9]. If $P_1$ and $\sigma$ can be found satisfying (14)-(15) then it can be shown (see [2] for example) that there exists a $K$ such that

$$P_1(A_{11}-A_{12}KC_1)+(A_{11}-A_{12}KC_1)^{\mathrm{T}}P_1+2dP_1<0 \quad (17)$$

holds. This implies that $A_{11}-A_{12}KC_1$ has a stability margin of $d$.

Remark: It can be shown [5] that if $r > 0$ then $\lambda(A_{11}^o) \subset \lambda(A_{11}-A_{12}KC_1)$ for all $K$ and the invariant zeros of $(A,B,C)$ always appear in the sliding mode dynamics.

### 3.2 The control strategy

To facilitate the development of the control law, an additional switching function dependent coordinate transformation will be made. Let $x \mapsto T_K x = \hat{x}$ where

$$T_K := \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix} \quad (18)$$

and $C_1$ is defined in (13). In this new coordinate system, the system triple $(\hat{A},\hat{B},F\hat{C})$ has the property that

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \quad F\hat{C} = \begin{bmatrix} 0 & I_m \end{bmatrix} \quad (19)$$

where $\hat{A}_{11} = A_{11}-A_{12}KC_1$ which is stable by choice of $K$. In addition

$$\hat{C} = \begin{bmatrix} 0_{p\times(n-p)} & \hat{T} \end{bmatrix} \quad (20)$$

where

$$\hat{T} := \begin{bmatrix} (T_1-T_2K) & T_2 \end{bmatrix} \quad (21)$$

and $T_1$ and $T_2$ represent the first $p-m$ columns and last $m$ columns of the matrix $T$ associated with the output distribution matrix from equation (10). Notice that $\hat{T}$ is nonsingular. Without loss of generality the gain matrix in the control law (5) can be written as

$$G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}\hat{T}^{-1} \quad (22)$$

where $\hat{T}$ is the component of the output distribution matrix from equation (21) and $G_1 \in \mathbb{R}^{m\times(p-m)}$ and $G_2 \in \mathbb{R}^{m\times m}$. These matrices will be synthesised so that the closed loop system matrix $A_c = \hat{A}-\hat{B}G\hat{C}$ is stable and

$$PA_c + A_c^{\mathrm{T}}P < 0 \quad (23)$$

where the symmetric positive definite Lyapunov matrix $P$ is block diagonal

$$P := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0 \quad (24)$$

with $P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $P_2 \in \mathbb{R}^{m\times m}$. The block diagonal structure, which also appears in [5], will be explained in the sequel.

The nonlinear component in the control law given in (5) is defined as

$$u_n(t) = \begin{cases} -\rho(u,y)P_2^{-1}\frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy(t)\neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

where $\rho(u,y)$ is the positive scalar function

$$\rho(u,y) = (k_1\|Gy\| + \alpha(t,y) + \eta)/(1-k_1) \quad (26)$$

where $\eta$ is a positive design scalar which will be shown to define the region in which sliding takes place. Because of the block diagonal structure of $P$ in (24) and the structure of $\hat{B}$ and $F\hat{C}$ in (19) it follows that

$$P\hat{B} = \begin{bmatrix} 0 \\ P_2 \end{bmatrix} = (F\hat{C})^{\mathrm{T}}P_2 \quad (27)$$

This equation will be used to prove the following:

**Proposition 1** *If it is possible to find a matrix $P$ satisfying (24) and a gain matrix $G$ so that (23) holds, then the control law given in (5) will induce a sliding motion on the surface $\mathcal{S}$.*

**Proof**:

Consider $V(t) := \hat{x}(t)^{\mathrm{T}}P\hat{x}(t)$. Taking derivatives along the trajectories

$$\begin{aligned} \dot{V} &= \hat{x}^{\mathrm{T}}(PA_c+A_c^{\mathrm{T}}P)\hat{x} + 2\hat{x}^{\mathrm{T}}P\hat{B}u_n \\ &= \hat{x}^{\mathrm{T}}(PA_c+A_c^{\mathrm{T}}P)\hat{x} + 2\hat{x}^{\mathrm{T}}(F\hat{C})^{\mathrm{T}}P_2u_n \\ &\leq x^{\mathrm{T}}(PA_c+A_c^{\mathrm{T}}P)\hat{x} - 2\eta\|Fy\| < 0 \end{aligned} \quad (28)$$

for $\hat{x} \neq 0$ and thus *quadratic stability* is proved. To show that a sliding motion is attained on $\mathcal{S}$ in finite time, first define $[\ \hat{x}_1^{\mathrm{T}} \quad \hat{x}_2^{\mathrm{T}}\ ]^{\mathrm{T}}$ to be a partition of the state vector $\hat{x}$ commensurate with the partition of the state matrices. Notice from (19) that $Fy = \hat{x}_2$. Also since $PA_c + A_c^{\mathrm{T}}P < 0$ it follows from the properties of quadratic forms that

$$\hat{x}_2^{\mathrm{T}} \left( P_2(\hat{A}_{22} - G_2) + (\hat{A}_{22} - G_2)^{\mathrm{T}} P_2 \right) \hat{x}_2 < 0 \qquad (29)$$

Define

$$V_s(t) := \hat{x}_2(t)^{\mathrm{T}} P_2 \hat{x}_2(t) \qquad (30)$$

Taking derivatives along the trajectories

$$\begin{aligned} \dot{V}_s &= 2\hat{x}_2^{\mathrm{T}} P_2(\hat{A}_{21} - G_1C_1)\hat{x}_1 + 2\hat{x}_2^{\mathrm{T}} P_2 u_n \\ &\quad + \hat{x}_2^{\mathrm{T}}(P_2(\hat{A}_{22} - G_2) + (\hat{A}_{22} - G_2)^{\mathrm{T}} P_2)\hat{x}_2 \\ &< 2\hat{x}_2^{\mathrm{T}} P_2(\hat{A}_{21} - G_1C_1)\hat{x}_1 - 2\eta\|\hat{x}_2\| \qquad (31) \end{aligned}$$

This inequality means that sliding takes place inside the domain

$$\Omega = \{(\hat{x}_1, \hat{x}_2) \ : \ \|\hat{x}_1\| < \eta\gamma^{-1}\}$$

where $\gamma = \|P_2(\hat{A}_{21} - G_1C_1)\|$. Since the closed loop system is quadratically stable, in finite time the state trajectories enter and remain in $\Omega$, and hence a sliding motion takes place in finite time. ∎

From the point of view of control law design, a requirement is to make

$$\|P_2(\hat{A}_{21} - G_1C_1)\| \qquad (32)$$

small to make the domain $\Omega$ large.

It can be shown that

$$PA_c + A_c^{\mathrm{T}}P = \begin{bmatrix} P_1\hat{A}_{11} + \hat{A}_{11}^{\mathrm{T}}P_1 & P_1\hat{A}_{12} + (\hat{A}_{21} - G_1C_1)^{\mathrm{T}}P_2 \\ P_2(\hat{A}_{21} - G_1C_1) + \hat{A}_{12}^{\mathrm{T}}P_1 & P_2\hat{A}_{22} + \hat{A}_{22}^{\mathrm{T}}P_2 - P_2G_2 - (P_2G_2)^{\mathrm{T}} \end{bmatrix} \qquad (33)$$

and hence by defining $L_1 = P_2G_1$ and $L_2 = P_2G_2$, it follows that $PA_c + A_c^{\mathrm{T}}P$ is affine in the parameters $P_1, P_2, L_1$ and $L_2$. Also

$$P_2(\hat{A}_{21} - G_1C_1) = P_2\hat{A}_{12} - L_1C_1 \qquad (34)$$

is affine in $P_2$ and $L_1$. This suggests that from the point of view of developing a numerical algorithm to synthesise the matrices $P_1, P_2, L_1$ and $L_2$, a Linear Matrix Inequality (LMI) approach can be adopted [3].

Rather than merely guaranteeing the eigenvalues of $A_c$ lie in the open left half plane, additional requirements can be incorporated within an LMI framework [4]. For instance a plausible restriction is that the eigenvalues of $A_c$ lie in a disc. This can be written as the LMI

$$\begin{bmatrix} -r_dP & PA_c + qP \\ qP + A_c^{\mathrm{T}}P & -r_dP \end{bmatrix} < 0 \qquad (35)$$

where the point $(-q, 0)$ represents the centre of the circle and $r_d$ represents the radius. Additionally, a stability margin can be incorporated via the LMI

$$P(A_c - h_2I) + (A_c^{\mathrm{T}} - h_2I)P < 0 \qquad (36)$$

which ensures all the eigenvalues of $A_c$ lie to the left of vertical line which cuts the horizontal axis at $(h_2, 0)$. Notice that both (35) and (36) are affine in $L_1, L_2, P_1$ and $P_2$. Furthermore if $P$ is any symmetric positive definite matrix which satisfies (35) and (36) then $\alpha P$ also satisfies (35) and (36) if $\alpha$ is any positive scalar. In this way, without loss of generality, it can be assumed that $P$ from (24) is a positive definite matrix subject to the constraint

$$P_2 > I_m \qquad (37)$$

The advantage of this is that the minimization of (32) must be achieved through the choice of $G_1$ rather than just by making $P_2$ small. In addition (37) implies from (25) that $\|u_n(t)\| < \rho(u, y)$. Let $\hat{A}_{211}$ be a partition of $\hat{A}_{21}$ from (19) so that

$$\hat{A}_{21} =: \begin{bmatrix} \overset{n-p}{\leftrightarrow} & \overset{p-m}{\leftrightarrow} \\ \hat{A}_{211} & \hat{A}_{212} \end{bmatrix} \qquad (38)$$

*The idea is then to select any $\gamma > \|\hat{A}_{211}\|$ and to synthesize $L_1, L_2, P_1$ and $P_2$ to minimise $k$ subject to*

$$\begin{bmatrix} -kI & [\ L_1 \quad L_2\ ]\hat{T}^{-1} \\ ([\ L_1 \quad L_2\ ]\hat{T}^{-1})^{\mathrm{T}} & -kI \end{bmatrix} < 0 \quad (39)$$

$$\begin{bmatrix} -\gamma I & P_2\hat{A}_{21} - L_1C_1 \\ \hat{A}_{21}^{\mathrm{T}}P_2 - C_1^{\mathrm{T}}L_1^{\mathrm{T}} & -\gamma I \end{bmatrix} < 0 \quad (40)$$

*and (35) – (37). If this problem has a feasible solution then the conditions of Proposition 1 are satisfied and the controller defined by $G_1, G_2$ and $P_2$ will induce a sliding motion.*

By the Schur expansion, (40) is equivalent to

$$\|P_2\hat{A}_{21} - L_1C_1\| < \gamma \qquad (41)$$

and (39) is equivalent to

$$\|\ [\ L_1 \quad L_2\ ]\hat{T}^{-1}\| < k \qquad (42)$$

Since $P_2 > I_m$ it follows that

$$\|G\| \leq \|P_2\ [\ G_1 \quad G_2\ ]\hat{T}^{-1}\| = \|\ [\ L_1 \quad L_2\ ]\hat{T}^{-1}\| < k$$

The proposed formulation maybe interpreted as one of finding a gain matrix $G$ of minimum norm so that the eigenvalues of $A_c$ lie in a defined convex sub-set of $\mathbb{C}_-$ subject to a defined tolerance on the maximum region of sliding $\gamma$.

## 4 Example

Consider the following model of a large transport aircraft taken from [13]. The model includes the usual longitudinal modes together with two elastic modes. The usual control surface for the longitudinal dynamics is the elevator. Here feedback from the ailerons is also used to alleviate the symmetric loads due to vertical gusts acting on the aircraft. The state vector is

$$x = \begin{bmatrix} u \\ w \\ q \\ \theta \\ e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} \quad \begin{array}{l} \text{forward velocity(m/s)} \\ \text{vertical velocity (m/s)} \\ \text{pitch rate (rad/s)} \\ \text{pitch attitude (rad)} \\ \text{displacement (first elastic mode)(m)} \\ \text{velocity (first elastic mode)(m/s)} \\ \text{displacement (second elastic mode)(m)} \\ \text{velocity (second elastic mode)(m/s)} \end{array}$$

and the inputs $u$ are the elevator and aileron deflections in radians. As well as taking measurements of the vertical velocity and the pitching movements, it is assumed that measurements of the aircraft's normal acceleration at the wing tip (NZW) and the centre of gravity (NZB) are available. Let

$$y_a = \begin{bmatrix} NZW \\ NZB \end{bmatrix} \quad \begin{array}{l} \text{acceleration at the wing tip (g)} \\ \text{acceleration at the c.g. (g)} \end{array}$$

It can be verified that the damping ratios of the open-loop poles of the system are 0.054, 0.168, 0.430 and 0.116 respectively [13]. The transfer function realization $(A_a, B_a, C_a, D_a)$ from $u \mapsto y_a$ is such that $D_a$ is nonsingular. The accelerometer measurements as in [13] have been filtered by first order lags with poles at $-10$. Let $A_f = \text{diag}\{-10, -10\}$ and define an augmented system as

$$A = \begin{bmatrix} A_a & 0 \\ -A_f C_a & A_f \end{bmatrix} \quad B = \begin{bmatrix} B_a \\ -A_f D_a \end{bmatrix} \tag{43}$$

This results in a 10th order system. The output distribution matrix is then given by

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}$$

Physically this means the pitch rate, pitch angle and the vertical velocity, together with the filtered accelerometer signals will be used in the control law. In order to improve the numerical conditioning of the system matrices the outputs were scaled before the (output dependent) transformations in Lemma 1 were deployed to generate the canonical form. In the design that follows the following scaling was used:

$$S_c = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} \tag{45}$$

Using the approach outlined in Section 3 with $\epsilon = 0.1$ and with a relative stability margin of $d = 0.00001$, solving (14)-(17) yields

$$K = \begin{bmatrix} 6.1068 & 2.6272 & -7.2116 \\ 7.7409 & 5.2007 & 6.6144 \end{bmatrix}$$

This gives sliding mode poles at $\{-0.0376 \pm 0.0495i, -10.2742 \pm 3.2139i, -3.7532 \pm 20.1860i, -32.4355 \pm 51.8040i\}$ and corresponding damping ratios of 0.589, 0.9542, 0.184 and 0.528. It is immediately seen that the prescribed ideal sliding mode dynamics yield an improvement in system damping.

The associated switching function from (11) is

$$F = \begin{bmatrix} -0.1616 & 0.0213 & -0.2982 & 0.4745 & 0.9412 \\ 0.1784 & -0.7903 & 0.5039 & -0.7052 & -0.9206 \end{bmatrix}$$

A controller must now be determined which will ensure the desired sliding motion is attained. It can be verified that $\|\hat{A}_{211}\| = 122.8159$. In the synthesis procedure $\gamma = 250$ and $r_d = 100$ and $q = 0$ and $h_2 = 0.001$. The controller design procedure from Section 3.2 gives

$$P_2 = \begin{bmatrix} 31.0904 & 19.9471 \\ 19.9471 & 14.2393 \end{bmatrix}$$

and a gain matrix

$$G = \begin{bmatrix} -9.292 & 63.082 & -46.041 & 59.323 & 93.519 \\ 8.502 & -111.855 & 66.281 & -83.069 & -119.744 \end{bmatrix}$$

The following set of results evaluate the response of the closed loop system to the vortex wind gust shown in Figure 1. The unit vector structure has been smoothed in the usual way to remove the discontinuity. The response of the vertical velocity is shown in Figure 2 where the solid line indicates the response under sliding mode control and the dashed line gives the open loop response for comparison. It is seen that the control strategy exhibits the robustness associated with sliding mode controllers; the effect of the gust on the rigid body dynamics is significantly reduced. The response of the normal acceleration as measured at the wing tip is shown in Figure 3 where again the solid line shows the response under sliding mode control and the dashed line gives the open loop response for comparison. It is seen that the control strategy significantly improves the excitation level of the elastic modes in the presence of a wind gust. The variation in the switching function is shown in Figure 4. It is seen that the required sliding mode is attained and maintained.

## 5 Conclusion

The design of output feedback sliding mode controllers, using an LMI based numerical approach, has been considered in this paper. LMI's have been used for both the design of the switching surface and for the computation of the control law gains. The class of systems to which this approach is applicable is broader than comparable existing schemes. The controller gains can be synthesized numerically in a straight forward manner and are not high gain in nature. The efficacy has been demonstrated on a non-trivial design example from the literature.

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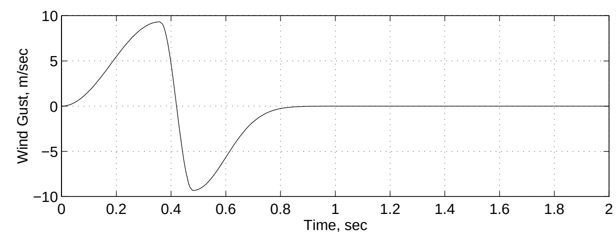


**Figure 1:** The wind gust representation

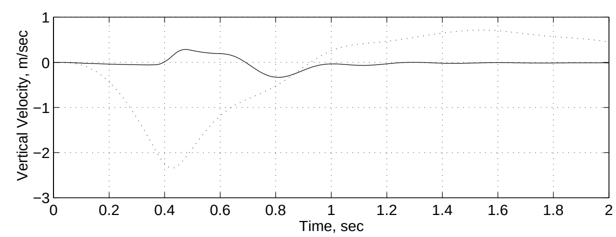


**Figure 2:** Open and closed-loop response of the vertical velocity

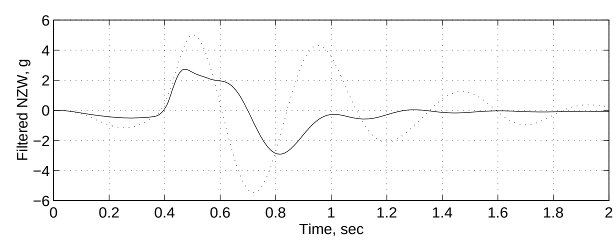


**Figure 3:** Open and closed-loop response of the normal acceleration at the wing tip

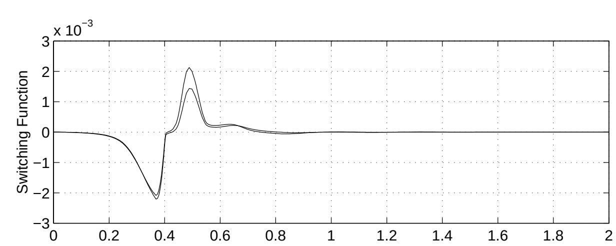


**Figure 4:** Evolution of the switching function