

HIGHER ORDER SLIDING: DIFFERENTIATION AND BLACK-BOX CONTROL

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Abstract. Sliding modes describe motions on discontinuity sets of dynamic systems and are provided by a persistent system switching with theoretically infinite frequency. The standard sliding modes are applicable to control output variables with relative degree 1. Having preserved or generalized the main properties of standard sliding mode, higher order sliding modes (HOSM) may be applied with any relative degree and, when properly used, totally remove the chattering effect. That allows full real-time control of the output variables, when only the relative degree of the dynamic system is known, and the system is actually considered as a “black box”. The HOSM controllers being based on the use of real-time higher-order output derivatives, robust exact differentiation becomes the key problem of the HOSM theory. Fortunately, the HOSM technique may also be applied to solve the differentiation problem. Differentiation usage in HOSM control is demonstrated by computer simulation of model and real-life examples.

1. Introduction

Control under heavy uncertainty conditions is one of the most important modern control problems. A natural way to withstand the uncertainty is to keep some constraints by “brutal force”. The obvious idea is to keep a constraint reacting immediately to any deviation of the system and steering it back to the constraint by a sufficiently energetic effort. That approach leads to the so-called sliding modes, which became the main operation modes in the VSS theory [19, 22]. Having proved their high accuracy and robustness with respect to internal and external disturbances, standard sliding modes also reveal their main drawback in the so-called chattering effect. Preserving the main advantages of the original approach, higher order sliding modes (HOSM) remove the chattering effect and provide for higher accuracy in realization.

HOSM [16, 5, 10, 1] generalize the basic sliding mode idea, acting on the higher order time derivatives of the system deviation from the constraint instead of influencing the first deviation derivative, as it happens in standard sliding modes. Let the constraint to be kept be defined by an equality $\sigma = 0$, where σ is a system output described by a smooth function. While successively differentiating σ along trajectories of a discontinuous system, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes $\sigma \equiv 0$ may be classified by the number r of the first total time derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist for some reason like trajectory nonuniqueness. That number is called sliding order, and it characterizes the dynamics smoothness degree in the vicinity of the sliding mode $\sigma \equiv 0$. Hence, the r th order sliding mode is

determined by the equalities $\sigma = \dot{\sigma} = \ddot{\sigma} = \dots = \sigma^{(r-1)} = 0$, where $\sigma, \dot{\sigma}, \ddot{\sigma}, \dots, \sigma^{(r-1)}$ are continuous functions. That imposes an r -dimensional condition on the state of the dynamic system. The discontinuous dynamic system being understood here in the Filippov sense [6], the sliding mode motions formally also satisfy the equality $\sigma^{(r)} = 0$, which is actually a result of some averaging process. The formal HOSM definition may be found in [7, 10].

The words “ r th order sliding” are often abridged to “ r -sliding”. The standard sliding modes are of the first order. Trivial cases of asymptotically stable HOSM are easily found in many classic VSSs. For example, there is an asymptotically stable 2-sliding mode with respect to the constraint $x = 0$ at the origin $x = \dot{x} = 0$ (at one point only) of a 2-dimensional VSS keeping the constraint $x + \dot{x} = 0$ in a standard 1-sliding mode. HOSMs attracting trajectories in finite time are not trivial and require special consideration for each controller [1, 5, 10, 12].

Standard sliding mode may be directly used for full real-time control of a “black box” system with relative degree 1. Nevertheless, that remarkable feature cannot be promptly applied, for in reality the relative degree of mechanical systems usually varies within the range from 2 to 4. On the other hand, higher order sliding controllers may be directly implemented for any relative degree [12, 13]. Unfortunately these controllers need the derivatives $\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}$ to be available in real time. Hence, real-time differentiation becomes the main problem of the output-based sliding controller design [11, 3].

A scheme of differentiator design is listed in this paper. First and second order asymptotically-optimal robust exact differentiators are presented. Computer simulation of model and real-life examples of closed-loop differentiator usage are demonstrated.

2. Black-box control problem

The task is to compel the output of an uncertain SISO dynamic system to track some real-time measured signal unknown in advance. Only the output is measured. The convergence time is required to be finite, the tracking is to be robust with respect to measurement errors and exact in their absence.

In order to solve such a problem some assumptions are still needed. Consider a system of the form

$$\dot{x} = f(t, x, u), \quad x \in X, \quad u \in \mathbf{R}, \quad \sigma = \sigma(t, x) \in \mathbf{R}, \quad (1)$$

where X is some uncertain smooth manifold, f, σ are some unknown smooth functions. Here σ is the deviation of the

output from some measured real-time reference signal and is considered as a new output. Thus, the task is to make it vanish in finite time.

Having introduced a new control $v = \dot{u}$ and considering u as an additional state variable, achieve some *linear-on-control dynamic system*. At the same time the chattering effect is removed for the real control u and any possibility is eliminated of ill-defined sliding modes common to systems non-linear on control [19, 4]. The frequent requirement to keep u within some given range may be treated by an artificial saturation effect as in [10] and is ignored here. Hence, general black-box control problem (1) is reduced to a linear-on-control black-box control problem with redefined control and system state

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \quad (2)$$

where $x \in X$, a , b , σ are unknown smooth functions. The relative degree r of the system is assumed to be constant and known. That means that total time derivatives $\sigma^{(i)}$ do not depend on u with $i = 1, \dots, r-1$, and $\frac{\partial}{\partial u} \sigma^{(r)} \neq 0$. The requirement for relative-degree constancy is often true in applications.

As follows from [9] $\sigma^{(r)} = h(t,x) + g(t,x)u$, $g(t,x) \neq 0$. It is easy to see that $g = L_b L_a^{r-1} \sigma = \frac{\partial}{\partial u} \sigma^{(r)}$, and $h = L_a^r \sigma$ is the r th total time derivative of σ calculated with $u = 0$. In other words, functions h and g may be defined in terms of input-output relations.

The problem is to make σ vanish in finite time and to keep $\sigma \equiv 0$ afterwards. Provided by some discontinuous feedback $u = U(t, x)$, the resulting r -sliding motion would satisfy the zero-dynamics equations $\sigma = \dot{\sigma} = \dots = \sigma^{(r)} = 0$ [9, 7]. Thus, the black-box control problem is reformulated in the sliding mode terms. The new controller has to generalize the standard 1-sliding relay controller $u = -K \text{sign } \sigma$ applicable with $r = 1$. Hence, $g(t,y)$ and $h(t,y)$ are to be bounded, $h > 0$. The resulting condition is that for some $K_m, K_M, C > 0$

$$0 < K_m \leq \frac{\partial}{\partial u} \sigma^{(r)} \leq K_M, \quad |L_a^r \sigma| \leq C. \quad (3)$$

Mark that in practice all variables are *a-priori* bounded, which implies boundedness condition (3). In the worst case condition (3) is only locally true, which allows for the local application of the presented method.

The last condition is that the system not explode when any admissible control is applied. In particular it means that the zero-dynamics is stable.

3. Arbitrary order sliding controllers

A family of r -sliding controllers with finite time convergence for any natural number r was presented in [12, 13]. Being constructed by simple recursive procedure, these controllers solve control problem (2), (3) and have the form

$$u = -\alpha \text{sign}(\varphi(\beta_1, \dots, \beta_{r-1}, \sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})), \quad (4)$$

where the positive constant parameters β_i are to be chosen sufficiently large in the index order. While each proper choice of $\beta_1, \dots, \beta_{r-1}$ is applicable to any system of the form (2), (3), the parameter $\alpha > 0$ is to be chosen specifically for each fixed C, K_m, K_M . Another important feature of these controllers is that with discrete measurements they provide for the accuracy $|\sigma^{(k)}| \sim \tau^{r-k}$, $k = 0, 1, \dots, r$, where $\sigma^{(0)} = \sigma$, $\tau > 0$ is the measurement (switching) time interval. That accuracy was proved to be the best possible with discontinuous $\sigma^{(r)}$ [10].

The number of choices of β_i is infinite. Here are a few examples [12, 13] with β_i tested for $r \leq 4$. The first is the relay controller, the second is listed in [5, 10, 20].

1. $u = -\alpha \text{sign } \sigma$,
2. $u = -\alpha \text{sign}(\dot{\sigma} + |\sigma|^{1/2} \text{sign } \sigma)$,
3. $u = -\alpha \text{sign}(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + \sigma^2)^{1/6} \text{sign}(\dot{\sigma} + |\sigma|^{2/3} \text{sign } \sigma))$,
4. $u = -\alpha \text{sign}\{\ddot{\sigma} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + |\sigma|^{3/12}) \text{sign}[\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^{3/6}) \text{sign}(\dot{\sigma} + 0.5|\sigma|^{3/4} \text{sign } \sigma)]\}$,
5. $u = -\alpha \text{sign}(\sigma^{(4)} + \beta_4(\sigma^{12} + |\dot{\sigma}|^{15} + \ddot{\sigma}^{20} + \ddot{\sigma}^{30})^{1/60} \text{sign}(\ddot{\sigma} + \beta_3(\sigma^{12} + |\dot{\sigma}|^{15} + \ddot{\sigma}^{20})^{1/30} \text{sign}(\dot{\sigma} + \beta_2(\sigma^{12} + |\dot{\sigma}|^{15})^{1/20} \text{sign}(\dot{\sigma} + \beta_1|\sigma|^{4/5} \text{sign } \sigma))))$.

Another conceptually close family of controllers called “terminal sliding modes” was published recently [20]. Unfortunately, the resulting closed-loop systems have locally unbounded right-hand sides, which prevents implementation of the Filippov theory. Trajectories are to start from a specified state space sector, full knowledge of the system is required.

Dynamic sliding modes [18] are also to be mentioned here as an effective approach. The corresponding sliding modes are provided by control implicitly defined from the equality $\frac{d}{dt} [P_{r-1}(\frac{d}{dt})\sigma] = -k \text{sign} [P_{r-1}(\frac{d}{dt})\sigma]$, where $P_{r-1}(\cdot)$ is a stable polynomial, $k > 0$. Trajectories transfer in finite time into a 1-sliding mode on the manifold $P_{r-1}(\frac{d}{dt})\sigma = 0$ and then exponentially converge to the r -sliding mode $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$. Compared with the described arbitrary-order sliding controllers, dynamic sliding modes feature infinite-time convergence, lower accuracy and globally unbounded control.

The *chattering effect* manifests itself in dangerous system vibrations, when control u in (3) has some real physical sense like spatial coordinate or force. It is eliminated if the goal $\sigma = 0$ is achieved by means of an l -smooth control. Introducing new coordinates $\dot{u}, \ddot{u}, \dots, u^{(l-1)}$ and new control $u^{(l)}$ transforms the problem into problem (2), (3) with higher relative degree.

4. Differentiation problem

As follows from (4), availability of $\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}$ is sufficient to solve the stated control problem (2), (3).

Let input signal $\eta(t)$ be a Lebesgue-measurable locally bounded function defined on $[0, \infty)$ and let it consist of an

unknown differentiable base signal $\eta_0(t)$ and a bounded measurable noise $N(t)$. The problem is to find an estimation of $\eta_0^{(i)}(t)$, continuously depending on $N(t)$ and being exact if $N(t) \equiv 0$. Such differentiator is called robust and exact [11]. Mark that it has to produce some output for any measurable locally bounded input. The problem is obviously incorrect if there is no *a-priori* information on $\eta_0(t)$. On the contrary, if it were, for example, known that $\eta_0(t) = \text{const}$, the problem would be trivial. Consider the case when $\eta_0(t)$ is supposed to have a continuous k th order derivative with known Lipschitz's constant $C > 0$. It is easy to see that no differentiator may provide for differentiation accuracy of the i th derivative better than $C^{i/k} \varepsilon^{(k-i)/k}$, $i = 0, 1, \dots, k$, for any input of the above structure (it is sufficient to consider the noise $N(t) = \varepsilon \sin(C/\varepsilon)^{1/k} t$ and $\eta_0(t) = 0$). It is proved in [11] that there is a differentiator providing for accuracy proportional to that listed here. Unfortunately, that is only a pure existence theorem. Note that the marked asymptotics may be further improved if additional restrictions are imposed on $\eta_0(t)$.

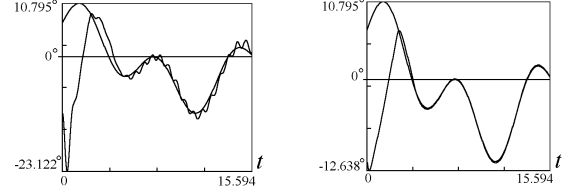
The standard approach for building a differentiator [8, 21] is as follows. Define an auxiliary dynamic system $\dot{x} = v$ and $\sigma(t, x) = x - \eta(t)$. Then, provided $\sigma = 0$ is kept in sliding mode, $v = \dot{\eta}$. That equality holds in average in 1-sliding mode and exactly in 2-sliding mode and higher.

Let $\eta_0(t)$ have a continuous derivative with known Lipschitz's constant $C > 0$. Following is a slightly modified first-order real-time exact robust differentiator [11]:

$$\begin{aligned} \dot{\zeta} &= v, & v &= \xi - \lambda |\zeta - \eta(t)|^{1/2} \text{sign}(\zeta - \eta(t)), \\ \dot{\xi} &= -\mu \text{sign}(\zeta - \eta(t)), & \dot{\omega} &= -\kappa \text{sign}(\omega - v) \end{aligned} \quad (5)$$

where $\mu, \lambda, \kappa > 0$ are constant parameters. Both $v(t)$ and $\omega(t)$ may be considered as the output of the differentiator. ω is a result of non-linear smoothing of v , carried out without loss of accuracy, $\kappa > C$. Solutions of the system are understood in the Filippov sense. Parameters may be chosen in the form $\mu = 1.1C$, $\lambda = 1.5C^{1/2}$ (see [11] for other choices) With $\sup |N(t)| = \varepsilon$ that differentiator features finite-time convergence and provides for the asymptotically best accuracy proportional to $C^{1/2} \varepsilon^{1/2}$. Therefore, having been k times successively implemented, that differentiator will provide for k th order differentiation accuracy of the order of $\varepsilon^{(2-k)}$.

Consider system (2), (3) closed by the described r -sliding controller. Since the output σ has a Lipschitzian $(r-1)$ th total time derivative, an asymptotically-optimal $(r-1)$ th-order differentiator may be applied in the feedback. Moreover, it would provide for the overall accuracy $\sup |\sigma| \sim \varepsilon$ [11]. Differentiator (5) may also be successively implemented resulting in a robust exact system. Thus, full real-time robust control of the output is possible, using only output measurements and knowledge of the relative degree.



a. Differentiator (5) is used b. Direct derivative sampling
Fig. 1: Pitch control

Consider a real life example of closed-loop aircraft pitch control [15]. The problem is to get the pitch angle θ of vehicle "Delilah" (Israel Military Industries Ltd.) to track some signal θ_c given in real time when undergoing unmeasured external disturbances. The actual nonlinear dynamic system is given by its linear 5-dimensional vertical plane numerical linear approximations based on wind tunnel experiments. The corresponding parameters of the linear systems as well as the current coordinates are not available in real time, θ is the measured output. The relative degree is 2, which means that the control (the elevator angle) appears explicitly only in $\ddot{\theta}$. Some delay and noise are also present in the measurements, and the system contains an actuator whose reaction involves both delay and discretization. The idea [15] was to keep a constraint $\sigma = \lambda(\theta - \theta_c) + \frac{d}{dt}(\theta - \theta_c) = 0$ by a 2-sliding controller, the derivative being obtained by a special sensor or by differentiator (5). The controller implemented here uses $\text{sign} \sigma$ which was replaced by the sign of the first difference $\Delta \sigma$. Tracking performance with the differentiator (simulation) and with special sensors (simulation confirmed by flight experiments) is demonstrated in Fig. 1. The vibration seen in Fig. 1a is caused by a large sampling period (1/64 s).

4. Second order robust exact differentiator

Let input signal $f(t)$ be a function defined on $[0, \infty)$ and let it consist of an unknown base signal $f_0(t)$ having a second derivative with known Lipschitz's constant $C > 0$ and a bounded measurable noise. The problem is to find real-time robust estimations of $\dot{f}_0(t)$ and $\ddot{f}_0(t)$ being exact in the absence of the measurement noise. The following differentiator is partially based on the approach from [2]:

$$\dot{x} = v, \quad v = -\lambda_1 |x - f(t)|^{2/3} \text{sign}(x - f(t)) + v_1 + z, \quad (6)$$

$$\dot{v}_1 = -\alpha_1 |x - f(t)|^{1/3} \text{sign}(x - f(t)), \quad (7)$$

$$\dot{z} = z_1, \quad z_1 = -\lambda_2 |z - v|^{1/2} \text{sign}(z - v) + z_2, \quad (8)$$

$$\dot{z}_2 = -\alpha_2 \text{sign}(z - v). \quad (9)$$

The system provides for keeping $\sigma = x - f(t) = 0$ and $\sigma_1 = z - \dot{f}(t) = 0$ in finite-time convergent 2-sliding mode, which also means that $\dot{\sigma} = v - \dot{f}(t) = 0$ and $\dot{\sigma}_1 = z_1 - \ddot{f}(t) = 0$. Thus the outputs v, z are considered as estimations of $\dot{f}_0(t)$, while z_1 is an estimation of $\ddot{f}_0(t)$. The output z_1 of the system may be non-linearly smoothed without any accuracy loss:

$$\dot{w} = -\delta \text{sign}(w - z_1), \quad \delta > C. \quad (10)$$

In the presence of measurement errors with maximal magnitude ε , differentiator (6) - (10) provides after a finite-time transient process for the proportionality of $\sup |v - \dot{f}_0(t)|$, $\sup |z - \dot{f}_0(t)|$ and $\sup |z_1 - \ddot{f}_0(t)|$, $\sup |w - \ddot{f}_0(t)|$ to $C^{1/3} \varepsilon^{2/3}$ and $C^{2/3} \varepsilon^{1/3}$ respectively. Recall that asymptotically it is the best possible accuracy. In the absence of measurement noises the differentiator provides with discrete measurements for $\sup |x - f(t)|$, $\sup |v - \dot{f}(t)|$ (also $\sup |z - \dot{f}(t)|$) and $\sup |z_1 - \ddot{f}(t)|$ (also $\sup |w - \ddot{f}(t)|$) being proportional to τ^3 , τ^2 and τ respectively. Note that the differentiator is really robust and the measurement step τ does not depend on the noise parameters. Moreover, the smaller τ , the better the differentiator performance.

The parameters $\lambda_1, \alpha_1, \lambda_2, \alpha_2, \delta > 0$ may be chosen separately for each C . It is easy to check that if $\lambda_1, \alpha_1, \lambda_2 = \lambda_{20}, \alpha_2 = \alpha_{20}, \delta = \delta_0$ are acceptable for $C = 1$, then $\lambda_1, \alpha_1, \lambda_2 = \lambda_{20} C^{1/2}, \alpha_2 = \alpha_{20} C, \delta = \delta_0 C$ are acceptable for each $C > 0$.

Let the base signal be $f_0(t) = \sin 0.5t + \cos t$. The author has tested the values $\lambda_1 = 6.325, \alpha_1 = 22, \lambda_2 = 4, \alpha_2 = 2.2, \delta = 3$. The differentiator performance in the absence of measurement noises with $\tau = 10^{-4}$ is demonstrated in Fig. 2. The outputs z and w were taken for $\dot{f}(t)$ and $\ddot{f}(t)$ respectively. The attained accuracies are $1.30 \cdot 10^{-10}$, $3.50 \cdot 10^{-5}$ and 0.0063 for tracking $f(t)$, the first and the second derivatives respectively in the absence of noise. With $\tau = 10^{-3}$ the accuracies change to $1.1 \cdot 10^{-7}$, 0.00039 , 0.041 respectively.

The differentiator performance was checked for different noises. Being robust, it is at the same time more sensitive to the low frequency noises due to its ability to differentiate them exactly if their third derivative is small enough. Let the noise have magnitude 0.001 . The graphs in Fig. 3, 4 are received with periodic non-differentiable noise having frequency close to 10 , $\tau = 0.001$. The attained accuracies are 0.00146 , 0.0512 and 0.421 for tracking $f_0(t)$, the first and the second derivatives respectively. Remind that the best attainable accuracy here is about 0.01 for the first derivative and 0.1 for the second. The differentiator outputs for noise frequency 10000 are shown in Fig. 5. The resulting accuracies are slightly less, but do not change significantly. It is seen from Fig. 4 that while output w of filter (10) improves the accuracy and smoothes z_1 considerably, it introduces an infinitesimal but systematic error. Changing the noise magnitude from 0.001 to 0.008 achieves new accuracies 0.170 and 0.910 of the first and second derivative estimations. Note that theoretically they are to be achieved from the previously measured accuracies for noise magnitude 0.001 multiplied by 4 and 2 respectively.

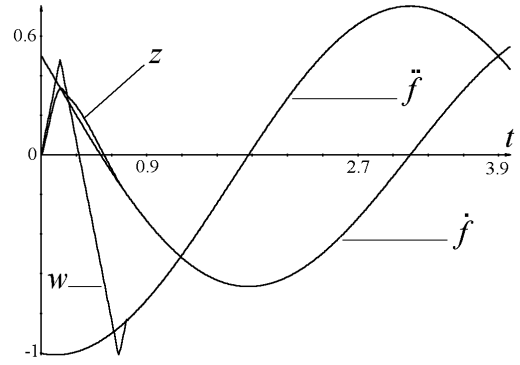


Fig. 2: Differentiation: 2 derivatives without noise

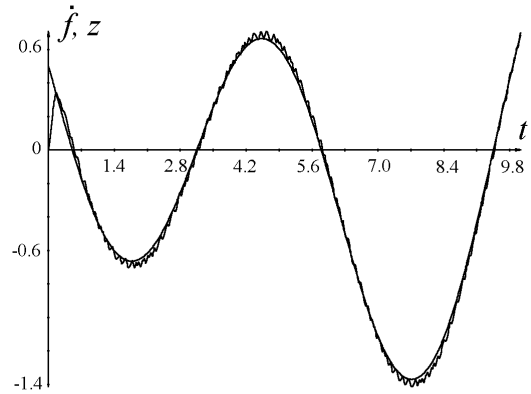


Fig. 3: The first derivative with noise

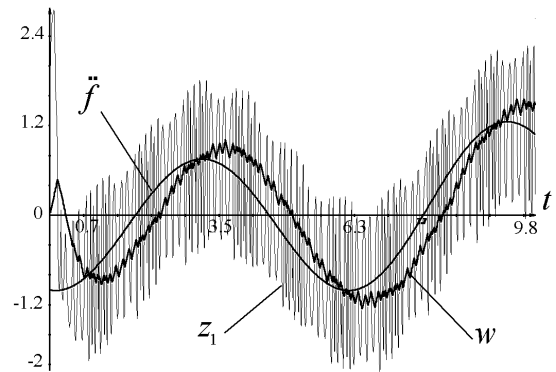


Fig. 4: The 2nd derivative: low frequency noise

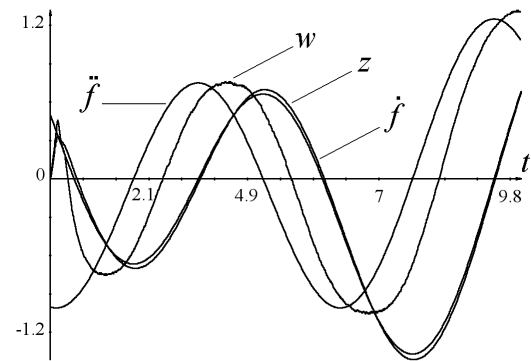


Fig. 5: 2 derivatives with high frequency noise

5. Model example: closed loop 3-sliding car control

Consider a simple kinematic model of car control [17]

$$\dot{y}_1 = v \cos \varphi, \quad \dot{y}_2 = v \sin \varphi, \quad \dot{\varphi} = v/l \tan \theta, \quad \dot{\theta} = u, \quad (11)$$

where y_1 and y_2 are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axes and θ is the steering angle (Fig. 6). The task is to steer the car from a given initial position to the trajectory given by the equation $y_2 = g(y_1)$, while y_1 and y_2 are assumed to be measured in real time. Define

$$s = y_2 - g(y_1),$$

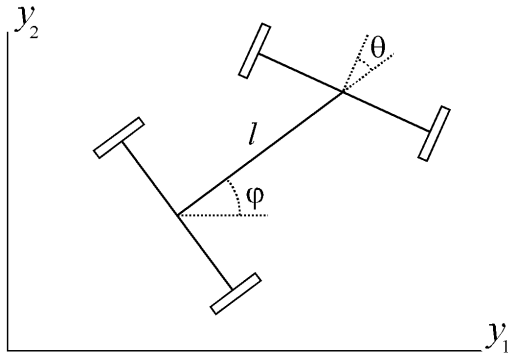


Fig. 6 Kinematic car model

Let $v = \text{const} = 10$ m/s, $l = 5$ m, $g(y_1) = 10 \sin 0.05y_1 + 5$, $y_1 = y_2 = \varphi = \theta = 0$ at $t = 0$. The relative degree of the system is 3 and 3-sliding controller $N^{\circ 3}$ may be applied here. A regularized 3-sliding controller [13] is implemented here:

$$u = -20 \text{sign}(\ddot{s} + 2(|\dot{s}|^3 + |s|^2)^{1/6} \max[-1, \min(1, 10|s|)^{-2/3} (\dot{s} + |s|^{2/3} \text{sign } s)]). \quad (12)$$

That controller provides for the appearance of a 1-sliding mode on a continuous piece-wise smooth surface and features more smooth convergence to the 3-sliding mode $\dot{s} = \ddot{s} = s = 0$ than the standard one. Substitute in (12) estimations of \ddot{s} and \dot{s} achieved by differentiator (6) - (10) with the input

$$f(t) = s = y_2 - g(y_1), \quad (13)$$

Consider two main choices of the substitutions:

$$u = -20 \text{sign}(z_1 + 2(|v|^3 + |f|^2)^{1/6} \max[-1, \min(1, 10|f|)^{-2/3} (v + |f|^{2/3} \text{sign } f)]), \quad (14)$$

$$u = -20 \text{sign}(w + 2(|v|^3 + |f|^2)^{1/6} \max[-1, \min(1, 10|f|)^{-2/3} (v + |f|^{2/3} \text{sign } f)]). \quad (15)$$

Let $\lambda_1 = 20$, $\alpha_1 = 100$, $\lambda_2 = 10$, $\alpha_2 = 25$, $\delta = 10$. Consider the closed-loop system (6)-(9), (11) - (14). The accuracies $|s| \leq 0.002214$, $|\dot{s}| \leq 0.348$, $|\ddot{s}| \leq 4.451$ are achieved with the measurement step $\tau = 0.001$. With $\tau = 0.0001$ they change to $|s| \leq 3.27 \cdot 10^{-6}$, $|\dot{s}| \leq 0.000502$ and $|\ddot{s}| \leq 0.436$.

The trajectory of the car is shown in Fig. 7, mutual graph of s , \dot{s} and \ddot{s} is shown in Fig. 8.

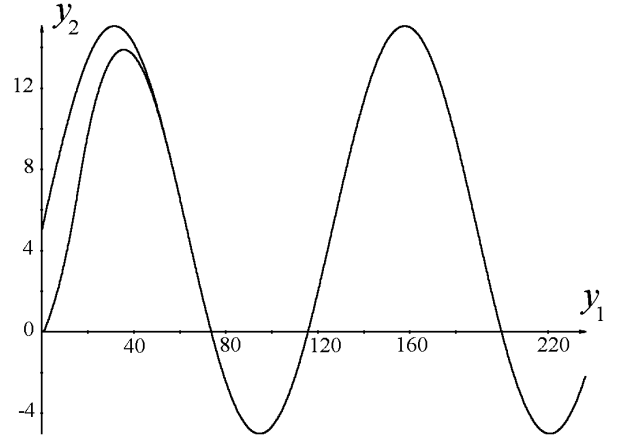


Fig. 7 Trajectory of the car with measurement error 0.

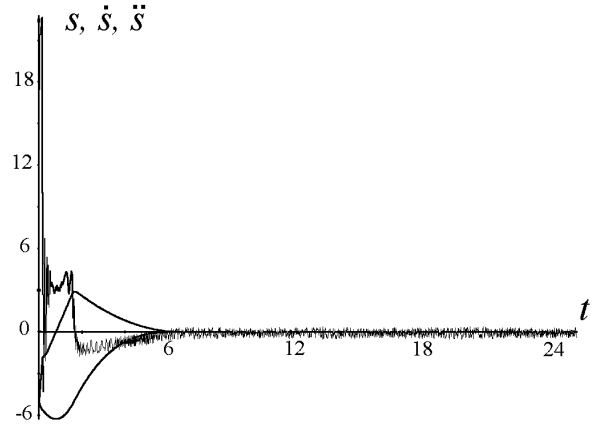


Fig. 8 Deviation and its derivatives

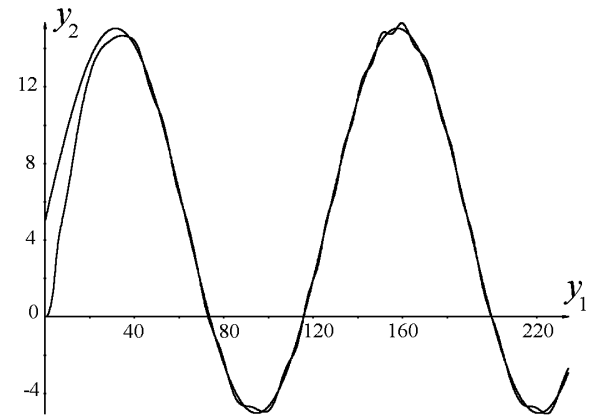


Fig. 9 Car trajectory with measurement error 0.1m.

Being robust with respect to the measurement noises, that controller nevertheless exceeds deviation $\sup|s| = 1.5$ from the given trajectory already with $\varepsilon = 0.01$. At the same time controller (6) - (10), (13), (15) provides only for $|s| \leq 0.035$ in the absence of noises, it is, however, less sensitive to measurements errors. In particular with $\varepsilon = 0.1$ deviation 0.21 is achieved (Fig. 9).

6. Conclusions

Most of the practically important problems in output control are covered by the cases when relative degree r equals 2, 3 and 4. The r -sliding controller requires availability of $\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}$. The cases $r = 2, 3$ are readily provided by the described differentiators. Only the sign of an expression of the form $\ddot{\sigma} - \Psi(\sigma, \dot{\sigma}, \ddot{\sigma})$ (see the above controller list) is actually required with $r = 4$. Thus, it may be replaced by $\text{sign}(\Delta \ddot{\sigma} - \tau \Psi)$, where $\Delta \ddot{\sigma}$ is the first difference between successive discrete measurements, τ is the measurement step. Nevertheless, large noise or a small sampling step lead to the total disappearance of any information from that expression. The solution is to make the measurement step τ vary with respect to the current measured values of $\sigma, \dot{\sigma}, \ddot{\sigma}$ [14]. Hence, the proposed second order differentiator together with the measurement-step feedback will supposedly provide for a robust 4-sliding mode realization.

No knowledge on the exact mathematical model of the process is needed. The smoothness degree of the control may be chosen arbitrarily. The transient process may be shortened as needed. Nevertheless, practical design of such high-performance controllers involves an inevitable compromise between a few contradictory requirements: higher smoothness degree requires more real-time information which leads to extensive use of observers and differentiators. In its turn, higher-order differentiation (even optimal) inevitably leads in practice to fast error accumulation, and direct measurement of the derivatives is preferable. At the same time the experience shows that it is possible to design a controller satisfying any reasonable (i.e. practically attainable) requirements.

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