Convergence and Error Analysis for a Max–Plus Algorithm

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Abstract

We consider max-plus based algorithms for the solution of nonlinear H_∞ problems. This class of algorithms has been described for several problem types such as nonlinear H_∞ filtering, nonlinear H_∞ control and nonlinear H_∞ control under partial information. Previous treatments have been oriented towards the general introduction of the algorithms. It has been noted that the corresponding convergence analysis was lacking in those papers. Here we demonstrate, in the case of nonlinear H_∞ control, that the errors introduced by the truncation to a finite number of max-plus basis functions go to zero as the number of basis functions increases. Some error bounds are also obtained.

1 Introduction

A class of max–plus algebra based algorithms has been proposed as a means for reducing computational costs in the solution of nonlinear H_∞ problems. It has been applied to nonlinear Robust/H_∞ filtering problems [4], [3] [10], [11], nonlinear H_∞ control problems [13], [12], [9], and nonlinear H_∞ control under partial information [6]. We will concentrate on the nonlinear H_∞ control problem here as the application of the algorithm, although most of the material corresponds to the case of partial information as well. It has previously been shown that if the H_∞ value function had a max–plus basis expansion (to be described below as well as in the references) consisting of a finite number of basis functions, then the computation of the value function reduces to a max–plus eigenvector computation for a max–plus eigenvalue of zero (the max–plus multiplicative identity). It was also demonstrated that for the matrix corresponding to the H_∞ problem, there is only one eigenvalue (zero) and a unique eigenvector corresponding to this eigenvalue. Further, this unique eigenvector yields the "correct" viscosity solution of the HJB PDE (among many).

However, in reality, the value function would not have a *finite* max–plus expansion in any but the most unusual case. In the previous papers (with the exception of [6]), this point was not addressed since the main goal was to present this new class of algorithms. In [6], the question was addressed in a broad sense. In this paper however, we will show that as the number of basis functions increases, the approximation obtained by the algorithm converges to the true value function. We will also obtain some simple error estimates for the size of the errors introduced by this basis truncation.

This leaves one final point which will not be fully addressed in this extended abstract. Specifically, there is an additional source of error in that typically, one would only compute the matrix entries approximately. We will indicate briefly the sub-algorithm being used to compute these entries. However, the error analysis for this part is still in progress.

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2 Review of the Max–Plus Based Algorithm

As noted in the introduction, we will consider here the infinite time–horizon H_∞ problem. We will restrict our attention to the fixed–feedback case where the control is built into the choice of dynamics. The case of active control computation (i.e. the game case) is discussed in [9], [13], [13]. Consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{1}$$

where X is the state taking values in $\mathbf{R}^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^m : w \in L_2[0,T]\ \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance. Let $l(x)$ be the running cost.

We will not waste space in the abstract with detailed assumptions. The following are sufficient. We will assume that all the functions f, σ and l (given below) are smooth. We assume f, σ are globally Lipschitz, and that σ is bounded. We will assume that l is locally Lipschitz, zero at the origin, and satisfies a quadratic growth bound. We will assume that there exist $c \in (0,\infty)$ such that

$$(x-y)^T(f(x)-f(y)) \le -c|x-y|^2 \qquad \forall\, x,y \in \mathbf{R}^n \quad (A1)$$

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Note that this assumption automatically implies the closed–loop stability criterion of H_∞ control.

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function, also referred to as the value function in the sequel, $W(x)$, such that

$$W(x) = \sup_{w\in\mathcal{W}} \sup_{T<\infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt. \qquad (2)$$

The corresponding PDE is

$$0 = -\Big[\frac{1}{2\gamma^2}\nabla W^T\sigma(x)\sigma^T(x)\nabla W + f^T(x)\nabla W + l(x)\Big]. \qquad (3)$$

We are looking for a solution satisfying $W(0) = 0$. Under a certain assumption on the size of γ, one can show that there is a unique solution of the DPE in the class of (viscosity) solutions which do not grow faster than a certain rate. See [16].

One can show that W is a fixed point of the solution operator to the DPE, where this solution operator has the representation $S_\tau[W(\cdot)](x) = \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\}$. More specifically, using the dynamic programming principle for an associated finite time horizon problem, one obtains

Theorem 2.1 *For any $\tau \in [0,\infty)$, W given by (2) satisfies $S_\tau[W] = W$, and further, it is the unique solution in the class mentioned above.*

Note that we will not provide proofs for the results in this review section. The proofs can be found in the references given earlier.

We remark that W is a fixed point of S_τ for any τ, which provides some freedom in the choice of problem we wish to solve.

Theorem 2.2 *The solution operator, S_τ, is linear in the max–plus algebra.*

A function ϕ is called **semiconvex** if for every $R < \infty$, there exists C_R such that

$$\widehat{\phi}(x) \doteq \phi(x) + \frac{C_R}{2}|x|^2$$

is convex on the ball $B_R \doteq \{x \in \mathbf{R}^n : |x| \le R\}$. The infimum over such C_R will be known as the semiconvexity constant for ϕ over B_R. We denote the space of semiconvex functions by $\mathcal{S}$, and the space of semiconvex functions which have semiconvexity constant C_R over B_R by $\mathcal{S}_{C_R}$

Theorem 2.3 *W lies in $\mathcal{S}$.*

The following max–plus basis can be derived from convex duality; see [4] for details. Let $\phi \in \mathcal{S}$. Let $\{x_i\}$ be a countable, dense set over B_R, and let $c > C_R$. Define

$$\psi_i(x) \doteq \frac{-c}{2}|x - x_i|^2 \qquad (4)$$

for each i. Then,

$$\phi(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \qquad \forall\, x \in B_R \qquad (5)$$

where $a_i \doteq -\max_{x\in B_R}[\psi_i(x) - \phi(x)]$. This is a max–plus basis expansion.

For the remainder of the section, fix any $\tau \in (0,\infty)$. We assume throughout that one may choose the $c > C_R$ such that

$$S_\tau[\psi_i] \in \mathcal{S}_c \text{ for all } i. \qquad (A2)$$

We will not discuss this assumption here, but simply note that we have verified that this assumption holds for several problems where candidate values of c were obtained by requiring solution of a certain Riccati inequality everywhere in B_R.

Under the assumption that W had a max–plus basis expansion with a finite number of terms, we obtained the following result in the referenced papers. Let $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i$, $a^T = (a_1\ a_2 \cdots a_n)$, and $B_{j,i} = -\max_{x\in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$. Note that B actually depends on τ, but for this section we fix any value τ, and suppress the dependence in the notation.

Theorem 2.4 *$S_\tau[W] = W$ if and only if $a = B \otimes a$ where $B \otimes a$ represents max–plus matrix multiplication.*

Now, suppose that one has computed B (the construction of B will be discussed in a later section). Once one has an approximation to B, one must compute the max–plus eigenvector. We should note that B has a unique eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [2]. By the above results, this eigenvalue must be zero. We can compute the eigenvector via the power method; this has the added benefit that the convergence analysis to follow is performed in an analogous way. In the power method, one computes an eigenvector, a by

$$a = \lim_{N\to\infty} B^N \otimes \vec{0}$$

where the power is to be understood in the max–plus sense and $\vec{0}$ is the zero vector.

Define

$$H(x,y) \doteq S_\tau[W](x) - \sup_{w\in\mathcal{W}_y}\Big\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\Big\} \qquad (6)$$

where $X(0) = 0$ and $\mathcal{W}_y = \{w \in \mathcal{W} : X(\tau) = y\}$.

Lemma 2.5 *Let $w \in \mathcal{W}$.*

$$\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt \le W(x) - W(X(\tau)) - H(x, X(\tau))$$

where $X(0) = x$.

Now let the $\{x_j\}$ be such that $x_1 = 0$, i.e. so that the first basis function is centered at the origin.

Lemma 2.6 $B_{1,1} = 0$. *Also, there exists $\delta > 0$ such that for all $j \neq 1$, $B_{j,j} \le -\delta$.*

Theorem 2.7 *Let $N \in \mathcal{N}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \le k_i \le n$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $N = 1$, $k_1 = k_2 = 1$. Then*

$$\sum_{i=1}^{N} B_{k_i,k_{i+1}} \le -\delta.$$

Theorem 2.8 $\lim_{N\to\infty} B^N \otimes 0$ *exists, converges in a finite number of steps, and satisfies $e = B \otimes e$.*

Not only is the eigenvalue unique, but one also has

Corollary 2.9 *There is a unique eigenvector up to a max–plus multiplicative constant, and of course, this is the output of the above power method.*

3 Convergence of the Algorithm

In this section we consider the approximation due to using only a finite number of functions in the max–plus basis expansion. It will be shown that as the number of functions increases (in a reasonable way), the approximate solution obtained by the eigenvector computation of Section 2 converges from below to the value function. In the next section, some error bounds will be obtained, but since these bounds are likely quite conservative, and since the convergence analysis of this section is much cleaner, it seems useful to present this separate convergence analysis.

First, we establish some notation. Let us have sets of basis functions indexed by n, that is the *sets* are indexed by n. Let the cardinality of the n^{th} set be $\mathcal{I}^{(n)}$. For each n, let $\mathcal{X}^{(n)} \doteq \{x_i^{(n)}\}_{i=1}^{\mathcal{I}^{(n)}}$ and $\mathcal{X}^{(n)} \subset \mathcal{X}^{(n+1)}$. For instance, in the one–dimensional case, one might have $\mathcal{X}^{(1)} = \{0\}$, $\mathcal{X}^{(2)} = \{-1/2, 0, 1/2\}$, $\mathcal{X}^{(3)} = \{-3/4, -1/2, -1/4, 0, 1/4, 1/2, 3/4\}$, and so on. Further, we will let the basis functions be given by $\psi_i^{(n)} \doteq \frac{-c}{2}|x - x_i^{(n)}|^2$, and consider the sets of basis functions $\Psi^{(n)} \doteq \{\psi_i^{(n)} : i \in \mathcal{I}^{(n)}\}$. Then define the approximations to the semigroup operator, S_τ by

$$S_\tau^{(n)}[\phi](x) \doteq \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x) \tag{7}$$

where

$$a_i^{(n)} \doteq -\max_x \left[\psi_i^{(n)}(x) - S_\tau[\phi](x)\right]. \tag{8}$$

In other words, $S_\tau^{(n)}$ is the result of the application of the S_τ followed by the truncation due to a finite number of basis functions.

Lastly, we use the notation ${S_\tau}^N$ to indicate repeated application of S_τ N times. (Of course, by the semigroup property, ${S_\tau}^N = S_{N\tau}$.) Correspondingly, we use the notation ${S_\tau^{(n)}}^N$ to indicate the application of $S_\tau^{(n)}$ N times.

Define $\phi_0(x) \equiv 0$ and $\phi_0^{(n)}(x) \doteq \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x)$ with $a_i^{(n)} \doteq -\max_x \left[\psi_i^{(n)}(x) - \phi_0(x)\right]$. It is well-known that (see [12], [16] among many others) that

$$\lim_{N\to\infty} {S_\tau}^N[\phi_0] = W. \tag{9}$$

Also, note that since $\mathcal{X}^{(n)} \subset \mathcal{X}^{(n+1)}$, one has

$${S_\tau^{(n)}}^N[\phi_0^{(n)}](x) \le {S_\tau^{(n+1)}}^N[\phi_0^{(n+1)}](x) \le {S_\tau}^N[\phi_0](x) \tag{10}$$

for all $x \in B_R$.

By Theorem 2.8, for each n, there exists $\overline{N}(n)$ such that

$${S_\tau^{(n)}}^N[\phi_0^{(n)}] = {S_\tau^{(n)}}^{\overline{N}(n)}[\phi_0^{(n)}] \qquad \forall\, N \ge \overline{N}(n). \tag{11}$$

Defining

$${W^{(n)}}^\infty \doteq {S_\tau^{(n)}}^{\overline{N}(n)}[\phi_0^{(n)}], \tag{12}$$

we further find that the limit is the fixed point. That is,

$$S_\tau^{(n)}[{W^{(n)}}^\infty] = {W^{(n)}}^\infty. \tag{13}$$

Then, by (9), (10) and (12), we find that

$${W^{(n)}}^\infty \text{ is monotonically increasing in } n \tag{14}$$

and

$${W^{(n)}}^\infty \le W. \tag{15}$$

Therefore, there exists ${W^\infty}^\infty \le W$ such that

$${W^{(n)}}^\infty \uparrow {W^\infty}^\infty. \tag{16}$$

Under Assumption $(A2)$, it is not difficult to show that given $\varepsilon > 0$, there exists $n_\varepsilon < \infty$ such that

$${W^{(n)}}^\infty(x) = S_\tau^{(n)}[{W^{(n)}}^\infty](x) \ge S_\tau[{W^{(n)}}^\infty](x) - \varepsilon$$

for all $x \in B_R$. On the other hand, one always has

$$S_\tau^{(n)}[\phi] \le S_\tau[\phi].$$

Combining these last two inequalities, one obtains

$$\begin{aligned} W^{(n)^\infty} &= S_\tau^{(n)}[W^{(n)^\infty}] \le S_\tau[W^{(n)^\infty}] \\ &\le S_\tau^{(n)}[W^{(n)^\infty}] - \varepsilon = W^{(n)^\infty} - \varepsilon. \end{aligned}$$

Combining this with (16), one finds

Theorem 3.1

$$W^{\infty\infty} = S_\tau[W^{\infty\infty}], \qquad (17)$$

or in other words, $W^{\infty\infty}$ is a fixed point of S_τ.

On the other hand, it is now known that (see [17], [16]) W is the smallest, nonnegative fixed point

$$W = S_\tau[W], \qquad (18)$$

and of course even more as mentioned in Theorem 2.1. Also, by (15),

$$W^{\infty\infty} \le W. \qquad (19)$$

Next we assemble several results. First, define a variant of S_τ which requires the terminal state to be in $B_\delta(0)$ as follows

$$\mathcal{S}_\tau^\delta[\phi(\cdot)](x) = \sup_{w \in \mathcal{W}^\delta} \{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \phi(X(\tau))\}$$

where $\mathcal{W}^\delta = \{w \in \mathcal{W} : |X(\tau)| \le \delta|\}$. From [16], we know that given $R < \infty$ and $\delta > 0$, there exists $\widehat{N} < \infty$ such that for all $N \ge \widehat{N}$,

$$S_{N\tau}[W](x) = \mathcal{S}_{N\tau}^\delta[W](x) \qquad \forall\, x \in B_R. \qquad (20)$$

Further, by (19) and [16], one also obtains

$$S_{N\tau}[W^{\infty\infty}](x) = \mathcal{S}_{N\tau}^\delta[W^{\infty\infty}](x) \qquad \forall\, x \in B_R. \quad (21)$$

Also, by Theorem 2.1, there exists $K_\gamma < \infty$ such that

$$0 \le W(x) \le K_\gamma |x|^2 \qquad \forall\, x \in \mathbf{R}^n. \qquad (22)$$

Finally, by Lemma 2.6, Theorem 2.8 and the choice of the center of the first basis function as $x_1 = 0$ (which we assume for all n), one finds

$$W^{\infty\infty}(x) \ge -\frac{c}{2}|x|^2. \qquad (23)$$

We will use these assembled results to prove the next theorem.

Theorem 3.2

$$W^{\infty\infty}(x) = W(x) \qquad \forall\, x \in B_R.$$

Proof: Choose N sufficiently large to satisfy the above condition. By Theorem 3.1,

$$W^{\infty\infty}(x) = S_\tau{}^N[W^{\infty\infty}](x) = S_{N\tau}[W^{\infty\infty}](x)$$

which by (21)

$$= \mathcal{S}_{N\tau}^\delta[W^{\infty\infty}](x) \qquad \forall\, x \in B_R. \qquad (24)$$

But, by (22) and (23),

$$W^{\infty\infty}(z) \ge W(z) - (K_\gamma + c/2)|z|^2 \qquad \forall\, z \in \mathbf{R}^n.$$

Combining this with (24) and the fact that $|X(N\tau)| \le \delta$ for the terminal state corresponding to any $w \in \mathcal{W}^\delta$ yields

$$\begin{aligned} W^{\infty\infty}(x) &\ge S_{N\tau}[W](x) - (K_\gamma + c/2)\delta^2 \\ &= W(x) - (K_\gamma + c/2)\delta^2 \qquad \forall\, x \in B_R. \end{aligned}$$

Since this is true for all $\delta > 0$, one has the result. □ ■

4 Errors Due to Truncation

The previous section demonstrated convergence of the algorithm to the value function as the basis function density increased. Here we outline one approach to obtaining specific error estimates. However since the analysis is more technical, not all the details will be included. The estimates may be rather conservative due to the form of the truncation error bound used; this question will become more clear below. Note that these are only the errors due to truncation to a finite number of basis functions; as noted above, analysis of the errors due to approximation of the entries in the B matrix will be delayed to the final paper.

We use the same notation as in the previous section. Recall that we choose the basis functions throughout such that $x_1^{(n)} = 0$, or in other words, $\psi_1^{(n)}(x) = \frac{-c}{2}|x|^2$ for all n.

We will use the notation

$$W_{N,\tau}^{(n)}(x) \doteq S_\tau{}^{(n)^N}[\phi_0^{(n)}](x)$$

where the N superscript indicates repeated application of the operator N times. We will not include the proof of the following which is essentially a statement about the nature of the truncation errors at each step.

Theorem 4.1 *Given $\varepsilon > 0$, there exists $n_\varepsilon < \infty$ such that for all $n \ge n_\varepsilon$, $N < \infty$ and $x \in B_R$,*

$$\begin{aligned} S_\tau[W_{N,\tau}^{(n)}(\cdot)](x) - \varepsilon|x| &\le S_\tau{}^{(n)}[W_{N,\tau}^{(n)}(\cdot)](x) \\ &\le S_\tau[W_{N,\tau}^{(n)}(\cdot)](x). \end{aligned}$$

Fix $\varepsilon > 0$ and choose $n > n_\varepsilon$ so that one also has $\phi_0(x) - \varepsilon|x| \le \phi_0^{(n)}(x) \le \phi_0(x)$.

Also fix $\overline{\varepsilon} > 0$, and let $w^{1,\overline{\varepsilon}}$ be $\overline{\varepsilon}/2$–optimal for $S_\tau[\phi_0]$, and let $X^{1,\overline{\varepsilon}}$ be the corresponding trajectory. Then,

$$0 \le S_\tau[\phi_0](x) - S_\tau[\phi_0^{(n)}](x) \le \varepsilon|X^{1,\overline{\varepsilon}}(\tau)| + \frac{\overline{\varepsilon}}{2}.$$

This implies

$$0 \le S_\tau[\phi_0](x) - S_\tau{}^{(n)}[\phi_0^{(n)}](x) \le \varepsilon\big(|x| + |X^{1,\overline{\varepsilon}}(\tau)|\big) + \frac{\overline{\varepsilon}}{2}.$$

Repeating this process (with $\overline{\varepsilon}/2^j$–optimal $w^{j,\overline{\varepsilon}}$ at the j^{th} step), one obtains the following result; we do not include the proof.

Theorem 4.2 *For any $x \in B_R$ and $N < \infty$,*

$$\begin{aligned} 0 &\le S_\tau{}^N[\phi_0](x) - S_\tau{}^{(n)\,N}[\phi_0^{(n)}](x) \\ &\le \varepsilon\Big[|x| + \sum_{j=1}^{N} |\overline{X}^{\overline{\varepsilon}}(j\tau)|\Big] + \overline{\varepsilon} \end{aligned}$$

where $\overline{X}^{\overline{\varepsilon}}$ is $\overline{\varepsilon}$–optimal for $S_{N\tau}[\phi_0]$.

From [16], we can also obtain that there exist $m, c \in (0,\infty)$ such that

$$|\overline{X}^{\overline{\varepsilon}}(t)| \le |x| e^{m-ct} \qquad \forall\, t \in [0, N\tau]. \tag{25}$$

Combining Theorem 4.2 and (25), one obtains

$$\begin{aligned} 0 &\le S_\tau{}^N[\phi_0](x) - S_\tau{}^{(n)\,N}[\phi_0^{(n)}](x) \\ &\le \varepsilon\Big(\frac{e^m}{1 - e^{-c\tau}}\Big)|x| + \overline{\varepsilon}. \end{aligned}$$

Noting that $\overline{\varepsilon} > 0$ was arbitrary, this then yields our bound on the errors introduced by truncation.

Theorem 4.3 *For $n \ge n_\varepsilon$,*

$$0 \le W(x) - W^{(n)\,\infty}(x) \le \varepsilon\Big(\frac{e^m}{1 - e^{-c\tau}}\Big)|x|$$

for all $x \in B_R$.

5 Approximation of B

The feasibility of the algorithm is dependent upon a feasible approximation algorithm for B. One approach is a Taylor series (in t) approximation to $S_t[\psi_i](x)$. More specifically, letting $V(t,x) = S_t[\psi_i](x)$, so that V satisfies

$$\begin{aligned} V_t &= f \cdot \nabla V + l + \nabla V^T \sigma\sigma^T \nabla V \\ V(0,x) &= \psi_i(x) \end{aligned} \tag{26}$$

one may approximate V as

$$V(t,x) = V_0(x) + V_1(x)t + \frac{1}{2}V_2(x)t^2 + \dots. \tag{27}$$

Here $V_0(x) = \psi_i(x)$ and V_1 is the right hand side of (26) with ψ_i replacing V. The higher order terms are computed by differentiating (26), and we do not include them. Then

$$B_{j,i} = -\max_x \left\{ \psi_j(x) - [V_0 + V_1\tau + \frac{1}{2}V_2\tau^2 + \dots](x) \right\}.$$

This method was applied but suffers from a problem which we describe only briefly. Note that the approximation of V via, say three terms, in the Taylor series at x out to time τ can be improved by reducing τ. However, the argmax moves off toward "∞" as $\tau \downarrow 0$, and the Taylor series approximation degrades as x moves off to ∞! Consequently, the approximation of $B_{j,i}$ does not tend to improve as $\tau \downarrow 0$, and so this approach was abandoned.

In its stead we are using a means of approximately tracking the argmax for small time intervals. Let

$$\overline{\psi}_j(t,x) = -\frac{1}{2}(x - \xi(t))^T C(x - \xi(t)$$

where

$$\xi(t) = x_i + (t/\tau)(x_j - x_i).$$

Let $V(t,x)$ be as above (with ψ_i as initial condition). Let

$$\overline{X}(t) = \operatorname{argmax}\left\{\overline{\psi}_j(t,x) - V(t,x)\right\}.$$

Note that

$$\overline{X}(\tau) = \operatorname{argmax}\left\{\psi_j(x) - V(\tau,x)\right\}$$

which is the desired quantity. The replacement of $\psi_j(\cdot)$ by $\overline{\psi}_j(t,\cdot)$ prevents the argmax from "blowing up" at $t \downarrow 0$. We use the Taylor expansions

$$\begin{aligned} V(t,x) =&\ V_0(t) + V_1(t)(x - \overline{X}(t)) \\ &+ \frac{1}{2}(x - \overline{X}(t))^T V_2(t)(x - \overline{X}(t)) + \dots \\ f_k(x) =&\ f_{k0} + f_{k1}(x - \overline{X}(t)) + \dots \end{aligned}$$

(where the k subscripts in the f expansion indicate components). One then obtains the following sequence of ordinary differential equations for the propagation of $\overline{X}$ leading to a computation of $B_{j,i}$:

$$\begin{aligned} \dot{\overline{X}}_k =&\ (V_2 - C)^{-1}\Big\{\frac{1}{\tau}\sum_l C_{k,l}(x_{il} - x_{jl}) \\ &- \big[\sum_l ((f_{l1})_k V_{1x_l} + f_{l0}V_{2x_k,x_l}) \\ &- 2\sum_{l,m}(V_{1x_l}a_{l,m}V_{2x_l,x_k}) + 2\overline{X}_k\big]\Big\} \\ \dot{V}_0 =&\ \sum_k \Big[\overline{\psi}_{jx_k}\big(f_{k0} + \overline{X}_k - \sum_l (a_{k,l}\overline{\psi}_{jx_l})\big)\Big] + |\overline{X}|^2 \end{aligned}$$

plus higher order equations which we do not include here. It will be shown that the error induced by cutting off this series at a finite number of differential equations can be bounded for small time. We also note that the initial conditions for this propagation are easily obtained, although we do not include them here.

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