Risk-sensitive portfolio optimization with partial information

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1 Introduction

There have been several works applying the idea of risk-sensitive control to problems of mathematical finance. Among them, Fleming [5], Fleming and Sheu [7] and Lefebre, Montulet [8] and Bielecki and Pliska [3] have studied risk-senstitive control problems arising from portforio management. In particular, Bielecki and Pliska [3], which treats risk-sensitive asset management by taking up a factor model, motivates to the present paper and we introduce here their formulation of the factor model and expose the relationships between theirs and the present one.

Let us consider the following market model consisting of m securities and n factors, whose prices and levels are defined as the solutions of the stochastic differential equations:

(1.1)

$$\begin{aligned} dS^i(t) &= S^i(t)\{(a+AX_t)^i dt + \textstyle\sum_{k=1}^{n+m} \sigma^i_k dW^k_t\}, \\ S^i(0) &= s^i, \;\; i=1,...,m \end{aligned}$$

and

$$dX_t = (b+BX_t)dt + \Lambda dW_t, \quad X(0) = x \in R^n. \tag{1.2}$$

We here denote by $S^i(t)$ the price of the i-th security and by X_t the factor process whose component X^j_t is the level of the j-th factor at time t. $W_t = (W^k_t)$ is an R^{m+n} valued standard Brownian motion process. The set of securities are considered to include stocks, bonds, cash and derivative securities and factors to include dividend yields, price-earning ratios, short -term interest rates, the rate of inflation, etc.. Take up R^m valued investment strategy $h(t)$ among m secutities such that

i) $h(t) \in Z \subset R^m, \;\; \sum_{i=1}^m h^i(t) = 1$

ii) $h(t)$ is $\sigma(S(s), X(s), 0 \le s \le t)$ measurable

and that it is locally integrable in t. Then, for each strategy h one can define the stochastic process $V_t = V_t(h)$ governed by the stochastic differential equation:

$$\begin{aligned} dV_t &= \textstyle\sum_{i=1}^m h^i(t)V(t)\{(a+AX_t)^i dt \\ &\quad + \textstyle\sum_{k=1}^{m+n} \sigma^i_k dW^k_t\} \\ V_0 &= v > 0, \end{aligned} \tag{1.3}$$

which represents investor's capital at time t.

Now let us consider the following problem: for $\theta \in (0,\infty)$ maximize the risk sensitized expected growth rate per unit time

(1.4)

$$J_\infty(v,x;h) = \liminf_{T\to\infty}\left(-\frac{2}{\theta T}\right)\log E[e^{-(2/\theta)\log V_T(h)}].$$

where h ranges over the set of all strategy satisfying i) and ii). Concerning the problem Bielecki and Pliska have constructed the optimal investment strategy from the solution of relevant ergodic type Bellman equation of risk sensitive control.

On the other hand in the present paper we shall relax the measurability condition for investment strategy $h(t)$ as $\sigma(S(s), 0 \le s \le t)$ measurable one in place of above defined ii), namely our strategy is to be selected without using past informations of the factor process $X(t)$. In place we confine ourselves to the case of $Z = R^m$ and consider the problem on finite time horizon. It seems to be very natural to decide the investment strategy by using only the past informations of securities $S(t)$ that one intends to invest. Then there arise a kind of risk-sensitive stochastic control problem with partial observation. We shall formulate our problem by regarding the factor process as the system process and price process of secuties as the observation process in terms of stochastic control of partially observable systems (cf.[1]). Note that in our problem the system noises and the observation noises are correlated. Furthermore our criterion is a type of exponential of integral performance index, which includes a stochastic integral. For such risk-sensiive stochastic control problem of the partially observable system we shall obtain the optimal strategy which has the explicit represention by the solutions of ordinary differential equations. Finally we note that our problem relates to the LEQG problem of a partially observable system studied by Bensoussan and Van Schuppen [2]. Difference from the work lies in that our noises are correlated and the performance index includes a stochastic integral.

2 Setting up

In the present section we shall introduce a factor model slightly modifying the one studied by Bielecki and Pliska [3] and then set up our problem arising from dynamic asset management for the factor model as a risk sensitive optimal control problem with partial observation. We consider a market with $m+1 \geq 2$ securities and $n \geq 1$ factors. We assume that the set of securities includes one bond, whose price is defined by ordinary differential equation:

$$dS^0(t) = r(t)S^0(t)dt, \quad S^0(0) = s^0, \tag{2.1}$$

where $r(t)$ is a deterministic function of t. The other secutity prices and factors are assumed to satsfy the following stochastic differential equations:

$$\begin{aligned} dS^i(t) &= S^i(t)\{(a+AX_t)^i dt + \textstyle\sum_{k=1}^{n+m} \sigma^i_k dW^k_t\}, \\ S^i(0) &= s^i, \quad i = 1, ..., m \end{aligned} \tag{2.2}$$

and

$$dX_t = (b + BX_t)dt + \Lambda dW_t, \quad X(0) = x \in R^n, \tag{2.3}$$

where $W_t = (W^k_t)_{k=1,..,(n+m)}$ is a $m+n$ dimensional standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. Here A, B, Λ are respectively $m \times n$, $n \times n$, $n \times (m+n)$ constant matrices and $a \in R^m$, $b \in R^n$. The constant matrix $(\sigma^i_k)_{i=1,2..,m;\ k=1,2..,(n+m)}$ will be often denoted by Σ in what follows. In the present paper we always assume that

$$\Sigma\Sigma^* > 0. \tag{2.4}$$

Let us denote investment strategy to i-th security $S^i(t)$ by $h^i(t)$, $i = 0, 1, ..., m$ and set

$$S(t) = (S^1(t), S^2(t), ..., S^m(t))^*,$$
$$h(t) = (h^1(t), h^2(t), ..., , h^m(t))^*$$

and

$$\mathcal{G}_t = \sigma(S(u); u \leq t).$$

Here S^* stands for transposed matrix of S.

Definition 2.1. *$(h^0(t), h(t)^*)_{0\leq t\leq T}$ is said an invetment strategy if the following conditions are satisfied*

i) $h(t)$ is a R^m valued $\mathcal{G}_t$ progressively measurable stochastic process such that

$$\sum_{i=1}^m h^i(t) + h^0(t) = 1 \tag{2.5}$$

ii)

$$P(\exists c(\omega) \text{ such that } |h(s)| \leq c(\omega),\ 0 \leq s \leq T) = 1.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. For simplicity when $(h^0(t), h(t)^*)_{0\leq t\leq T} \in \mathcal{H}(T)$ we will often write $h \in \mathcal{H}(T)$ since h^0 is determined by (2.5) if we have R^m valued $\mathcal{G}_t$- progressively measurable process h satisfying ii) in Definition 2.1. For given $h \in \mathcal{H}(T)$ the process $V_t = V_t(h)$ representing the investor's capital at time t is determined by the stochastic differential equation:

$$\begin{aligned} \frac{dV_t}{V_t} &= \textstyle\sum_{i=0}^m h^i(t) \frac{dS^i(t)}{S^i(t)} \\ &= h^0(t)r(t)dt \\ &\quad + \textstyle\sum_{i=1}^m h^i(t)\{(a+AX_t)^i dt + \sum_{k=1}^{m+n} \sigma^i_k dW^k_t\} \\ V_0 &= v. \end{aligned}$$

Then, taking (2.5) into account it turns out to be a solution of

$$\begin{aligned} \frac{dV_t}{V_t} &= r(t)dt + h(t)^*(a + AX_t - r(t)\mathbf{1})dt \\ &\quad + h(t)^*\Sigma dW_t, \\ V_0 &= v, \end{aligned} \tag{2.6}$$

where $\mathbf{1} = (1, 1, ..., 1)^*$. Now we consider the following problem. For a given constant $\theta > -2$, $\theta \neq 0$ maximize the following risk-sensitized expected growth rate up to time horizon T:

$$J(v, x; h; T) = -\frac{2}{\theta} \log E[e^{-\frac{\theta}{2}\log V_T(h)}], \tag{2.7}$$

where h ranges over the set of all admissible investment strategy $\mathcal{H}(T)$ prescribed above. Note that in our problem a strategy h is to be chosen as $\sigma(S(u); u \leq t)$ measurable process, different from the case of Bielecki-Pliska where it is $\sigma((S(u), X_u), u \leq t)$ measurable. Namely, in our case the strategy is to be selected without using past informations of the factor process X_t.

Since V_t satisfies (2.6) we have

$$\begin{aligned} V_t^{-\theta/2} &= v^{-\theta/2} \exp\{\textstyle\int_0^t \Phi(X_s, h_s, r(s); \theta)ds \\ &\quad - \frac{\theta}{2} \textstyle\int_0^t h^*_s \Sigma dW_s - \frac{1}{2}(\frac{\theta}{2})^2 \int_0^t h^*_s \Sigma\Sigma^* h_s ds\}, \end{aligned} \tag{2.8}$$

where

$$\Phi(x, h, r; \theta) = \frac{\theta}{4}(\frac{\theta}{2}+1)h^*\Sigma\Sigma^* h - \frac{\theta}{2}r - \frac{\theta}{2}h^*(a + Ax - r\mathbf{1}).$$

Therefore, if $\theta > 0$ (resp. $-2 < \theta < 0$) our problem maximizing $J(v, x; h; T)$ is reduced to the one minimizing (resp. maximizing) the following criterion:

$$\begin{aligned} I(x, h; T) &= v^{-\theta/2} E[\exp\{\textstyle\int_0^t \Phi(X_s, h_s, r(s); \theta)ds \\ &\quad - \frac{\theta}{2} \textstyle\int_0^t h^*_s \Sigma dW_s - \frac{1}{2}(\frac{\theta}{2})^2 \int_0^t h^*_s \Sigma\Sigma^* h_s ds\}]. \end{aligned} \tag{2.9}$$

Now we shall reformulate the above problem as a partially observable risk-sensitive stochastic control problem. For that we set

$$Y^i_t = \log S^i(t),$$

then we can see that $Y_t = (Y_t^1, ..., Y_t^m)^*$ satisfies the following stochastic diferential

$$(2.10)\quad dY_t^i = \{a^i - \frac{1}{2}(\Sigma\Sigma^*)^{ii} + (AX_t)^i\}dt + \sum_{k=1}^{m+n} \sigma_k^i dW_t^k,$$

$i = 1, ..., m$, by using Itô formula. So, setting $d = (d^i) \equiv (a^i - \frac{1}{2}(\Sigma\Sigma^*)^{ii})$, we have

$$(2.11)\qquad dY_t = (d + AX_t)dt + \Sigma dW_t,$$

which we shall regard as the SDE defining the observation process in terms of stochastic control with partial observation. On the other hand, X_t defined by (2.3) is regarded as a system process. In the present setting system noise ΛdW_t and observation noise ΣdW_t are correlated in general. Note that $\sigma(Y_u, ; u \le t) = \sigma(S(u); u \le t)$ holds since log is a strictly increasing function, so our problem is to minimize (or maximize) the criterion (2.9) while looking at the observation process Y_t and choosing a $\sigma(Y_u, ; u \le t)$ measurable strategy $h(t)$. Though there is no control in the SDE (2.3) defining system process X_t criterion $I(x, h; T)$ is defined as a functional of the strategy $h(t)$ measurable with respect to observation and the problem is the one of stochstic control with partial observation.

Now let us introduce a new probability measure $\hat{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\frac{d\hat{P}}{dP} = \rho_T,$$

where

$$(2.12)\qquad \rho_t = e^{-\int_0^t (d+AX_s)^*(\Sigma\Sigma^*)^{-1}\Sigma dW_s} \times e^{-\frac{1}{2}\int_0^t (d+AX_s)^*(\Sigma\Sigma^*)^{-1}(d+AX_s)ds}.$$

We see that $\hat{P}$ is a probability measure since it can be seen by standard arguments (cf. [1]) that ρ_t is a martingale and $E[\rho_T] = 1$. Moreover, according to Girsanov theorem,

$$(2.13)\qquad \hat{W}_t = W_t + \int_0^t \Sigma^*(\Sigma\Sigma^*)^{-1}(d + AX_s)ds$$

turns out to be a standard Brownian motion process under the probability measure $\hat{P}$ and we have

$$(2.14)\qquad dY_t = \Sigma d\hat{W}_t$$

(2.15)

$$dX_t = \{b + BX_t - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(d + AX_t)\}dt + \Lambda d\hat{W}_t.$$

Set $\eta_t = \frac{1}{\rho_t}$, then we have

$$(2.16)\qquad \eta_t = e^{\int_0^t (d+AX_s)^*(\Sigma\Sigma^*)^{-1}dY_s} \times e^{-\frac{1}{2}\int_0^t (d+AX_s)^*(\Sigma\Sigma^*)^{-1}(d+AX_s)ds}.$$

Let us rewrite our criterion $I(x, h; T)$ by using new probability measure $\hat{P}$. We have

(2.17)

$$\begin{aligned} I(x,h;T) &= v^{-\theta/2}\hat{E}[\eta_T \exp\{\int_0^t \Phi(X_s, h_s, r(s); \theta)ds \\ &-\tfrac{\theta}{2}\int_0^t h_s^*\Sigma dW_s - \tfrac{1}{2}(\tfrac{\theta}{2})^2\int_0^t h_s^*\Sigma\Sigma^* h_s ds\}] \\ &= v^{-\theta/2}\hat{E}[\exp\{\int_0^T \Phi(X_s, h_s; r(s); \theta)ds \\ &+ \int_0^T Q(X_s, h_s)^* dY_s \\ &-\tfrac{1}{2}\int_0^T Q(X_s, h_s)^*(\Sigma\Sigma^*)Q(X_s, h_s)ds\}] \\ &= \hat{E}[\hat{E}[\exp\{\int_0^T \Phi(X_s, h_s; r(s); \theta)ds\}\Psi_T|\mathcal{G}_T]], \end{aligned}$$

where

$$\Psi_t = \exp\{\int_0^t Q(X_s, h_s)^* dY_s\} \times \exp\{-\tfrac{1}{2}\int_0^t Q(X_s, h_s)^*(\Sigma\Sigma^*)Q(X_s, h_s)ds\}$$

and

$$\begin{aligned} Q(x,h) &= (\Sigma\Sigma^*)^{-1}(Ax + d) - \tfrac{\theta}{2}h \\ &= (\Sigma\Sigma^*)^{-1}\{(Ax + d) - \tfrac{\theta}{2}(\Sigma\Sigma^*)h\}. \end{aligned}$$

Set

(2.18)

$$\begin{aligned} &q^h(t)(\varphi(t)) \\ &= \hat{E}[\exp\{\int_0^t \Phi(X_s, h_s; r(s); \theta)ds\}\Psi_t\varphi(t, X_t)|\mathcal{G}_t], \end{aligned}$$

then (2.17) reads

$$(2.19)\qquad I(x, h; T) = v^{-\theta/2}\hat{E}[q^h(T)(1)]$$

Hence, if $\theta > 0$ (resp. $-2 < \theta < 0$) our problem is reduced to minimize (resp. maximize) I of (2.19) when taking h over $\mathcal{H}(T)$.

3 Modified Zakai equation and explicit representation of its solution

Let us set

$$(3.1)\qquad L\varphi = \frac{1}{2}(\Lambda\Lambda^*)^{ij}D_{ij}\varphi + (b + Bx)^i D_i\varphi,$$

Then, the following proposition can be obtained by using Ito calculus in a standard way.

Proposition 3.1. *$q(t)(\varphi(t)) \equiv q^h(t)(\varphi(t))$ satisfies the following stochastic partial differential equation (SPDE):*

(3.2)

$$\begin{aligned} &q(t)(\varphi(t)) = q(0)(\varphi(0)) \\ &+ \int_0^t q(s)(\tfrac{\partial\varphi}{\partial t}(s,\cdot) + L\varphi(s,\cdot) - \tfrac{\theta}{2}h_s^*\Sigma\Lambda^* D\varphi(s,\cdot) \\ &+\Phi_s(\cdot)\varphi(s,\cdot))ds + \int_0^t q(s)(\varphi(s,\cdot)Q(\cdot, h_s))dY_s \\ &+ \int_0^t q(s)((D\varphi)^*\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1})dY_s, \end{aligned}$$

where $\Phi_s(\cdot) = \Phi(\cdot, h_s; r(s); \theta)$.

Remark. It is convenient to write (3.2) as

$$\begin{aligned}&\hat{E}[\xi_t q(t)(\varphi(t))] - q(0)(\varphi(0))\\&= \hat{E}[\int_0^t \xi_s q(s)(\frac{\partial\varphi}{\partial s} + L\varphi - \frac{\theta}{2}h^*\Sigma\Lambda^* D\varphi\\&+\Phi_s\varphi + \sqrt{-1}\beta_s^* Q(\cdot, h_s)\varphi\\&+\sqrt{-1}\beta_s^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^* D\varphi)ds]\end{aligned}$$

In fact, we can see that the solution of (3.2) is in the space

$$\begin{aligned}&\{q(\cdot); q(t) \in \mathcal{L}(B^*; L^1(\Omega, \mathcal{F}_t, \hat{P})^*),\ \forall t,\\&p(\cdot) \in \mathcal{L}(\hat{L}^\infty_{\mathcal{G}_t}(0,T;B_2)^*; \hat{L}^1_{\mathcal{G}_t}(0,T)^*)\},\end{aligned}$$

where B is the set of all bonded Borel measurable functions on R^n and $B_2 = \{\varphi; \frac{\varphi}{1+|x|^2} \in B\}$. The space B equipped wih weak topology, namely $\varphi_n \to \varphi$ in B^* if $\| \varphi_n \|$ is bounded $\varphi_n(x) \to \varphi(x),\ \forall x$. Weak topology for $\hat{L}^\infty_{\mathcal{G}_t}(0,T;B_2)$ is defined as $\| \varphi_n \|_{L^\infty(\Omega\times(0,T);B_2)}$ is bounded and for a.e. (ω,t), $\varphi_n(\omega,t,x) \to \varphi(\omega,t,x)\ \forall x$ and denoted by $\hat{L}^\infty_{\mathcal{G}_t}(0,T;B_2)^*$. Moreover, $\hat{L}^1_{\mathcal{G}_t}(0,T)$ means the set of all $\mathcal{G}_t$ adapted stochastic pocesses integrable with respect to the probabiliy measure $\hat{P}$, which is denoted by $\hat{L}^1_{\mathcal{G}_t}(0,T)^*$ when equipped with weak topology.

Now let us give the explicit representation to the solution of SPDE (3.2). For that let us introduce matrix Riccati equation

$$\begin{aligned}&\dot{\Pi} + (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)\\&\quad - \Lambda\Lambda^* - B\Pi - \Pi B^* = 0,\\&\Pi(0) = 0.\end{aligned} \tag{3.3}$$

and stochastic differential equation:
(3.4)

$$\begin{aligned}&d\gamma_t\\&\quad = \{B\gamma_t + b - (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\gamma_t + d)\}dt\\&\quad + (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}dY_t\\&\gamma_0 = x.\end{aligned}$$

The following theorem can be seen by using the methods developed in [1].

Theorem 3.1. *The solution of SPDE (3.2) with* $q(0)(\varphi(0)) = \varphi(0,x)$ *has the following representation:*

$$q(t)(\varphi(t)) = \alpha_t \int \varphi(t, \gamma_t + \Pi_t^{\frac{1}{2}} z) \frac{1}{(2\pi)^{n/2}} e^{-\frac{|z|^2}{2}} dz,$$

where

$$\begin{aligned}\alpha_t \quad = \quad &\exp\{\int_0^t Q(\gamma_s, h_s)^* dY_s\\&- \frac{1}{2}\int_0^t Q(\gamma_s, h_s)^*(\Sigma\Sigma^*)Q(\gamma_s, h_s)ds\\&+ \int_0^t \Phi(\gamma_s, h_s; r(s); \theta)ds\}.\end{aligned}$$

Remark It is known that (3.9) has a unique solution (cf. [4],[6]).

4 Optimal strategy

In the present section we shall construct optimal strategy minimizing (resp. maximizing) the criterion (2.19) for $\theta > 0$ (resp. $-2 < \theta < 0$). Because of Theorem 3.1 (2.19) reads

$$I(x,h;T) = v^{-\theta/2}\hat{E}[\alpha_T]. \tag{4.1}$$

Let us introduce the following $n \times n$ matrix Riccati differential equation.
(4.2)

$$\begin{aligned}&\dot{W} + W\{B - \frac{\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\}\\&\quad + \{B^* - \frac{\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)\}W\\&\quad + \frac{2\theta}{\theta+2}W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)W\\&\quad - \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0,\\&W(T) = 0.\end{aligned}$$

When we have a solution W of (4.2) we get a solution g of the following linear differential equation on R^n.
(4.3)

$$\begin{aligned}&\dot{g} + B^* g\\&+\frac{2\theta}{\theta+2}W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g\\&-\frac{\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g + Wb\\&-\frac{1}{\theta+2}\{A + \theta(A\Pi + \Sigma\Lambda^*)W\}^*(\Sigma\Sigma^*)^{-1}(a - r(t)\mathbf{1})\\&= 0,\\&g(T) = 0.\end{aligned}$$

Furthermore, for given solutions W of (4.2) and g of (4.3) we have a solution k of the following differential equation.
(4.4)

$$\begin{aligned}&\dot{k} - r(t)\\&\quad + \mathrm{tr}[W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)]\\&\quad + \theta g^*(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g\\&\quad + g^* b - \frac{1}{\theta+2}c_t^*(\Sigma\Sigma^*)^{-1}c_t = 0,\\&k(T) = 0,\end{aligned}$$

where

$$c_t = a - r(t)\mathbf{1} + \theta(A\Pi + \Sigma\Lambda^*)g.$$

Let us denote by $\mathcal{A}(T)$ the set of all investment strategy satisfying

$$\hat{E}[e^{\int_0^T \Xi_s^*(h)dY_s - \frac{1}{2}\int_0^T \Xi_s(h)\Sigma\Sigma^*\Xi_s(h)ds}] = 1$$

where

$$\begin{aligned}&\Xi_t^* = [(\gamma_t^* A^* + d^*) + \theta(\gamma_t^* W + g^*)(\Pi A^* + \Lambda\Sigma^*)\\&\quad - \frac{\theta}{2}h_t^*(\Sigma\Sigma^*)](\Sigma\Sigma^*)^{-1}.\end{aligned}$$

Theorem 4.1. *If (4.2) has a solution W, then there exists an optimal strategy $\hat{h} \in \mathcal{A}(T)$ maximizing the criterion (2.7) and it is explicitly represented as*
(4.5)

$$\begin{aligned}&\hat{h}_t = \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}[a - r(t)\mathbf{1} + \theta(A\Pi + \Sigma\Lambda^*)g\\&\quad + \{A + \theta(A\Pi + \Sigma\Lambda^*)W\}\gamma_t]\end{aligned}$$

where g is a solution of (4.3) and Π (resp. γ_t) is the one of (3.3) (resp. 3.4). Moreover

$$J(v,x;\hat{h};T) = \sup_{h\in\mathcal{A}(T)} J(v,x;h;T) = \log v - x^*W(0)x - 2g^*(0)x - k(0) \tag{4.6}$$

where k is a solution of (4.4).

Proof. Let us introduce $n\times n$ matrix valued function $W(t)$, $g(t)\in R^n$ and a scalar function $k(t)\in R^1$ and set

$$\chi_t = \frac{\theta}{2}\gamma_t^* W(t)\gamma_t + \theta g(t)^*\gamma_t + \frac{\theta}{2}k(t).$$

Then we have

(4.7)
$$\begin{aligned} d\chi_t &= (\tfrac{\theta}{2}\gamma_t^*\dot{W}\gamma_t + \theta\dot{g}^*\gamma_t + \tfrac{\theta}{2}\dot{k})dt \\ &+ \theta(\gamma_t^* W + g^*)d\gamma_t \\ &+ \tfrac{\theta}{2}\mathrm{tr}[W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)]dt. \end{aligned}$$

Therefore

(4.8)
$$\begin{aligned} de^{\chi_t} &= e^{\chi_t}[\theta(\gamma_t^*W + g^*)d\gamma_t \\ &+ \{\tfrac{\theta}{2}\gamma_t^*\dot{W}\gamma_t + \theta\dot{g}^*\gamma_t + \tfrac{\theta}{2}\dot{k} \\ &+ \tfrac{\theta}{2}\mathrm{tr}[W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)] \\ &+ \tfrac{\theta^2}{2}(\gamma_t^*W + g^*)(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1} \\ &(A\Pi + \Sigma\Lambda^*)(W\gamma_t + g)\}dt]. \end{aligned}$$

Since α_t satisfies

$$\begin{aligned} d\alpha_t &= \alpha_t\{[(\gamma_t^*A^* + d^*) - \tfrac{\theta}{2}h_t^*(\Sigma\Sigma^*)](\Sigma\Sigma^*)^{-1}dY_t \\ &+ \Phi(\gamma_t, h_t; r(t);\theta)dt\}. \end{aligned}$$

we obtain

(4.9)
$$\begin{aligned} d(\alpha_t e^{\chi_t}) &= \alpha_t e^{\chi_t}\times \\ &\{[(\gamma_t^*A^* + d^*) + \theta(\gamma_t^*W + g^*)(\Pi A^* + \Lambda\Sigma^*) \\ &- \tfrac{\theta}{2}h_t^*(\Sigma\Sigma^*)](\Sigma\Sigma^*)^{-1}dY_t \\ &+ \tfrac{\theta}{2}\gamma_t^*(\dot{W} + WB + B^*W \\ &+ \theta W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)W)\gamma_t dt \\ &+ \theta[\dot{g}^* + b^*W + g^*B \\ &+ \theta g^*(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)W]\gamma_t dt \\ &+ \tfrac{\theta}{2}\{\dot{k} - r(t) + g^*b \\ &+ \mathrm{tr}[W(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)] \\ &+ \theta g^*(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g\}dt \\ &+ \tfrac{\theta}{4}\{(\tfrac{\theta}{2}+1)h_t^*\Sigma\Sigma^*h_t \\ &- 2[(a + A\gamma_t - r(t)\mathbf{1})^* \\ &+ \theta(\gamma_t^*W + g)(\Pi A^* + \Lambda\Sigma^*)]h_t\}dt\} \\ &\equiv \alpha_t e^{\chi_t}\{I_1 + I_2 + I_3 + I_4 + I_5\}. \end{aligned}$$

We rewrite I_5 as

(4.10)
$$\begin{aligned} I_5 &= \{\tfrac{\theta}{4}(\tfrac{\theta}{2}+1)\Gamma_t^*(\Sigma\Sigma^*)\Gamma_t \\ &\quad - \tfrac{\theta}{2(\theta+2)}\gamma_t^*F^*(\Sigma\Sigma^*)^{-1}F\gamma_t \\ &\quad - \tfrac{\theta}{\theta+2}c_t^*(\Sigma\Sigma^*)^{-1}F\gamma_t - \tfrac{\theta}{2(\theta+2)}c_t^*(\Sigma\Sigma^*)^{-1}c_t\}dt \\ &\equiv II_1 + II_2 + II_3 + II_4, \end{aligned}$$

where

$$F = A + \theta(A\Pi + \Sigma\Lambda^*)W$$

and

$$\Gamma_t = \{h_t - \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}(c_t + F\gamma_t)\}.$$

When $W(t)$, $g(t)$ and $k(t)$ satisfy respectively (4.2),(4.3) and (4.4) we see that

$$I_2 + II_2 = I_3 + II_3 = I_4 + II_4 = 0$$

and therefore

$$d(\alpha_t e^{\chi_t}) = \alpha_t\chi_t\{\Xi_t dY_t + \frac{\theta}{4}(\frac{\theta}{2}+1)\Gamma_t^*(\Sigma\Sigma^*)\Gamma_t dt\}.$$

Namely

$$\alpha_T e^{\chi_T} = e^{\chi_0 + \int_0^T \Xi_t^* dY_t - \frac{1}{2}\int_0^T \Xi_t^*\Sigma\Sigma^*\Xi_t dt} \times e^{\frac{\theta}{4}(\frac{\theta}{2}+1)\int_0^T \Gamma_t^*(\Sigma\Sigma^*)\Gamma_t dt}. \tag{4.11}$$

Since $\chi_T = 0$ by (4.2),(4.3) and (4.4) we obtain for $\theta > 0$

$$\alpha_T \geq e^{\chi_0 + \int_0^T \Xi_t^* dY_t - \frac{1}{2}\int_0^T \Xi_t^*\Sigma\Sigma^*\Xi_t dt} \tag{4.12}$$

and moreover if we take $\Gamma_t = 0$, namely

$$h_t = \hat{h}_t \equiv \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}(c_t + F\gamma_t) \tag{4.13}$$

equality holds in (4.12). It is easy to see that

$$e^{\int_0^T \Xi_t^* dY_t - \frac{1}{2}\int_0^T \Xi_t^*\Sigma\Sigma^*\Xi_t dt}$$

is a martingale with respect to $\hat{P}$ by standard arguments. Therefore we see that

$$\hat{E}[\alpha_T] \geq e^{\chi_o} \equiv e^{\frac{\theta}{2}x^*W(0)x + \theta g^*(0)x + \frac{\theta}{2}k(0)}$$

and equality holds for $\hat{h}$. On the other hand, if $-2 < \theta < 0$, then we have converse inequality of (4.12). Hence similar arguments to the above lead us to our our theorem.

□

Remark It is known that (4.2) has a unique solution if $\theta > 0$ (cf. [4],[6]).

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