Averaging Methods for Force Controlled and Acceleration Controlled Lagrangian Systems

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Abstract

Recent research has shown that for the class of controlled Lagrangian systems having fewer control inputs than configuration variables, one may blur the distinction between directly controlled states and the corresponding input variables in analyzing the response to oscillatory forcing. Following this approach, stable responses are associated with local minima of an energy-like quantity which we have called the *averaged potential*. Construction of the averaged potential involves first constructing a reduced Lagrangian to which a Hamiltonian is associated by means of a *restricted Legendre transformation*. The Hamiltonian is time varying, but by simple averaging one obtains a canonical *averaged Hamiltonian* from which the averaged potential is immediately determined. It is also possible to an averaging analysis of the full (unreduced) system under high-frequency oscillatory forcing. Under suitable symmetry conditions, the averaged effect of an oscillatory input may also be studied in terms of a certain *averaged potential* which in general differs from the one obtained for the reduced system. In the present paper we discuss the differences between these two approaches and the resulting averaged potentials.

1 Introduction

Recently, interest has emerged in a new approach to nonlinear control design for Lagrangian systems which, following Bloch *et al.*, ([10]), might be called *controlled Lagrangian methods*. The idea is to use structured control inputs to modify the form of the Lagrangian in order to obtain a parameterized family of so-called *controlled Lagrangians*. Since the controlled system dynamics are governed by the Euler-Lagrange equations of the *controlled Lagrangian*, energy methods can be used to prescribe and analyze stable controlled motions. While [10] treats feedback designs, other work on oscillation mediated control of Lagrangian systems ([1]-[11]) can also be studied within this framework. The present paper is aimed at understanding the difference between two similar but essentially distinct approaches to averaging oscillation controlled Lagrangian systems—one of which is direct and one which uses an intermediate reduction step. The work draws heavily on a recent paper of Bullo ([11]) and our own recent work ([9]) on the role of symmetries in the averaging theory of controlled Lagrangian systems.

We treat controlled Lagrangian systems in which not every configuration variable is directly actuated. Such (*super-articulated*) systems are characterized by a Lagrangian $L(r,q,\dot r,\dot q)$ together with equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = u, \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0. \tag{2}$$

While the general study of such systems assumes that the configuration variables take values in a differentiable manifold, in the present paper we shall work with local coordinates so that we may assume $(r,q) \in \mathbb{R}^m \times \mathbb{R}^n \approx \mathbb{R}^{m+n}$.

The common approach in [1]-[11] involves averaging methods applied either directly to the entire system (1)-(2) or to the q–dynamics (2) alone with the configuration variable r playing the role of the high-frequency input. Either approach may be tested in the laboratory, and assuming a reasonable level of control authority, one expects the q–response to prescribed high-frequency oscillations in either the $u(\cdot)$ or the $r(\cdot)$ variables to be essentially the same. It is the the goal of the present paper to examine the differences between these two (*force-controlled* and *acceleration-controlled*) cases.

[1]Support from the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures (through Harvard University) is gratefully acknowledged.

2 Averaged Potential Theory

2.1 The full system, force-controlled case

Throughout the paper we shall assume that the Lagrangian appearing in (1)-(2) has the form

$$L(q;\dot r,\dot q)=\frac{1}{2}(\dot r^T,\dot q^T)\begin{pmatrix}\mathcal{N}(q) & \mathcal{A}(q)\\ \mathcal{A}(q)^T & \mathcal{M}(q)\end{pmatrix}\begin{pmatrix}\dot r\\ \dot q\end{pmatrix} -V(q). \quad (3)$$

The directly-controlled configuration variables r are thus assumed to be cyclic variables. This is a simplifying assumption which may be relaxed as indicated in [6] and [8]. Using the shorthand notation $y^T=(r^T,q^T)$, and $M=\begin{pmatrix}\mathcal{N} & \mathcal{A}\\ \mathcal{A}^T & \mathcal{M}\end{pmatrix}$, we rewrite (1)-(2) as

$$M(y)\ddot y+\hat\Gamma(y,\dot y)+\begin{pmatrix}0\\ \frac{\partial V}{\partial q}\end{pmatrix}=\begin{pmatrix}u\\ 0\end{pmatrix}, \quad (4)$$

where

$$\hat\Gamma(y,\dot y)=\begin{pmatrix}\hat\Gamma_1(y,\dot y), & \ldots, & \hat\Gamma_n(y,\dot y)\end{pmatrix}^T,$$

with

$$\hat\Gamma_k(y,\dot y)=\sum_{i,j}\Gamma_{ijk}\dot y_i\dot y_j,$$

and

$$\Gamma_{ijk}=\frac{1}{2}\left(\frac{\partial m_{ki}}{\partial y_j}+\frac{\partial m_{kj}}{\partial y_i}-\frac{\partial m_{ij}}{\partial y_k}\right).$$

Classical averaging methods may be used to characterize the dynamics of first-order differential equations of the form

$$\dot x=\epsilon f(x,t,\epsilon);\quad x(0)=x_0.$$

The theory describes the degree to which trajectories of this equation may be approximated by trajectories of the associated *autonomous averaged system*

$$\dot\xi=\epsilon\bar f(\xi),$$

where

$$\bar f(\xi)=\frac{1}{T}\int_0^T f(\xi,t,0)\,dt.$$

Roughly what the theory shows is that under mild regularity assumptions, trajectories of the x and ξ equations that start at the same initial value $x(0)=\xi(0)$ remain "close" over the time interval $0\le t<\mathcal{O}(1/\epsilon)$. We refer to Sanders and Verhulst, [14], for details.

To rewrite (4) in the desired form, we render it as a first-order system by letting $y_1=y$, $y_2=\dot y$. Then

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix}=$$

$$\begin{pmatrix}y_2\\ -M^{-1}(y_1)\hat\Gamma_k(y_1,y_2)-M^{-1}(y_1)\frac{\partial V}{\partial y_1}\end{pmatrix}+$$

$$\begin{pmatrix}0\\ M^{-1}(y_1)\begin{pmatrix}u\\ 0\end{pmatrix}\end{pmatrix}. \quad (5)$$

Recall that $y_1=\begin{pmatrix}r\\ q\end{pmatrix}$ and that $V(\cdot)$ depends only on the last n q-components of y_1. Hence $\frac{\partial V}{\partial y_1}=\begin{pmatrix}0\\ \frac{\partial V}{\partial q}\end{pmatrix}$. Let $\varphi(t,z)$ be the flow associated with the differential equation

$$\begin{pmatrix}\dot z_1\\ \dot z_2\end{pmatrix}=\begin{pmatrix}0\\ M^{-1}(z_1)\begin{pmatrix}u\\ 0\end{pmatrix}\end{pmatrix}.$$

In this case, noting that z_1 is constant, φ may be explicitly written

$$\varphi(t,z)=\begin{pmatrix}z_1\\ z_2+M^{-1}(z_1)\begin{pmatrix}v(t)\\ 0\end{pmatrix}\end{pmatrix},$$

where $v(t)=\int^t u(s)\,ds$, and $z=(z_1,z_2)^T$. To apply classical averaging techniques to the system (5), suppose $w(t)$ is any piecewise continuous periodic function. We consider inputs to (4) of the form $u(t)=(1/\epsilon)\,w(t/\epsilon)$, and we apply a time-varying change of coordinates $y(t)=varphi(t,x(t))$ to define $x(t)$. In terms of the "slow" time variable $\tau=t/\epsilon$, we may write $\tilde x(tau)=x(t)$ and derive the equation to which we apply averaging theory

$$\frac{dx}{d\tau}=\epsilon\,\frac{\partial\varphi}{\partial x}^{-1}_{|(\tau,x(\tau))}f(\varphi(\tau,x(\tau))), \quad (6)$$

where $f(y)$ is the vectorfield defined on $\mathbb{R}^{2(m+n)}$ by

$$f(y_1,y_2)=\begin{pmatrix}y_2\\ -M^{-1}(y_1)\left(\hat\Gamma(y_1,y_2)+\frac{\partial V}{\partial y_1}\right)\end{pmatrix}.$$

If we assume that the integrated input $\tilde v(\tau)=\int^\tau w(s)\,ds$, is a zero-mean periodic function of period $T>0$, then the *autonomous averaged system* associated with (6) may be explicitly written

$$\frac{d}{d\tau}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\epsilon\left\{\begin{pmatrix}x_2\\ -M^{-1}(x_1)\left(\hat\Gamma(x_1,x_2)+\frac{\partial V}{\partial q}\right)\end{pmatrix}\right.$$

$$\left.+\begin{pmatrix}0\\ \overline{-\left\{\frac{\partial}{\partial x_1}\left[M^{-1}(x_1)\begin{pmatrix}\tilde v\\ 0\end{pmatrix}\right]\right\}\cdot M^{-1}(x_1)\begin{pmatrix}\tilde v\\ 0\end{pmatrix}}\end{pmatrix}\right\}, \quad (7)$$

where the overbar indicates the result of simple averaging over one period. (Specifically, if $g(x,\tau)$ is periodic in τ with fundamental period $T>0$, we write

$\overline{g(x)} = \overline{g(x,\tau)} = (1/T)\int_0^T g(x,\tau)\,d\tau$.) The term

$$\overline{\left\{ \frac{\partial}{\partial x_1} \left[M^{-1}(x_1) \begin{pmatrix} v \\ 0 \end{pmatrix} \right] \right\} \cdot M^{-1}(x_1) \begin{pmatrix} v \\ 0 \end{pmatrix}} \quad (8)$$

accounts for the net averaged effect of the oscillatory input u on the dynamics (4).

Additional insight is obtained by writing the averaged system as a (Lagrangian) second-order differential equation. This may be done using the following Lemma.

Lemma 1 *The terms in (7) which are quadratic in $v(\tau)$ and which give rise to (8) may be rewritten*

$$\left\{ \tfrac{\partial}{\partial x_1} \left[M^{-1}(x_1) \begin{pmatrix} v \\ 0 \end{pmatrix} \right] \right\} \cdot M^{-1}(x_1) \begin{pmatrix} v \\ 0 \end{pmatrix} = -M^{-1}(x_1) \left\langle M^{-1}(x_1) \begin{pmatrix} v \\ 0 \end{pmatrix} \vdots M^{-1}(x_1) \begin{pmatrix} v \\ 0 \end{pmatrix} \right\rangle,$$

where we define the bilinear form $\left\langle \cdot \vdots \cdot \right\rangle : T_y^*Y \times T_y^*Y \to T_y^*Y$ *by*

$$\left\langle v \vdots w \right\rangle = \begin{pmatrix} \frac{1}{2}\sum_{i,j} \left(\frac{\partial m_{1,i}}{\partial q_j} + \frac{\partial m_{1,j}}{\partial q_i} \right) v_i w_j \\ \vdots \\ \frac{1}{2}\sum_{i,j} \left(\frac{\partial m_{n,i}}{\partial q_j} + \frac{\partial m_{n,j}}{\partial q_i} \right) v_i w_j \end{pmatrix}.$$

□

The result is that trajectories $y(t) = (r(t), q(t))$ defined by (4) with $u(t) = (1/\epsilon)w(t/\epsilon)$ can be approximated by trajectories $x(t) = (r_a(t), y_a(t))$ of the *averaged Lagrangian system*

$$M(x)\ddot{x} + \hat{\Gamma}(x,\dot{x}) + \begin{pmatrix} 0 \\ \frac{\partial V}{\partial q} \end{pmatrix} - \left(\overline{\left\langle M^{-1}(x) \begin{pmatrix} v \\ 0 \end{pmatrix} \vdots M^{-1}(x) \begin{pmatrix} v \\ 0 \end{pmatrix} \right\rangle} \right) = 0, \quad (9)$$

where the bracket term is defined in Lemma 1, $v(t) = \int^t u(s)\,ds$, and the overbar indicates that a simple average over one time period T has been computed as indicated above. The averaged term in this expression is related to the *symmetric product* which was studied by Crouch, [12], and later, in the context of small-time local controllability and configuration controllability for mechanical systems, by Lewis and Murray, [13]. For further details on the role of the symmetric product in the theory of averaged Lagrangian systems, we refer to Bullo, [11].

Remark 1 *With no further assumptions, classical averaging theory implies that trajectories of (9) approximate trajectories of the nonautonomous system (4) on the time interval $0 \le t < \mathcal{O}(1)$. (See [14].) This approximation may be extended to the semi-infinite time interval $0 \le t \le \infty$ through the introduction of dissipation into the models (as was done in [2] and [11]) or by a Floquet-type analysis (as carried out in [3]).*

Remark 2 *With $v(t) = \int^t u(s)\,ds$, the mapping $\rho : T_r^*R \to T_y^*Y$ defined by*

$$u \longmapsto \overline{\left\langle M^{-1}(x) \begin{pmatrix} v \\ 0 \end{pmatrix} \vdots M^{-1}(x) \begin{pmatrix} v \\ 0 \end{pmatrix} \right\rangle}$$

provides a succinct summary description of the way in which high-frequency inputs $u(\cdot)$ influence the dynamics of the q-variables in our system.

2.2 The reduced system, acceleration-controlled case

A widely studied alternative approach to oscillation mediated control of Lagrangian systems defined by (3)-(4) is to consider the *reduced Lagrangian*

$$\mathcal{L}(q,\dot{q};\dot{r}) = \frac{1}{2}\dot{q}^T\mathcal{M}(q)\dot{q} + \dot{r}^T\mathcal{A}(q)\dot{q} - \mathcal{V}_a(q), \quad (10)$$

where $\mathcal{V}_a(q) = V(q) - (1/2)\dot{r}^T\mathcal{N}(q)\dot{r}$ is a time-varying *augmented potential.* (See [6] for more detail regarding this approach.) Note that the q-dynamics prescribed by (2) may also be written in terms of the reduced Lagrangian:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0, \quad (2')$$

and in this setting, we view the triple $(r, \dot{r}, \ddot{r})$ as a generalized input. The goal is to study the dynamic response of (2′) to high-frequency oscillations in $r(\cdot)$. We note that as the frequency ω is increased, the amplitude of $\dot{r}$ scales as $\omega \cdot \mathcal{O}(r)$ and $\ddot{r}$ scales as $\omega^2 \cdot \mathcal{O}(r)$. Acceleration terms clearly have the dominant influence on the system's response, and thus we refer to (2′) as an *acceleration-controlled* Lagrangian system. We refer to [6] for a discussion of invariance properties and conditions under which there is a transformation of coordinates which eliminates terms involving $\ddot{r}$.

To find the analog of the variational equation (6), to which averaging theory applies, we write the equations of motion in Hamiltonian form—in terms of a *restricted Legendre transformation* and associated conjugate momentum

$$\begin{aligned} p &= \tfrac{\partial \mathcal{L}}{\partial \dot{q}} \\ &= \mathcal{M}(q)\dot{q} + \mathcal{A}^T(q)\dot{r}. \end{aligned}$$

This gives rise to the noncanonical Hamiltonian

$$\begin{aligned} \mathcal{H}(q,p,t) &= p \cdot \dot{q} - \mathcal{L} \\ &= \tfrac{1}{2}(p^T - \dot{r}^T\mathcal{A})\mathcal{M}^{-1}(p - \mathcal{A}^T\dot{r}) + \mathcal{V}_a. \end{aligned}$$

Taking simple time averages of the periodic terms yields the *averaged Hamiltonian*

$$\bar{\mathcal{H}}(q,p) = \frac{1}{2}p^T\mathcal{M}^{-1}p + \frac{1}{2}\overline{\dot{r}^T\mathcal{A}\mathcal{M}^{-1}\mathcal{A}^T\dot{r}} + \mathcal{V}_a \quad (11)$$

in terms of which the equations of motion of the averaged system are given in canonical form

$$\dot{q} = \frac{\partial\bar{\mathcal{H}}}{\partial p}, \quad \dot{p} = -\frac{\partial\bar{\mathcal{H}}}{\partial q}.$$

The last two terms on the right hand side of (11) together comprise the *averaged potential*

$$\mathcal{V}_A(q) = \frac{1}{2}\overline{\dot{r}^T\mathcal{A}\mathcal{M}^{-1}\mathcal{A}^T\dot{r}} + \mathcal{V}_a. \quad (12)$$

The behavior of $(2')$ under the influence of a high-frequency oscillatory pseudo-input $\dot{r}(\cdot)$ is largely determined by the critical point structure of $\mathcal{V}_A$. More specifically, under any of a variety of regularity assumptions, an *averaging principle* holds which states that any trajectory of $(2')$ whose initial point is sufficiently close to a strict local minimum of $(2')$ will remain confined to a neighborhood of that local minimum for all time. We refer to [3] or [6] for a detailed discussion of averaged potential theory.

Corresponding to the averaged Hamiltonian (11) there is an *averaged Lagrangian*

$$\bar{\mathcal{L}} = \frac{1}{2}\dot{q}^T\mathcal{M}\dot{q} - \mathcal{V}_A(q).$$

The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial\bar{\mathcal{L}}}{\partial\dot{q}} - \frac{\partial\bar{\mathcal{L}}}{\partial q} = 0 \quad (13)$$

provide a simple description of the averaged system dynamics.

The principal aim of this paper is to compare the qualitative features of the averaged "full" force-controlled system (9) with those of the averaged "reduced" acceleration-controlled system (13). Within the present framework, the net averaged effect on (13) of a high-frequency oscillatory input $r(\cdot)$ is felt as a conservative (potential) force which is a component of $\frac{\partial\mathcal{V}_A}{\partial q}$. The net averaged effect on (2) of a high-frequency oscillatory input $u(\cdot)$ applied to (1), however, is

$$\frac{\partial V}{\partial q} + \overline{\left\langle M^{-1}(q)\begin{pmatrix} v \\ 0 \end{pmatrix} \vdots M^{-1}(q)\begin{pmatrix} v \\ 0 \end{pmatrix}\right\rangle}, \quad (14)$$

which may be represented as the gradient of a potential function provided certain symmetry conditions are satisfied. We refer to Bullo ([11]) for a discussion of such symmetry conditions.

3 The Pendulum on a Cart

A detailed comparison of oscillation mediated *force control* and *acceleration control* of super-articulated systems can be carried out for the cart-pendulum system depicted in the figure. Let s denote the position of

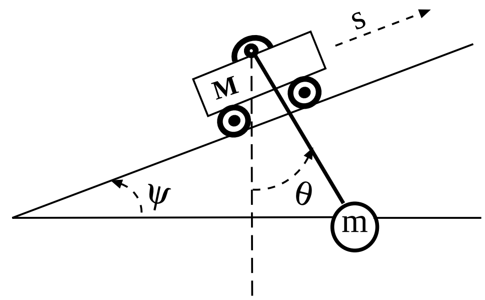

Figure 1: A simple frictionless pendulum is attached to a cart whose motion up an inclined plane is controlled.

the cart along the incline and let θ denote the angle of the pendulum with respect to the downward pointing vertical. The configuration space for this system is $Q = \mathbb{R}^1 \times S^1$, with the first factor being the cart position s and the second factor being the pendulum angle θ. We assume that s is directly controlled, while θ is only controlled indirectly through dynamic interactions within the mechanism. Clearly this can be cast in the form (1)-(2), and this is done by writing the *controlled Lagrangian* $L(s,\theta;\dot{s},\dot{\theta}) = (1/2)(\alpha\dot{\theta}^2 + 2\beta\cos(\theta-\psi)\dot{\theta}\dot{s} + \gamma\dot{s}^2) + D\cos\theta + su$. Here u denotes the control input, and

$$\alpha = m\ell^2, \quad \beta = m\ell, \quad \gamma = m + M, \quad D = mg\ell.$$

The (controlled) equations of motion are the corresponding Euler-Lagrange equations:

$$\begin{pmatrix} \gamma & \beta\cos(\theta-\psi) \\ \beta\cos(\theta-\psi) & \alpha \end{pmatrix}\begin{pmatrix} \ddot{s} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} -\beta\sin(\theta-\psi)\dot{\theta}^2 \\ D\sin\theta \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}. \quad (15)$$

Suppose the cart is subjected to a zero-mean periodic forcing $u(\cdot)$. Then the averaged equation (9) specializes to

$$M(\theta)\begin{pmatrix} \ddot{s} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} -\beta\sin(\theta-\psi)\dot{\theta}^2 \\ D\sin\theta \end{pmatrix} + M(\theta)G(\theta)\sigma^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (16)$$

where

$$M(\theta) = \begin{pmatrix} \gamma & \beta c(\theta) \\ \beta c(\theta) & \alpha \end{pmatrix},$$

$$G(\theta) = \begin{pmatrix} \frac{2\alpha\beta^3 s(\theta)c^2(\theta)}{[\alpha\gamma-\beta^2 c^2(\theta)]^3} \\ \\ \frac{-\beta^2 s(\theta)c(\theta)[\alpha\gamma+\beta^2 c^2(\theta)]}{[\alpha\gamma-\beta^2 c^2(\theta)]^3} \end{pmatrix},$$

$c(\theta) = \cos(\theta-\psi)$, and $s(\theta) = \sin(\theta-\psi)$. Here σ^2 is the rms-value of the periodic input:

$$\sigma^2 = \frac{1}{T}\int_0^T v(t)^2\, dt,$$

where $v(t) = \int^t u(\tau)\, d\tau$.

To facilitate comparison with the acceleration-controlled reduced Lagrangian, we multiply (16) through by $M^{-1}(\theta)$; this yields a second-order equation in θ which has no dependence on s:

$$\ddot\theta - \frac{\beta^2\sin(\theta-\psi)\cos(\theta-\psi)\dot\theta^2+\alpha D\sin\theta}{\alpha\gamma-\beta^2\cos^2(\theta-\psi)}$$

$$+\frac{\beta^2\sin(\theta-\psi)cos(\theta-\psi)[\alpha\gamma+\beta^2\cos^2(\theta-\psi)]}{\alpha\gamma-\beta^2\cos^2(\theta-\psi)}\sigma^2$$

$$= 0 \qquad (17)$$

Re-introducing the length and mass parameters, this equation may be rewritten

$$[M+m\sin^2(\theta-\psi)]\ddot\theta$$

$$+\, m\sin(\theta-\psi)\cos(\theta-\psi)\dot\theta^2 + mg\ell\sin\theta$$

$$-\frac{sin(\theta-\psi)\cos(\theta-\psi)[m+M+m\cos^2(\theta-\psi)]}{\ell^2[M+m\sin^2(\theta-\psi)]^2}\sigma^2$$

$$= 0,$$

which is an equation of the form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} \;=\; 0,$$

with

$$L(\theta,\dot\theta) = \frac{1}{2}\left[M+m\sin^2(\theta-\psi)\right]\dot\theta^2 - V_{A_1}(\theta),$$

where

$$\begin{aligned} V_{A_1}(\theta) \;&=\; -mg\ell\cos\theta \\ &+\; \tfrac{1}{\ell^2}\Big\{\tfrac{M+m}{m[M+m\sin^2(\theta-\psi)]} \\ &\quad + \tfrac{1}{2m}\log(2[M+m\sin^2(\theta-\psi)]\Big\}\,\sigma^2. \end{aligned}$$

The reduced Lagrangian (10) for the cart-pendulum model is

$$\mathcal{L}(\theta,\dot\theta;\dot s) = \frac{1}{2}\alpha\dot\theta^2 + \beta\cos(\theta-\psi)\dot s\dot\theta + D\cos\theta.$$

(17)

This gives rise to the acceleration-controlled θ-dynamics

$$\alpha\ddot\theta + \beta\cos(\theta-\psi)\ddot s + D\sin\theta = 0, \qquad (19)$$

where the acceleration variable $\ddot s$ is regarded as the control input. Looking at the general form of (12), we note that in (3) $\dot s$ plays the role of $\dot r$, $\mathcal{A}(\theta) = \beta\cos(\theta-\psi)$, $\mathcal{M}(\theta) = \alpha$, and $\mathcal{V}_a(\theta) = -D\cos\theta$. (There are no time-dependent terms in $\mathcal{V}_a$ in this case.) The averaged potential (12) thus specializes to

$$V_{A_2}(\theta) = \frac{1}{2}m\cos^2(\theta-\psi)\sigma^2 - mg\ell\cos\theta,$$

where now

$$\sigma^2 = \frac{1}{T}\int_0^T \dot s(t)^2\, dt.$$

Because the inputs in the force-controlled and acceleration-controlled systems ((15) and (19) resp.) are physically different, the rms parameter σ^2 does not have the same meaning in V_{A_1} and V_{A_2}. Nevertheless, in both cases, it plays a similar role as a bifurcation parameter. Suppose $\psi = \pi/2$, for instance. For σ^2 sufficiently large, $\theta = \pi$ is a local minimum of both averaged potentials, V_{A_1} and V_{A_2}. Accordingly, it can be shown that for all sufficiently high-frequency inputs ($u(\cdot)$ and $\ddot s(\cdot)$ resp.) the pendulum systems (15) and (19) will undergo stable motions in a neighborhood of the inverted equilibrium $\theta = \pi$.

For the cart-pendulum example, some of the qualitative differences between the force-controlled and acceleration controlled cases are displayed in Figure 2. What the figure suggests (which is born out by analysis) is that the σ-dependent bifurcations of V_{A_1} and V_{A_2} at $\theta = \pi$ are qualitatively different, with V_{A_1} exhibiting a subcritical pitchfork bifurcation and V_{A_2} exhibiting a supercritical pitchfork bifurcation.

A more detailed understanding of the relationship between the critical point structures of V_{A_1} and V_{A_2} and the respective nonautonomous systems may be obtained by considering inputs with the explicit form $u(t) = \alpha\sin\omega t$ ($s(t) = \alpha\sin\omega t$ resp.). To compare the two approaches in terms of input energy required to stabilize the inverted equilibrium ($\theta = \pi$), it makes sense to compare the L_2 norms of the actual input $u(\cdot)$ (not $s(\cdot)$) in both cases. Numerical studies indicate that for inputs of a given frequency, the integral cost

$$\int_0^T u(s)^2\, ds$$

over one period is between 25% and 50% more expensive in the acceleration-controlled case (to produce motions of θ confined to a given neighborhood of π). Nevertheless, the acceleration-control approach to design

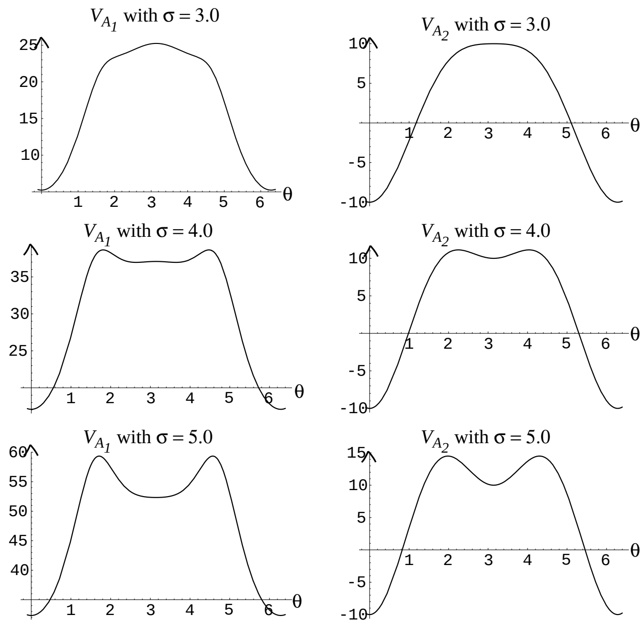

Figure 2: A comparison of parametric dependence (on σ^2) of the two *averaged potentials*, V_{A_1} (left column = force-controlled case) and V_{A_2} (right column = acceleration-controlled case). The figure illustrates the fact that in both cases for sufficiently large values of σ^2, the inverted pendulum equilibrium $\theta = \pi$ is a local minimum of the averaged potential. As described in the text, the qualitative features of the σ-dependent bifurcations differ, however.

has the advantage of automatically regulating the position $s(t)$ of the cart; indeed it is prescribed. Additional regulation of the cart position is required in the force-controlled case, and this additional input may limit or remove the energy advantage of the force-control design.

4 Conclusion

A number of interesting open questions remain regarding both the example and the general methods of the paper. It remains to produce a detailed analysis of the precise relationship among (i) the critical point structure of V_{A_1}, (ii) the dynamics of the averaged system, and (iii) the dynamics of the nonautonomous system (15) under periodic forcing. Such analysis for the acceleration-controlled system (19) and the corresponding averaged potential V_{A_2} has been carried out in [4]. One of the questions to be answered in such an analysis is how the design approaches change with respect to characteristic length-scales. In particular, it is known ([7]) that acceleration-control designs are effective for very small-scale devices. It remains unknown whether the theory will lead to practical designs (regulating both s and θ) at small length scales in the force-controlled case. Moreover, whether one approach becomes relatively more costly (in terms of the integrated u^2 cost) as characteristic length scales are reduced is also not known. A further question, regarding the general approach, is the extent to which the dynamic response of a force-controlled system (with oscillatory forcing) can be characterized in terms of an *averaged potential*. The question has been addressed in [11], but the conditions seem more severe than in the acceleration-controlled case.

Acknowledgment: It is a pleasure to acknowledge a number of stimulating discussions with Dr. K. Nonaka. I am also grateful for his critical reading of the manuscript.

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