Relative Equilibria and Stability of Rings of Satellites

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Abstract

The problem of control of rings of satellites is of current interest driven by applications in telecommunications and space science. The problem of stability of a ring is the subject of this paper. We use methods from geometric approaches to hamiltonian systems to treat this problem.

1 Introduction

The problem of control of rings of satellites is of current interest driven by applications in telecommunicatons and space science [7]. The problem of stability of a ring is the subject of this paper. In the context of the N-body problem of classical mechanics certain configurations known as central configurations have received serious attention in recent years [19][1][17]. While much remains to be understood regarding these special relative equilibria, the literature does shed light on certain limiting cases, noteable among these being the case of the (N+1)-body problem with one dominant mass [17][16]. This is the problem at the heart of Maxwell's study of Saturn's rings [13][6][4].

In part II of his essay [13] that won the Adams Prize for the year 1856, James Clerk Maxwell showed that if the mass of Saturn is sufficiently large, then a ring of discrete mutually gravitating particles in circular orbit around the planet would maintain a stable morphology. This was set against hypotheses of solid and fluid rings. For Maxwell, Saturn was a perfect homogeneous sphere and the problem of stability arises due to mutual attraction of orbiting particles. See also the later discussions by Cook and Franklin [6].

In our present (and future) era of large constellations of communication satellites supporting global coverage, there are interesting questions of maintaining constellations in stable configurations in the presence of disturbing influences (such as the terms in earth's gravitational potential due to oblateness). It has been suggested that setting up artificial potential fields governing the interactions of satellites is a way to solve these problems [14][7][15]. This approach has its roots in work in robotics on obstacle avoidance via artificial potential fields [8][10][18]. In the present paper, we discuss a model for a ring of earth satellites interacting via artificial potentials. We use the method of amended potentials (see [19][1][2][9][12][11]) to treat the stability question.

2 Kepler Problem

Ignoring the effects of oblateness of the earth (distorting the gravitational petential), the motion of an artificial earth satellite, treated as a point mass moving about a fixed center, is governed by the Lagrangian on $T(\mathbb{R}^3)$

$$L = \frac{1}{2}\mu \parallel \dot{x} \parallel^2 - V(\parallel x \parallel) \tag{1}$$

where $V(\parallel x \parallel) = -\frac{k}{\|x\|}$ is the Kepler-Newton potential, admitting the group $SO(3)$ of configuration space symmetries. Thus angular momentum $l = x \times p = x \times \mu\dot{x}$ is conserved. Additionally, for subtler reasons, the Laplace vector

$$A = p \times l - \mu k \frac{x}{\parallel x \parallel} \tag{2}$$

is also conserved, yielding the focal equation of the satellite orbit

$$\frac{1}{r} = \frac{\mu k}{\parallel l \parallel^2}(1 + e\cos(\theta)) \tag{3}$$

where $e = \frac{\|A\|}{\mu k} \geqslant 0$ is the eccentricity. Bounded (elliptical) orbits correspond to energy $E = \frac{1}{2}\mu \parallel \dot{x} \parallel^2 + V(\parallel x \parallel) < 0$. the components of the angular momentum vector, together with the components of the Runge-Lenz vector $D = \frac{A}{\sqrt{-2\mu E}}$, taken as functions on the submanifold of the phase space $T^*(\mathbb{R}^3)$ corresponding to $E < 0$, generate a full $SO(4)$ summetry as is well-known.

The conservation of l implies that one can write the dynamics in polar coordinates (r, θ) on $l^\perp$ with Lagrangian $L = \frac{r^2\dot{\theta}^2 + \dot{r}^2}{2}\mu - V(r)$, which is *independant of* θ. Thus the Kepler problem reduces to a one degree of freedom problem [19][1][2]:

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot r} &= \mu\ddot r = \frac{\partial L}{\partial r}\\ &= -\frac{\partial}{\partial r}\left(V+\frac{\|l\|^2}{2\mu r^2}\right)\end{aligned} \tag{4}$$

where $V_l = V + \frac{1}{2}\frac{\|l\|^2}{\mu r^2}$ is the amended potential. The evolution of the polar angle (Kepler's area theorem) is given by

$$\dot\theta = \| \, l \, \| \, / \mu r^2 \tag{5}$$

Relative equilibria (see[1]) for the Kepler problem are given by critical points of V_l and $\dot r = 0$, i.e. circular orbits, and hence equilibria for (4).

If the satellite is subject to additional forcing f^{ext} (e.g. disturbance, drag, control jets), satisfying

$$f^{ext}\cdot l = 0$$

for a fixed vector l, then $T(l^{\perp})$ is an invariant manifold for the *forced* system leading to the 2 d.o.f. problem:

$$\begin{aligned}\frac{d}{dt}\quad \frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} &= \quad \mu\ddot r + \frac{\partial V}{\partial r} - \mu\dot\theta^2 r = \quad f_r\\ \frac{d}{dt}\quad \frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} &= \quad \mu\ddot\theta r^2 + 2\dot\theta\dot r r\mu = \quad f_\theta\end{aligned} \tag{6}$$

Let $r(t) \equiv R$ and $\dot\theta(t) \equiv \omega = \| \, l \, \| \, / \mu R^2$ define a relative equilibrium for the unforced system. Setting $V = -\frac{k}{r}$ in (6) and linearizing about the relative equilibrium one obtains the dynamics [5]:

$$\begin{aligned}\ddot x - 2\omega\dot y - 3\omega^2 x &= a_r\\ \ddot y + 2\omega\dot x &= a_\theta\end{aligned} \tag{7}$$

where $a_r = f_r / \mu$ and $a_\theta = f_\theta / \mu R$.

The characteristic polynomial for (7) is $s^2\left(s^2+\omega^2\right)$ so one concludes relative stability module S^1 (see [1], [9]) for the relative equlibrium. But the (Rayleigh) dissipation $a_r = -\gamma\dot x \quad ; \quad a_\theta = -\delta\dot y \quad ,$ with $\gamma,\delta > 0$ destroys this since the corresponding characteristic polynomial is

$$s(s^3 + (\gamma+\delta)s^2 + (\omega^2+\gamma\delta)s - 3\omega^2\delta)$$

See [3].

We note further that the forced hamiltonian form of the equations (6) is

$$\begin{aligned}\dot r &= \frac{\partial H}{\partial p_r}\\ \dot\theta &= \frac{\partial H}{\partial p_\theta}\\ \dot p_r &= -\frac{\partial H}{\partial r} + f_r\\ \dot p_\theta &= -\frac{\partial H}{\partial \theta} + f_\theta\end{aligned} \tag{8}$$

and $H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} - \frac{k}{r}$ is the Kepler hamiltonian.

3 Rings

If we have N identical satellites, nominally in a ring of radius R with constant rotation rate ω, and if we let the control jets act according to the feedback laws:

$$\begin{aligned}f_{r_i} &= -\frac{\partial V^c}{\partial r_i}\\ f_\theta &= -\frac{\partial V^c}{\partial \theta_i} \qquad\qquad i = 1,2,\ldots,N\end{aligned} \tag{9}$$

where $V^c = V^c(r_1, r_2 \ldots, r_N \quad ; \quad \theta_1, \theta_2, \ldots, \theta_N)$ is a *synthetic control potential* statisfying certain conditions (to be determined below), then the closed-loop system is hamiltonian with

$$H^{total} = \sum_{i=1}^{N} H_i + V^c \tag{10}$$

where,

$$H_i = \frac{p_{r_i}^2}{2\mu} + \frac{p_{\theta_i}^2}{2\mu r_i^2} - \frac{k}{r_i} \tag{11}$$

In part II of his essay [13], Maxwell considered N identical mutally gravitating particles in a ring around Saturn. In our language, the potential V^c for Maxwell was the gravitational potential energy due to mutual attraction of the ring particles. (To be precise, one is dealing with the planar Maxwell problem.) It has a critical point for all N particles located at the vertices of a (steadily rotating) regular N-gon. In other words, the regular N-gon configuration, a planar central configuration [17] for the $(N+1)$-body problem, is a relative equilibrium for the $(N+1)$-body problem.

Maxwell showed that if the number of particles N is large enough and if

$$\text{Mass(Ring)} < \frac{\text{Mass(Saturn)}}{N^2} \quad 2.298$$

then the ring is *linearly stable*.

See also the analysis of Cook and Franklin [6] and refined investigations discussed in [16] [17], leading up to the requirement $N \geqslant 7$.

Returning to our setting, we note the following natural conditions for *small* artificial earth satellites (in a ring of nominal radius R).

- (i) ignore all mutual attraction between the satellites
- (ii)

  $$\frac{\partial V_c}{\partial r_i} \Big|_{\substack{r_i=R \\ i=1,2,\ldots,N}} = 0 \qquad \text{and Hessian} \quad \left[\left(\frac{\partial^2 Vc}{\partial r_i \partial r_j}\right)\right] > 0$$

- (iii)

  $$V_c(r_1, r_2, \ldots, r_N \quad ; \quad \theta_1, \theta_2, \ldots, \theta_N)$$
  $$= V_c(r_1, r_2, \ldots, r_N \quad ; \quad \theta_1+\beta, \theta_2+\beta, \ldots, \theta_N+\beta)$$
  $$\text{for all } \beta \quad (S^1 \text{ symmetry}).$$

- (iv) V_c is a global minimum at the regular N-gon configuration

  $$r_i = R, \qquad i = 1, 2, \ldots, N$$

  and

  $$\theta_{i+1} - \theta_i = \frac{2\pi}{N} \qquad i = 1, 2, \ldots, N-1$$

It is clear that the closed loop system with hamiltonian (10) has an S^1 symmetry and admits Casimir functions (see below) of the form $\Phi(| p_{\theta_1} + p_{\theta_2} + \ldots + p_{\theta_N} |^2)$.

4 Stability

Definition
$z_e \in S$ is a *relative equilibrium* for the dynamics X_h corresponding to a G-invariant hamiltonian h on a symplectic manifold (S, Ω). We say that z_e is *relatively stable modulo* G if $\pi(z_e)$ is a Lyapunov stable equilibrium for the Poisson-reduced dynamics $\hat{X}_h$ on S/G.

Relative Stability Theorem (Arnold)
$\pi(z_e)$ is an equilibrium point of $\hat{X}_h$ *if* it is a critical point of $\hat{h}_{|L}$ the restriction of $\hat{h}$ to the symplectic leaf through $\pi(z_e)$.

In that case $\pi(z_e)$ is Lyapunov stable if

- (i) Hessian $D^2\left(\hat{h} \mid L\right)(\pi(z_e))$ is definite
- (ii) the point $\pi(z_e)$ has a neighborhood W on which the rank of the Poisson structure $\{.\ ,\ .\}_{S/G}$ is constant (generic point).

Remark
At generic points, nontrivial (local) Casimir functions C_Φ (i.e. smooth functions of S/G that Poisson commute with all functions) exist.

One can verify condition (i) of Arnold by seeking a (local) Casimir function C_Φ such that $\pi(z_e)$ is an unconstrained critical point of $\hat{h} + C_\Phi$ *and* $D^2\left(\hat{h} + C_\Phi\right)$ at $\pi(z_e)$ is definite. This is the essence of the energy-Casimir method [12].

The function V^c, associated H^c, and Casimir function $\Phi\left(| p_{\theta_1} + p_{\theta_2} + \ldots + p_{\theta_N} |^2\right)$ of section (3) fulfill the requirements of Arnold's theorem. The discussion in [14] suggests examples of V^c.

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