Model Validation for Nonlinear Feedback Systems *

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Abstract

Model validation provides a useful means of assessing the ability of a model to account for a specific experimental observation, and has application to modeling, identification and fault detection. In a robust control framework norm-bounded perturbations are included to account for dynamic uncertainties in the system. We consider a discrete-time or sampled-data framework with a general linear fractional transformation (LFT) model structure which allows for the consideration of nonlinear feedback structures. Block structured, causal , time-varying perturbations are considered and we give a sufficient condition—necessary and sufficient in the single perturbation block case—for the model to be invalidated by the datum. The condition is testable by a convex LMI feasibility problem in which the matrix basis grows linearly in size with respect to the data length and the number of decision variables is equal to the number of perturbation blocks.

1. Introduction

Models for robust control design contain bounded uncertainties (perturbations and unknown noise/signals) with explicitly specified bounds. For work in the $\mathcal{H}_\infty$ domain considered here, see for example [1, 2] and the references therein. Model validation is the formal method for assessing such models with respect to an experimental datum. Robust control models are specified as sets, where the sizes of the perturbation and noise/disturbance bounds specify the boundary of the sets. In this context, model validation for robust control models can be stated as follows: Given a robust control model, is there a perturbation and noise/disturbance signal from the assumed sets which makes the model consistent with the experimental observation. No assumptions are made about the nature of the physical system. Rather, measurements are taken, and the assumption that the model describes the system is directly tested.

The robust control model validation problem was first considered in the frequency domain by Smith and Doyle [3]. Poolla et al. [4] considered the model validation problem for discrete-time models with time domain experimental data. Their formulation applied to a more restricted, although still common, class of perturbations. Zhou and Kimura [5] have considered a similar problem and addressed the issue of identifying certain system parameters in this framework.

More recent work has focused on the sampled-data framework and the emphasis has again been structures in which the norm-bounded perturbations affect the residual data/model mismatch linearly [6, 7].

In this paper we provide results for the general LFT framework with block-structured perturbations. In this case the perturbations affect the data/model residual in a fractional form, leading to non-convex problems in the general case. The fractional structure is the most general and will occur when we examine closed-loop systems, even when the open-loop system has a linear (additive or multiplicative) perturbation structure. We focus on linear time-varying systems which can be used to capture the effects of nonlinear perturbations. The discrete-time or sampled-data structure uses a representation in terms of the matrix of impulse response coefficients. For linear systems this is a Toeplitz matrix, but using the matrix representation allows for the consideration of time-varying impulse response coefficients and hence, nonlinear systems.

2. The Model Validation Problem

$\mathcal{H}_\infty$ Robust control models include norm bounded perturbations, Δ, to account for unmodeled dynamics. The linear fractional transformation (LFT) structure considered here is illustrated in Figure 1, and is defined by the equations,

$$\begin{aligned} z &= P_{11}v + P_{12}w + P_{13}u \\ y &= P_{21}v + P_{22}w + P_{23}u \\ v &= \Delta z. \end{aligned} \tag{1}$$

The model includes an assumed bound on the perturbation, $\|\Delta\|_\infty \leq \gamma$, and on the unknown exogenous inputs, $\|w\|_2 \leq \gamma$. The inclusion of the perturbations, Δ, in a feedback form allows this structure to be equally applicable to open- and closed-loop systems.

Model validation is the data based assessment of this model. Given measurements of the input, u, and output, y, we wish to determine whether or not there is a Δ and w, with $\|\Delta\|_\infty \leq \gamma$ and $\|w\|_2 \leq \gamma$, such that the model equations in (1) are satisfied. If no such Δ and w exist then the particular datum invalidates the model.

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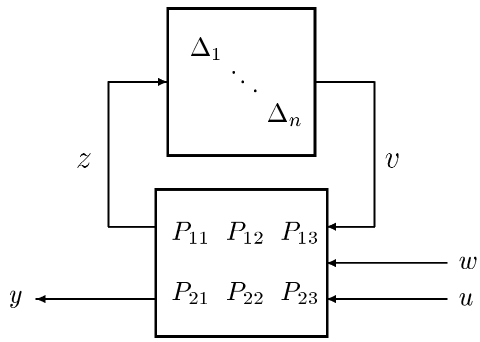

Figure 1: Generic LFT model structure for model validation

The model validation problem can be posed more formally in several frameworks. For simplicity in exposition we will assume that the measured data and model are in a discrete-time framework. Refer to [4] for more details of the discrete-time model validation problem. The results discussed here extend easily to the sampled-data case. This problem has been considered in [6] and [7] for a somewhat simpler structure than the general LFT model given here. A more complete overview of model validation in the frequency, discrete, and sampled-data domains is given in [8]. In the following we assume that the measurement consists of N discrete values $(y(k), u(k))$, $k = 0, \ldots, N-1$.

Additional structure can be imposed on the perturbation, Δ, in a number of ways. We can restrict Δ to belong to a specific class of operator. Common choices are:

LTV. Linear time-varying operators, $\Delta : l_2^{nz} \rightarrow l_2^{nv}$, $\|\Delta\|_\infty \leq \gamma$.

LTI. Linear time-invariant operators, $\Delta : l_2^{nz} \rightarrow l_2^{nv}$, $\|\Delta\|_\infty \leq \gamma$.

IQC. Integral quadratic constraints. The perturbation is characterized by an IQC on v and z.

The above notation assumes discrete signals v and z, which we will formalize in Section 3. In this paper we will focus on the LTV perturbation case.

We may also impose a block diagonal structure—with n blocks—on Δ;

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n),$$
$$\Delta_i : \mathcal{R}^{nzi} \rightarrow \mathcal{R}^{nvi}.$$

The individual block dimensions satisfy

$$\sum_{i=1}^{n} nzi = nz \quad \text{and} \quad \sum_{i=1}^{n} nvi = nv.$$

Corresponding analysis results are well established for each of the above perturbation assumptions. See for example [9] and the references therein.

3. Discrete-time Extension Conditions

The development of a computable model validation problem requires the characterization of the constraints on Δ in terms of the input and output signals z and v. Such characterizations are known as extension conditions.

Before stating the relevant extension conditions several preliminary definitions are required. Denote the truncation operator, Π, as follows. Given $x \in l_2$,

$$\Pi_l x = (x(0), x(1), \ldots, x(l), 0, \ldots).$$

We define a Toeplitz matrix from a discrete signal x, of length N, in the following manner.

$$\mathcal{T}(v) = \begin{bmatrix} v(0) & 0 & \ldots & & 0 \\ v(1) & v(0) & & & \\ \vdots & \ddots & \ddots & & \vdots \\ v(N-2) & & \ddots & v(0) & 0 \\ v(N-1) & v(N-2) & \ldots & v(1) & v(0) \end{bmatrix}.$$

We state the extension conditions for a single block Δ. The formulation for multiple blocks is obvious. The LTV Δ case is given by the following. See [4] for a proof.

LTV Extension Condition

If there exists a causal linear time-varying $\Delta : l_2 \rightarrow l_2$ with $\|\Delta\|_\infty \leq \gamma$ such that

$$\Pi_{N-1} v = \Pi_{N-1} \Delta \Pi_{N-1} z = \Pi_{N-1} \Delta z,$$

then $\|\Pi_T v\|_2^2 \leq \gamma^2 \|\Pi_T z\|_2^2$ for all $T \in [0, N-1]$.

The corresponding LTI Δ case is given below. Again a proof can be found in [4].

LTI Extension Condition

If there exists a causal linear time invariant $\Delta : l_2 \rightarrow l_2$ with $\|\Delta\|_\infty \leq \gamma$ such that

$$\Pi_{N-1} v = \Pi_{N-1} \Delta \Pi_{N-1} z = \Pi_{N-1} \Delta z,$$

then $\mathcal{T}(v)^T \mathcal{T}(v) \leq \gamma^2 \mathcal{T}(z)^T \mathcal{T}(z)$.

In the discrete time case the components of P are described by their impulse response coefficients. For example, in the case where P_{11} is LTI,

$$P_{11} = \begin{bmatrix} P_{11}(0) & 0 & \ldots & & 0 \\ P_{11}(1) & P_{11}(0) & & & \\ \vdots & \ddots & \ddots & & \vdots \\ P_{11}(N-2) & & \ddots & P_{11}(0) & 0 \\ P_{11}(N-1) & P_{11}(N-2) & \ldots & P_{11}(1) & P_{11}(0) \end{bmatrix}.$$

When expressed in this way, the model equations (1), hold as matrix equations.

The results presented in the sequel do not make use of the Toeplitz structure that arises in the case of an LTI model, P. They therefore hold for more general time-varying systems modeled in terms of the time-varying impulse response coefficients. In the LTI model case it is possible to exploit the Toeplitz structure in the numerical solution of the problem [10].

In the sampled-data model validation framework the extension conditions are the same as those given above. The formulation of the nominal system, P_{11}, etc., involves the use of a lifting theory. However, the problem is transformed into an equivalent discrete-time problem and the theory given here is directly applicable. For simplicity we state it only for the discrete-time case.

4. Model Validation for LTV Perturbations

We now develop a sufficient condition for an observed experimental datum, $(y(k),\, u(k))$, $k = 0, \ldots, N-1$, to invalidate a model. In the case where there is a single perturbation block ($n = 1$), this condition is both necessary and sufficient for invalidation.

The development follows several steps. We consider the model in terms of the unknown signals, $x = [v^T\, w^T]^T$, and then express the constraints on x that arise from the model, including the assumptions on the size of Δ and w. We then develop a computable condition that, if it holds, proves that the set of feasible x is empty.

The invalidation condition derived in this section must be applied to all truncations of the data vector: $(\Pi_T y,\, \Pi_T u)$ for all $T = 0, \ldots, N-1$. For clarity we drop the time truncation notation and consider the data vectors as $(y,\, u)$.

4.1. Preparametrization of the equality constraint

The nominal, noise-free, model is simply,

$$y_{nom} = P_{23} u.$$

In any reasonable experiment $y_{nom} \neq y$, and the residual will have to be accounted for by the unknown signal,

$$y - P_{23} u = [P_{21}\, P_{22}] \begin{bmatrix} v \\ w \end{bmatrix}. \tag{2}$$

This equality constraint is easily removed from the problem by parametrizing all $(v,\, w)$ satisfying (2) in the form,

$$\begin{aligned} \begin{bmatrix} v \\ w \end{bmatrix} &= \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} + R\zeta, \\ &= x_0 + R\zeta, \\ &= x(\zeta). \end{aligned}$$

Here, $(v_0,\, w_0)$ is any particular solution of (2) and $R\zeta$ spans the input null space of $[P_{21}\, P_{22}]$. A singular value decomposition is required to calculate R. Note that $x(\zeta)$ is affine in ζ.

4.2. Reparametrization of the exogenous signal constraint

We now consider the norm condition imposed by the model on the exogenous signal,

$$\|w\|_2 \leq \gamma. \tag{3}$$

All w meeting the equality constraint, (2), are given by,

$$\begin{aligned} w &= [0_{nv}\, I_{nw}] x(\zeta) \\ &= [0_{nv}\, I_{nw}](x_0 + R\zeta). \end{aligned}$$

It is useful to express the negation of the exogenous signal constraint: $\|w\|_2 > \gamma$, or equivalently $w^T w > \gamma^2$. All signals satisfying the equality constraint, (2), but failing the exogenous signal norm constraint, (3), are generated by ζ satisfying,

$$x(\zeta)^T \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} x(\zeta) - \gamma^2 > 0.$$

This condition is a quadratic form on ζ,

$$F_0(\gamma, \zeta) = \zeta^T A_0 \zeta + 2 b_0^T \zeta + c_0(\gamma) > 0,$$

where,

$$\begin{aligned} A_0 &= R^T \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} R, \\ b_0 &= R^T \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} x_0 \end{aligned}$$

and

$$c_0(\gamma) = x_0^T \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} x_0 - \gamma^2.$$

For technical reasons, we will actually develop a condition using $F_0(\gamma, \zeta) \geq 0$, which can be viewed as the condition that $\|w\|_2 \leq \gamma - \epsilon$ for arbitrarily small ϵ. The desired result is obtained in the limit and we drop the explicit inclusion of ϵ in the sequel.

4.3. Reparametrization of the perturbation block constraints

We now consider the constraints that arise from $\|\Delta\|_\infty \leq \gamma$. The above extension condition is used to formulate this in terms of ζ. Consider the ith block, Δ_i, with input and output signals z_i and v_i respectively. Define the projection from x to v_i by N_i,

$$\begin{aligned} v_i &= [0, \ldots, 0, I_{nvi}, 0, \ldots, 0, 0_{nw}]\, x \\ &= N_i\, x. \end{aligned}$$

Denote M_i as the projection from z to z_i,

$$\begin{aligned} z_i &= [0, \ldots, 0, I_{nzi}, 0, \ldots, 0]\, z \\ &= M_i\, z. \end{aligned}$$

All signals v_i satisfying the equality constraint, (2), are given by,

$$v_i = N_i(x_0 + R\zeta).$$

Similarly, the signals z_i satisfying (2) can be expressed as,

$$z_i = M_i[P_{11}\, P_{12}](x_0 + R\zeta) + M_i P_{13} u.$$

The LTV extension condition given in Section 3 is applied to each perturbation, Δ_i, and is equivalent to,

$$\gamma^2 \|z_i\|_2^2 - \|v_i\|_2^2 \geq 0. \tag{4}$$

By substituting v_i and z_i given above this can be reformulated as the following quadratic form on ζ.

The ζ that generate signals $(v,\, w)$ satisfying both the equality constraint (2), and the LTV extension condition, (4), are characterized by,

$$F_i(\gamma, \zeta) = \zeta^T A_i(\gamma) \zeta + 2 b_i^T(\gamma) \zeta + c_i(\gamma) \geq 0,$$

where,

$$A_i(\gamma) = \gamma^2 R^T \begin{bmatrix} P_{11}^T \\ P_{12}^T \end{bmatrix} M_i^T M_i [P_{11}\ P_{12}] R - R^T N_i^T N_i R,$$

$$b_i(\gamma) = \gamma^2 R^T \begin{bmatrix} P_{11}^T \\ P_{12}^T \end{bmatrix} M_i^T M_i P_{13} u - R^T N_i^T N_i x_0, + \gamma^2 R^T \begin{bmatrix} P_{11}^T \\ P_{12}^T \end{bmatrix} M_i^T M_i [P_{11}\ P_{12}] x_0$$

and

$$c_i(\gamma) = \gamma^2 u^T P_{13}^T M_i^T M_i P_{13} u - x_0^T N_i^T N_i x_0 + \gamma^2 x_0^T \begin{bmatrix} P_{11}^T \\ P_{12}^T \end{bmatrix} M_i^T M_i [P_{11}\ P_{12}] x_0$$

4.4. Model validation conditions

The following theorem follows simply from the above quadratic form definitions.

Theorem Given $\gamma > 0$. The LFT perturbation model, with $\|\Delta\|_\infty \leq \gamma$ and $\|w\|_2 \leq \gamma$, is invalidated by the measured datum, (y, u),
if and only if
there exists an $l \in [0, N-1]$ such that such that for all ζ satisfying $F_i(\gamma, \zeta) \geq 0$, $i = 1, \ldots, n$;

$$F_0(\gamma, \zeta) \geq 0.$$

Proof: Given a fixed $l \in [0, N-1]$ and from the truncated measurements $(\Pi_l y, \Pi_l u)$ consider the quadratic forms F_i, $i = 0, \ldots, n$ given in the previous section. For any $\hat{\zeta}$ such that $F_i(\gamma, \hat{\zeta}) \geq 0$, for all $i = 1, \ldots, n$, the signals

$$\begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix} = \hat{x} = x_0 + R\hat{\zeta},$$

satisfy both the residual equation (2) and the norm constraint $\|\Delta_i\|_\infty \leq \gamma$ for all n perturbation blocks. Furthermore all such signals satisfying these constraints can be generated in this way from a $\hat{\zeta}$ such that $F_i(\gamma, \hat{\zeta}) \geq 0$, for all $i = 1, \ldots, n$. In contrast, the condition that $F_0(\gamma, \hat{\zeta}) \geq 0$ is equivalent to $\hat{\zeta}$ generating signals that satisfy the equality constraints and fail the $\|\hat{w}\|_2 \leq \gamma$ condition. If this is true for all ζ then there is no feasible solution satisfying all constraints and the model is invalidated. If, on the other hand, there exists a $\tilde{\zeta}$ such that $F_i(\gamma, \tilde{\zeta}) \geq 0$, for all $i = 1, \ldots, n$, and $F_0(\gamma, \tilde{\zeta}) < 0$, then $\|\tilde{w}\|_2 < \gamma$ and the extension conditions implies that there exists Δ such that $\Pi_l v = \Delta \Pi_l z$ with $\|\Delta\|_\infty \leq \gamma$. If there exists such a $\tilde{\zeta}$ for all $l \in [0, N-1]$ then the model is not invalidated. □

4.5. Application of the $\mathcal{S}$-procedure

The $\mathcal{S}$-procedure can be applied directly to the condition to be tested in the above theorem. We state it in the appropriate form here.

Theorem ($\mathcal{S}$-procedure)
If there exist $\tau_i \geq 0$, $i = 1, \ldots, n$, such that

$$F_0(\gamma, \zeta) - \sum_{i=1}^{n} \tau_i F_i(\gamma, \zeta) \geq 0, \quad \textit{for all } \zeta, \qquad (5)$$

then, for all ζ such that $F_i(\gamma, \zeta) \geq 0$, $i = 1, \ldots, n$;

$$F_0(\gamma, \zeta) \geq 0.$$

If $n = 1$ then this condition is necessary and sufficient.

A sufficient condition for the invalidation of the model is the existence of τ_i, $i = 1, \ldots, n$ such that (5) is satisfied. In the case of a single perturbation block $(n = 1)$ this condition is both necessary and sufficient; see [13].

It is possible to apply this condition in a way that gives additional information. By performing a bisection (or similar) search on γ it is possible to find the smallest γ for which the datum does not invalidate the model. This can then be compared to size of Δ and w considered as reasonable in the development of the model to determine whether if the model is inadequate.

4.6. Formulation as an LMI problem

We now reformulate the search for τ_i, $i = 1, \ldots, n$ satisfying (5) as a linear matrix inequality (LMI) problem. This gives a computable convex sufficient condition for the invalidation of the model. In the single perturbation LFT case, this result shows that the LTV model validation problem is convex.

The following is a simple adaptation of a result given by Boyd [11, p. 23].

Theorem
The existence of τ_i, $i = 1, \ldots, n$ such that

$$F(\gamma, \zeta, \tau) = F_0(\gamma, \zeta) - \sum_{i=1}^{n} \tau_i F_i(\gamma, \zeta) \geq 0, \quad \textit{for all } \zeta \qquad (6)$$

is equivalent to the existence of τ_i, $i = 1, \ldots, n$ such that

$$\begin{bmatrix} A_0 - \sum_{i=1}^{n} \tau_i A_i(\gamma) & b_0 - \sum_{i=1}^{n} \tau_i b_i(\gamma) \\ b_0^T - \sum_{i=1}^{n} \tau_i b_i^T(\gamma) & c_0(\gamma) - \sum_{i=1}^{n} \tau_i c_i(\gamma) \end{bmatrix} := \begin{bmatrix} A(\tau, \gamma) & b(\tau, \gamma) \\ b^T(\tau, \gamma) & c(\tau, \gamma) \end{bmatrix} \geq 0. \qquad (7)$$

Proof:
If: Given $\hat{\tau}$ such that (7) is satisfied, implies that,

$$[\zeta^T\ 1] \begin{bmatrix} A(\hat{\tau}, \gamma) & b(\hat{\tau}, \gamma) \\ b^T(\hat{\tau}, \gamma) & c(\hat{\tau}, \gamma) \end{bmatrix} \begin{bmatrix} \zeta \\ 1 \end{bmatrix} = F(\gamma, \zeta, \hat{\tau}) \geq 0,$$

for all ζ.
Only if: Given $\hat{\tau}$ such that (6) holds for all ζ, consider

$$y^T \begin{bmatrix} A(\hat{\tau}, \gamma) & b(\hat{\tau}, \gamma) \\ b^T(\hat{\tau}, \gamma) & c(\hat{\tau}, \gamma) \end{bmatrix} y = [\zeta^T\ \eta] \begin{bmatrix} A(\hat{\tau}, \gamma) & b(\hat{\tau}, \gamma) \\ b^T(\hat{\tau}, \gamma) & c(\hat{\tau}, \gamma) \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix}.$$

If $\eta \neq 0$, then dividing y in the above by η makes it equivalent to (6) which is positive for all ζ. In the case where the last component of y is zero $(\eta = 0)$, we have

$$[\zeta^T\ 0] \begin{bmatrix} A(\hat{\tau}, \gamma) & b(\hat{\tau}, \gamma) \\ b^T(\hat{\tau}, \gamma) & c(\hat{\tau}, \gamma) \end{bmatrix} \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = \zeta^T A(\hat{\tau}, \gamma) \zeta.$$

The positive semidefiniteness of $A(\hat{\tau}, \gamma)$ is shown by noting that $F(\gamma, \zeta, \hat{\tau})$ is quadratic in ζ and for ζ sufficiently large is dominated by the $A(\hat{\tau}, \gamma)$ term. Therefore

$$y^T \begin{bmatrix} A(\hat{\tau}, \gamma) & b(\hat{\tau}, \gamma) \\ b^T(\hat{\tau}, \gamma) & c(\hat{\tau}, \gamma) \end{bmatrix} y \geq 0$$

for all y.□

The search for a τ such that (7) holds for all ζ is a linear matrix inequality (LMI) feasibility problem. Given a measured datum $(y(k),\ u(k))$, $k = 0, \ldots, N-1$, we would solve the LMI (7) for each truncation, $(\Pi_l y(k), \Pi_l u(k))$, $l = 0, \ldots, N-1$. Given $\gamma > 0$, if we find a truncation, $\hat{l}$, and a feasible $\hat{\tau}$, then the datum invalidates the model.

Note that the number of decision variables in the LMI is only the number of perturbation blocks, n, which is typically very low. In the case where $n = 1$ the LMI is necessary and sufficient for invalidation.

More importantly, the number of decision variables is independent of the data length. The complexity of the LMI problem does grow because the matrix size is linear in N. The computational growth for general purpose algorithms is expected to be between $O(N^2)$ and $O(N^3)$. Further detail on the formulation and solution of LMI problems can be found in [11] and [12].

5. Discussion

The significance of this work lies in the consideration of the block structured LFT framework. This makes it applicable to general robust control modeling problems, including those that arise from closed-loop systems. Prior work in the discrete-time and sampled-data domains focussed on model structures in which the perturbation linearly affected the output. Such structures give rise to convex optimization problems for model validation.

We show here, for the LTV perturbation case, that a single perturbation LFT structure also gives a convex LMI feasibility problem. The early work in the frequency domain case showed that the model validation problem could be transformed into a constrained structured singular value problem, which was also convex in the single perturbation block case. In light of this the result is not surprising that the discrete-time/sampled-data case is convex for LTV perturbations in the same LFT structure. It remains to be seen if it true for LTV perturbations and this is the subject of current research.

In the general block-structured LFT case we give a sufficient condition for invalidation which is then transformed into an LMI feasibility problem. The basis matrices in the LMI grow only linearly with the experiment data length. Furthermore the number of decision variables is very low; equal to the number of perturbation blocks which is frequently between one and five in practical problems. This contrasts with the existing results for LTI perturbations in this framework. In the LTI case, the number of decision variables grows linearly with the experiment data length. The complexity of the resulting LMI can be approximately $O(N^5)$ depending on the LMI algorithm used. The large order makes the solution of large data length problems infeasible and to alleviate this subsampling approaches have been used. It remains to be seen if the LTI problem can be reformulated in the manner shown here for the LTV case, and whether or not this will result in a reduced complexity LMI problem. This question will be pursued by choosing an IQC description of LTI perturbations.

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