Unfalsified Nonlinear Adaptive Control

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1 Introduction

In this paper we pursue the idea that *adaptive linear control* can be used to nullify deleterious effects of unknown nonlinearities.[1] As defined here, an adaptive linear control has the form,

$$u_t = C(z, \alpha_t)e_t \tag{1}$$

where u_t is the control applied at discrete time t, $\alpha_t \in \mathbf{R}^p$ is the *adaptive parameter* adjusted according to some rule based on past measurements, $C(z, \alpha_t)$ is a discrete-time (hence the z-transform variable) dynamical controller, dependent on α_t, with the property that for $\alpha_t = \alpha$, where α is a constant, $C(z, \alpha)$ is linear-time-invariant (LTI), and e_t is the error signal input to the controller, *e.g.*,

$$e_t = r_t - y_t \tag{2}$$

with reference r_t and measured plant output y_t. Of course *any* control is robust against some nonlinearities, *e.g.*, the small gain theorem, [2]. What makes (1) special is that there could exist a trajectory (or value) of α_t which is perfectly matched to handle some extreme nonlinear behavior of the system being controlled. There also could *not* exist such a trajectory or value. Discovering the class of nonlinearities for which this approach works is the essence of this paper.

This work is partly a continuation of some earlier efforts [4] to use adaptive linear feedforward control to control a periodically forced Duffing system, which is well known to exhibit complex behavior from periodic to chaotic, depending on some parameters, *e.g.*, [14], [12]. Simulation results showed that the adaptation continuously improved performance despite the complex system behavior, *i.e.*, the system state passed in and out of both chaotic and multi-periodic attractors, finally settling down to a "quiet" periodic orbit. An analysis was presented based on the method of averaging [1]. Under slow parameter adjustment it was shown that the source of the complex behavior was the nonlinearity in the system being controlled, not that introduced by the adaptation.

In this paper we use the *direct controller falsification* paradigm developed by Safonov *et al.*[11] to design an iterative adaptive linear control [5, 6] to control a nonlinear systems which exhibits multi-periodic and chaotic behavior. "Direct" adaptive control in general is different from the "indirect" approach of using measured data to estimate a parameter in a specified nonlinearity, or to establish the existence of a nonlinearity in a specified class, and then design the appropriate linear or nonlinear controller. It should be noted that trouble can arise not only from a nonlinearity present in the system itself, but also from the nonlinear effect of adaptation. Even in the case where the system is linear, adaptation can induce bifurcations and chaotic effects which may be unwanted, [9, 10]. In general, though, for slow adaptation, the plant nonlinearity dominates.

2 Chaotic Dynamics

The *logistic map*,

$$x_{t+1} = \mu x_t(1 - x_t) \tag{3}$$

is known to have complicated behavior dependent on the *scaling parameter* μ, *e.g.*, [12]:

$$\begin{array}{lcl} 0 < \mu < 1 & \Rightarrow & x = 0 \text{ is (locally) stable} \\ 1 < \mu < 3 & \Rightarrow & x = (\mu - 1)/\mu \text{ is globally stable} \\ 3 < \mu < 4 & \Rightarrow & \text{period doubling} \rightarrow \text{chaos as } \mu \text{ increases} \end{array} \tag{4}$$

Figure 1 shows the bifurcation diagram of the logistic map starting from the stable point $(\mu, x) = (2.9, 1.9/2.9 = 0.655172)$ on the left with μ, $2.9 < \mu < 3.9$, increasing to the right.

If (3) represents the behavior of an adaptively controlled system, and the goal of adaptation were to bring the system to a constant value, or even to $x = 0$, then it would be necessary to somehow effect the scaling properly to bring the system out of the chaotic region. From figure 1, this means that feedback must somehow effectively *reduce* the size of the scaling parameter. To see this more clearly, consider the bilinear system (the "plant") controlled,

$$y_{t+1} = u_t \; y_t \tag{5}$$

under feedback control,

$$u_t = \alpha_t r_t - \beta_t y_t \tag{6}$$

where (α_t, β_t) are the controller parameters to be adapted from the data $\{\, y_t, u_t \mid t = 1, \ldots, \ell \,\}$. If

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[1] The title should have been, "Unfalsified linear adaptive control of nonlinear systems," but it was too late to change.

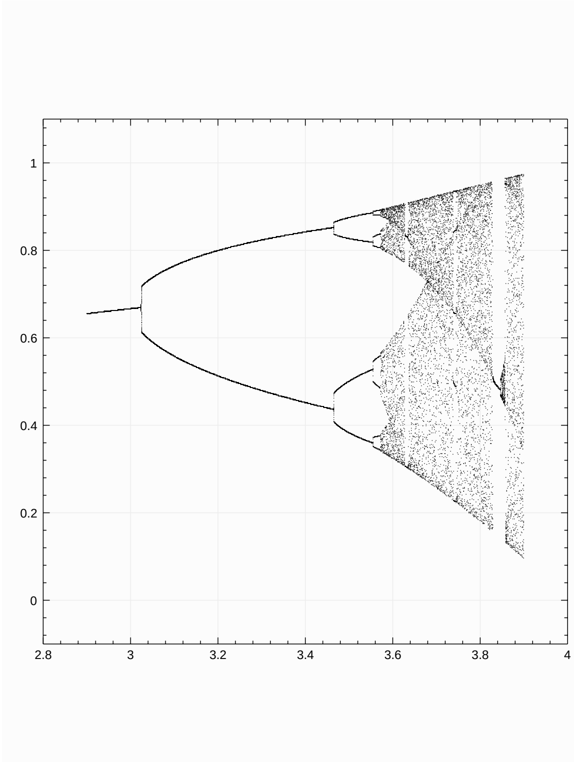

Figure 1: Bifurcation diagram (x_t vs. μ) for the logistic map (3).

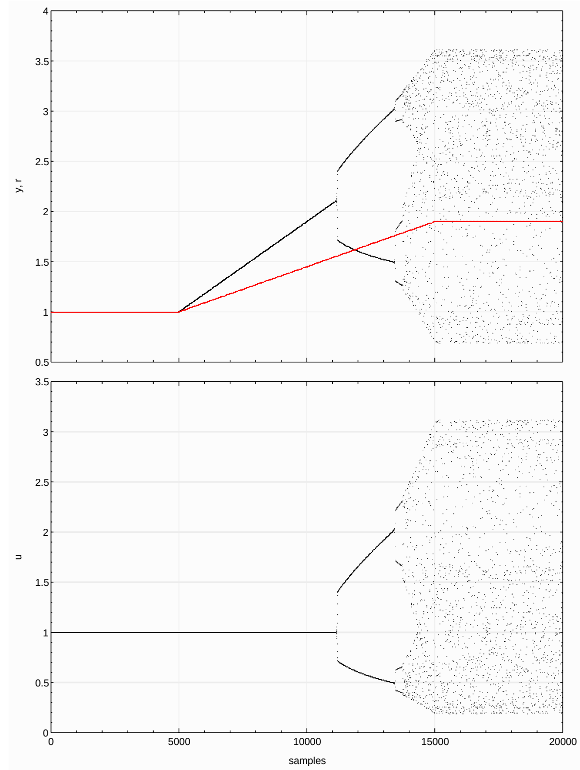

Figure 2: Response of feedback system (5)-(6) with parameters (9). **Row 1:** (y_t, r_t); **Row 2:** u_t.

(r_t, α_t, β_t) are constants (r, α, β), then (5)-(6) become,

$$\begin{array}{rcl} x_{t+1} & = & \alpha r \ x_t(1-x_t) \\ y_t & = & (\alpha r/\beta)\, x_t \\ u_t & = & \alpha r(1-x_t) \end{array} \tag{7}$$

Comparing to (4), the scaling parameter is now αr which interestingly depends on the feedforward parameter α and not the feedback parameter β. Moreover, if $1 < \alpha r < 3$, then $x = (\alpha r - 1)/\alpha r$ or $y = (\alpha r - 1)/\beta$ is globally stable. Thus, to achieve a globally stable $y = r$ requires that:

$$\begin{array}{l} \alpha = \beta + 1/r \\ 0 < \beta r < 2 \end{array} \tag{8}$$

Figure 2 shows a simulation of (5)-(6) under the following conditions:

$$r_t = \begin{cases} r_0, \ 0 \leq t < 5000 \\ r_0 + (r_f - r_0)(t - 5000), \ 5000 \leq t < 10000 \\ r_f, \ t \geq 10000 \end{cases}$$

$$\begin{array}{c} y_0 = (\alpha r_0 - 1)/\beta \\ r_0 = 1, \ r_f = 1.9 \\ \alpha = 2, \ \beta = 1 \end{array} \tag{9}$$

The system starts out at the fixed point $y_0 = 1$ which is stable because $\alpha r_0 = 2$. As the reference ramps up to $r_f = 1.9$ where $\alpha r_f = 3.8$ the system response becomes multi-periodic and eventually chaotic. If (α, β) were adapted from the data to make $y_t \to r_f$, then it would be desirable that (8) was satisfied, which in this case becomes $\alpha = \beta + 1/1.9$ and $0 < 1.9\beta < 2$.

What we pursue in this paper is to determine if direct unfalsified control can achieve the stable equilibrium conditions stated above using just the data without any prior assumptions about the system (5), *i.e.*, that it is bilinear. In the next section we present a brief overview of unfalsification and following that apply the approach to the system (5)-(6).

3 Direct Controller Unfalsification

The origin of the ideas for controller unfalsification are presented by Safonov and Tsao in [11] and the references therein; an application to a nonlinear robotic manipulator appears in [13] and to switching autopilots in [3]. Extensions to iterative adaptation are in [5, 6] and computations using convex programming with application to aerospace laboratory experiments is discussed in [15, 16].

As shown in [11], to falsify a controller, requires testing for inconsistencies between three sets that define the space of all signals of interest. The three sets of these signals are: (1) $\mathbf{M}_{\text{cntrl}}$, the set of candidate controllers, (2) $\mathbf{M}_{\text{data}}$, the data set, and (3) $\mathbf{M}_{\text{spec}}$, the closed-loop desired performance specification.

To illustrate the idea, consider the following example:

$$\mathbf{M}_{\mathrm{cntrl}}(\alpha) = \{\, y, u, r \mid u = C(\alpha)(r - y) \,\} \tag{10}$$

$$\begin{aligned} \mathbf{M}_{\mathrm{data}} &= \{\, y, u, r \mid y = \bar{y}, u = \bar{u} \,\} \\ (\bar{y}, \bar{u}) &= \{\, y_t, u_t \mid t = 1, \ldots, \ell \,\} \end{aligned} \tag{11}$$

$$\begin{gathered} \mathbf{M}_{\mathrm{spec}} = \{\, y, u, r \mid \|\varepsilon\|_{\mathrm{rms}} \leq \gamma \|r\|_{\mathrm{rms}}\,,\ \forall\ \|r\|_{\mathrm{rms}} < \infty \,\} \\ \varepsilon = \begin{bmatrix} W_1(r - y) \\ W_2 u \end{bmatrix} \end{gathered} \tag{12}$$

where r is a command, $C(\alpha)$ a candidate controller with parameter vector α, and $(\bar{u}, \bar{y})$ is the available *finite* input output sampled-data of length ℓ. The closed-loop specification says that the error signal, ε, consisting of weighted (filtered) versions of the tracking error, $r - y$, and the control signal, u, should be "small," as measured by γ, compared to the command r; W_1, W_2 are LTI weighting filters which determine what frequencies are emphasized. If the plant and controller are linear-time-invariant and disturbance-free, then the closed-loop specification is equivalent to one of the familiar $\mathbf{H}_\infty$ weighted performance criterion,

$$\begin{gathered} \left\| \begin{bmatrix} W_1 S(\alpha) \\ W_2 Q(\alpha) \end{bmatrix} \right\|_{\mathbf{H}_\infty} \leq \gamma \\ S(\alpha) = (1 + PC(\alpha))^{-1}, \quad Q(\alpha) = (1 + PC(\alpha))^{-1} C(\alpha) \end{gathered} \tag{13}$$

In general, the specification set says that the "rms gain" from r to ε, is bounded by γ. (Typically the controller is chosen to make γ as small as possible, and preferably smaller than unity if the weights are properly normalized.) The specification is also equivalent to,

$$\begin{aligned} \mathbf{M}_{\mathrm{spec}} &= \left\{ y, u, r \;\middle|\; \|r \mapsto \varepsilon\|_{\mathrm{rms\_gain}} \leq \gamma \right\} \\ &= \left\{ y, u, r \;\middle|\; \varepsilon = \Delta r, \quad \|\Delta\|_{\mathrm{rms\_gain}} \leq \gamma \right\} \end{aligned} \tag{14}$$

where Δ denotes the *closed-loop system* mapping $r \mapsto \varepsilon$, emphasizing the fact that the closed-loop specification set is equivalent to an *uncertainty model* of the closed-loop system.

Condition for controller unfalsification The set of unfalsified controllers, $\mathbf{M}^{\mathrm{unf}}_{\mathrm{cntrl}}(\alpha)$, are those which are consistent with the data and meet the performance specifications, *i.e.*,

$$\mathbf{M}^{\mathrm{unf}}_{\mathrm{cntrl}}(\alpha) = \{\, \mathbf{M}_{\mathrm{cntrl}}(\alpha) \mid \mathbf{M}_{\mathrm{cntrl}}(\alpha) \cap \mathbf{M}_{\mathrm{data}} \subset \mathbf{M}_{\mathrm{spec}} \,\}$$

As pointed out in [11], this method of direct adaptive control has several generic properties. Most significantly, $C(\alpha)$ can be tested for unfalsification *without implementation*, the data alone is used. In addition, the test for controller unfalsification is "plant-model free." No plant model is needed to test its conditions. It depends only on the data, the controller and the specification.

Computation
To calculate $\mathbf{M}^{\mathrm{unf}}_{\mathrm{cntrl}}(\alpha)$ we seek a command r which would have produced the finite ℓ length data $(\bar{u}, \bar{y})$ had the candidate controller $C(\alpha)$ been in the loop. As a practical matter the ℓ data might be recorded over a moving window of length ℓ. Specifically, let t denote the current time and define the recorded data as:

$$\begin{aligned} (\bar{y}, \bar{u}, \bar{r}) &= \{\, y_k, u_k, r_k \mid k \in T_\ell(t) \,\} \\ T_\ell(t) &= [t + 1 - \ell, \ldots, t] \end{aligned} \tag{15}$$

If $\ell = t$, then the data window grows with time,

$$T_t(t) = [1, \ldots, t]$$

If the data length ℓ is very large, then $T_\ell(t) \approx T_\infty(t)$ where,

$$T_\infty(t) = (-\infty, \ldots, t]$$

Suppose we seek controllers of the form

$$u = C(\alpha)(r - y) \tag{16}$$

Then the plant output/input data $(\bar{y}, \bar{u})$ would have been produced with controller $C(\alpha)$ if the reference had been

$$\hat{r}(\alpha) = \bar{y} + C(\alpha)^{-1} \bar{u} \tag{17}$$

Correspondingly, the performance error would have been

$$\hat{\varepsilon}(\alpha) = \begin{bmatrix} W_1 C(\alpha)^{-1} \bar{u} \\ W_2 \bar{u} \end{bmatrix} \tag{18}$$

Hence, for each candidate controller $C(\alpha)$, the closed-loop gain from $\hat{r}(\alpha)$ to $\hat{\varepsilon}(\alpha)$ based on available data would have been,

$$\hat{\gamma}_\ell(t, \alpha) = \sup_{k \in T_\ell(t)} \frac{\|\hat{\varepsilon}(\alpha)\|_{[t+1-\ell, k]}}{\|\hat{r}(\alpha)\|_{[t+1-\ell, k]}} \tag{19}$$

There are possibly many choices for replacing the existing controller. The most interesting ones are those for which the predicted (unfalsified) performance, $\hat{\gamma}_\ell(t, \alpha)$, is smaller than the measured performance based solely on the current ℓ-data $(\bar{y}, \bar{u}, \bar{r})$,

$$\bar{\gamma}_\ell(t) = \sup_{k \in T_\ell(t)} \frac{\|\bar{\varepsilon}\|_{[t+1-\ell, k]}}{\|\bar{r}\|_{[t+1-\ell, k]}} \tag{20}$$

where

$$\bar{\varepsilon} = \begin{bmatrix} W_1(\bar{r} - \bar{y}) \\ W_2 \bar{u} \end{bmatrix} \tag{21}$$

Adaptive controller switching
Suppose that the existing controller has parameters set to α^t. An algorithm for *cautious* switching to a new set of controller parameters is,

$$\begin{aligned} \alpha^{t+1} &= \begin{cases} \arg\sup \{\, \hat{\gamma}(t, \alpha) \mid \alpha \in \mathbf{A}_{\mathrm{unf}}(t, \epsilon) \,\} \\ \\ \alpha^t \text{ if } \mathbf{A}_{\mathrm{unf}}(t, \epsilon) \text{ is empty} \end{cases} \\ \mathbf{A}_{\mathrm{unf}}(t, \epsilon) &= \{\, \alpha \mid \hat{\gamma}_\ell(t, \alpha) \leq (1 - \epsilon) \bar{\gamma}_\ell(t) \,\} \end{aligned} \tag{22}$$

The parameter $\epsilon \in (0,1)$ sets a threshold for switching (adaptation) based on the measured performance. If ϵ is small the parameters may switch often, possibly even every sample. A larger value, say $\epsilon \in [.05, 10]$ may be more sensible allowing a reasonable accumulation of data before making a decision to switch. If the set is empty, then no new controller is implemented and more data is collected.

An algorithm for *aggressive* switching to a new set of controller parameters is,

$$\alpha^{t+1} = \begin{cases} \arg\inf \{ \hat{\gamma}(t,\alpha) \mid \alpha \in \mathbf{A}_{\mathrm{unf}}(t,\epsilon) \} \\ \alpha^t \text{ if } \mathbf{A}_{\mathrm{unf}}(t,\epsilon) \text{ is empty} \end{cases} \tag{23}$$

It is also possible that the data "suggests" re-defining the performance goals before switching. Suppose that $\mathbf{A}_{\mathrm{unf}}(t,\epsilon)$ is not empty and the distribution $\hat{\gamma}_\ell(t,\alpha) \leq (1-\epsilon)\bar{\gamma}_\ell(t)$ is weighted towards smaller ϵ values. Then it might make more sense to change parameters in the weighting filters and repeat the calculations. For example, following the "windsurfing approach to adaptation, the weights can be changed so as to reflect an increase in performance over a larger bandwidth than originally specified [7, 8].

Another procedure is to switch only after a fixed amount of data has been accumulated, *i.e.*, the controller parameters are held fixed over a prescribed interval. This is sometimes referred to as *iterative adaptation*, *e.g.*, [5, 6]. In this case either of the above switching algorithms, (22) or (23), can be used at the iterative switching times.

4 Simulation results

The feedback system (5)-(6) is simulated with the parameters as given by (9). An iterative adaptation algorithm is used to adjust only α_t in (6); $\beta_t = 1$ throughout. The simulation is run for 50000 samples and α is adapted every 5000 samples based on the data from the previous 5000 samples. The performance error in (12) is simplified to

$$\varepsilon = r - y \tag{24}$$

The adaptive algorithm is as follows:

- At the switching times

  $$t_k = 5000k, \ k = 1, \ldots, 10$$

  collect data

  $$(\bar{y}, \bar{u}, \bar{r}) = \{ y_t, u_t, r_t \mid t = 1 + t_{k-1}, \ldots, t_k \}$$

- compute measured performance

  $$\bar{\gamma} = \sup_{t \in [1+t_{k-1}, t_k]} \frac{\|\bar{y} - \bar{r}\|_{[1+t_{k-1}, t]}}{\|\bar{r}\|_{[1+t_{k-1}, t]}}$$

- compute predicted (unfalsified) performance

  $$\begin{aligned} \hat{\gamma}(\alpha) &= \sup_{t \in [1+t_{k-1}, t_k]} \frac{\|\bar{y} - \hat{r}(\alpha)\|_{[1+t_{k-1}, t]}}{\|\hat{r}(\alpha)\|_{[1+t_{k-1}, t]}} \\ \hat{r}(\alpha) &= (\bar{u} + \beta \bar{y})/\alpha \end{aligned}$$

- aggressive parameter adaptation

  – if there exists α, $\hat{\gamma}(\alpha) < \bar{\gamma}$, then set controller parameter to:

  $$\begin{aligned} \hat{\alpha} &= \arg\inf_{\alpha} \hat{\gamma}(\alpha) \\ &= \arg\inf_{\alpha} \sup_{t \in [1+t_{k-1}, t_k]} \frac{\|\alpha \bar{y} - \bar{v}\|_{[1+t_{k-1}, t]}}{\|\bar{v}\|_{[1+t_{k-1}, t]}} \\ & \quad (\bar{v} = \bar{u} + \beta \bar{y}) \\ &= \arg\inf_{\alpha_\tau} \sup_{\tau \in [1+t_{k-1}, t_k]} \frac{\|\alpha_\tau \bar{y} - \bar{v}\|_{[1+t_{k-1}, \tau]}}{\|\bar{v}\|_{[1+t_{k-1}, \tau]}} \\ \alpha_\tau &= \arg\inf_{\alpha} \|\alpha \bar{y} - \bar{v}\|_{[1+t_{k-1}, \tau]} \\ &= \frac{\langle \bar{y}, \bar{v} \rangle_{[1+t_{k-1}, \tau]}}{\|\bar{y}\|^2_{[1+t_{k-1}, \tau]}} \end{aligned}$$

  – if no α exosts such that $\hat{\gamma}(\alpha) < \bar{\gamma}$, then set collect more data.

Figure 3 shows a comparison with and without adaptation. The left column shows $(y, u, \hat{\alpha})$; $\hat{\alpha}$ is estimated as described above but is not implemented; the control remains fixed at $u_t = \alpha_0 r_t - \beta y_t$. The right column shows $(y, u, \hat{\alpha})$ with $\hat{\alpha}$ implemented at the switching times, hence, $u_t = \hat{\alpha}_t r_t - \beta y_t$. (In all cases $\hat{\gamma} < \bar{\gamma}$.) The final value of $\hat{\alpha}$ is $\hat{\alpha}_f = 1.52639$ which is very close to the optimal value for perfect tracking, $\alpha_f = \beta + 1/r_f = 1 + 1/1.9 = 1.52632$.

It is important to note that because $\beta = 1$ in this example it is possible to only adapt α and achieve the stable tracking condition (8). In general both parameters would have to be adjusted.

5 Concluding Remarks

A direct adaptive algorithm based on unfalsified control ideas has been shown to correctly adjust a control parameter without any knowledge of the plant, which in this case is bilinear and with a linear feedback controller is easily coaxed into multi-periodic and chaotic behavior. Is there a general conclusion to be drawn? No, not from a single example, but it is encouraging. At present no method of analysis is apparent from which one can predict the (nonlinear) systems for which this method of adaptation will likely be successful. As mentioned in the introduction, averaging analysis of slowly adapting systems can be used to draw some inferences, provided

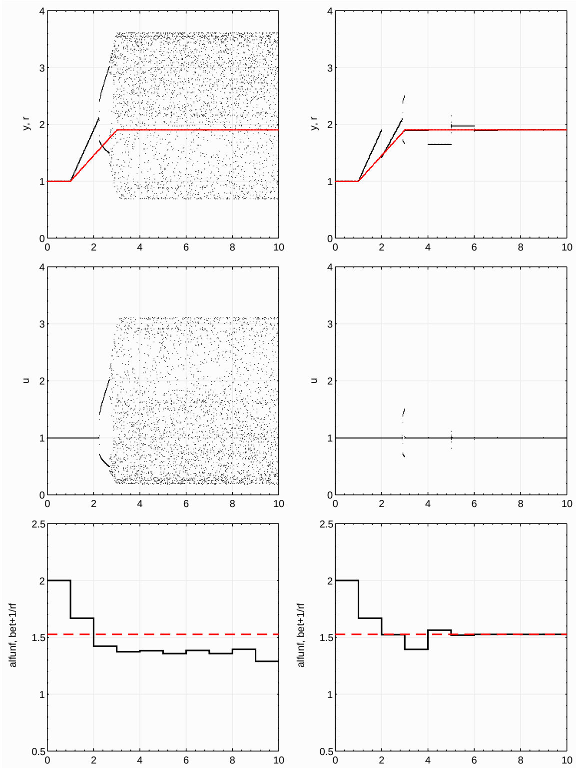

Figure 3: Comparison with and without adaptation of $(y_t, u_t, \hat{\alpha}_t)$ vs $t/5000$. **Left column:** adaptation off, **Right column:** adaptation on. **Row 1:** (y_t, r_t); **Row 2:** u_t; **Row 3:** $(\hat{\alpha}_t, \alpha_f)$.

the method presented here is analogous to slow adaptation, It may be so, because the parameters are iteratively adjusted after accumulating a reasonable amount of data which most likely has an averaging effect.

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