On the Identifiability of Nonlinear Maps in a General Interconnected System *

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Abstract

In this paper we investigate the problem of identifying static nonlinear maps as part of a general, structured interconnected system. The static nonlinear maps that require identification are not naturally parameterized via basis functions or other expansions. Thus, we are interested in non-parametric identification of these nonlinear maps.

The particular focus of this paper is the issue of "identifiability". Loosely speaking, the static nonlinear maps in an interconnected system are identifiable if it is possible to determine them uniquely on the basis of input-output experiments. As is well known, identifiability concepts are of fundamental importance in system identification [15].

In this paper, we focus on the case where only the static nonlinearity needs to be identified and the linear components of the interconnection are known. We offer a readily computable test for identifiability in the case that the inputs to every nonlinear map are measured. This test reduces to a matrix positivity computation.

1 Introduction

In the field of nonlinear system identification many open problems still remain. We categorize the available results using two criteria.

First we distinguish between a parametric or non-parametric approach to the problem of identifying static nonlinear maps. Much of the previous literature uses a parametric approach where nonlinear maps are represented using small sets of basis functions. Non-parametric methods include Volterra kernel expansions, neural networks, radial basis function expansions, and Fourier series [2, 7, 13, 20, 11, 14]. These studies offer asymptotic analyses, and local convergence results.

Second, we differentiate between structured and unstructured problems. Unstructured problems treat the most general case but frequently *a priori* structural information about the system to be identified can be incorporated [23, 17, 22], resulting in a model with structure. Further, with unstructured methods it is typically difficult to ascribe physical interpretation to the resulting models, and they are usually not well suited for subsequent analysis and design. They do however have the advantage of being general purpose. The most thoroughly investigated structured identification problems are Hammerstein and Weiner system [3, 17, 19]. However, many of the simplest problems here remain open.

This paper is concerned with identification problems of interconnected structured nonlinear systems. We believe that it is fruitful to concentrate on a specific class of structured nonlinear systems. Our broad objective is to develop and analyze appropriate identification algorithms for structured models. In particular, we are concerned with problems in which the nonlinear elements to be identified are *non-parametric*. By this we mean that these elements do not have a natural parameterization that is known or suggested from an analytical understanding of the underlying process. Such problems are particularly common in process control applications, or in nonlinear model reduction problems where an approximation is sought for a subsystem containing complex nonlinear dynamics.

In this paper we focus on the issue of identifiability. Loosely speaking, the nonlinear maps in an interconnected system are identifiable if it is possible to determine them uniquely on the basis of input-output experiments. Identifiability is a fundamental concept in system identification. Much of the available literature on identifiability covers *linear* model structures only. This includes work on non-parametric linear models [15], and on parametric (including nonlinearly parameterized) linear models [10, 11, 12, 24, 25]. For nonlinear model structures much less is known. Notable here are the research of [16, 9] which offer results for *parametric* nonlinear model structures based on differential ideals.

Identifiability issues of nonlinear maps are still unexplored. There is little previous work on identifiability of non-parametric nonlinear systems with the exception of [4, 5] which treat one single-input single-output

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static nonlinearity in an interconnected system. Our result in [8] extends the work of [4, 5] to several multivariable nonlinearities. In this paper we explore this issue further and provide a readily computable test for identifiability.

The remainder of this paper is organized as follows. Section 2 establishes our notation. In Section 3, we define the class of model structures under consideration. In Section 4, we make some (fairly standard) definitions of identifiability. Section 5 contains our main results and the computational aspects. These are for the special case where the inputs to the nonlinear elements are measured. The proofs of our main results may be found in the Appendix.

2 Notation

We deal exclusively with one-sided sequence spaces. We denote a sequence as $w = (w_0, w_1, \dots)$, that is w_t is the value of sequence w at time t. Associated with the sequence w will be the multiplication operator $\{w_t\}$ with action $(\{w_t\}(u))_t = w_t u_t$. Let $\mathcal{S}$ denote the set of one-sided vector-valued real sequences. Let ℓ_2 denote the Hilbert space of one-sided sequences equipped with the usual norm.

Let z denote the *left-shift (advance) operator* on $\mathcal{S}$, i.e.

$$z(u_0, u_1, \dots) = (u_1, u_2, \dots)$$

and z^{-1} is the *right-shift (delay) operator*

$$z^{-1}(u_0, u_1, \dots) = (0, u_0, u_1, \dots).$$

Note that $zz^{-1} = I$ but $z^{-1}z \neq I$ and that only z^{-1} commutes with any linear time-invariant operator. An *input-output operator* is a map $H : \ell_2 \to \mathcal{S}$, and will be called *time-invariant* if it commutes with the right-shift operator, i.e., $Hz^{-1} = z^{-1}H$. The operator H is called static if it is time-invariant and $(Hu)_t$ depends only on u_t.

In the interest of readability, we supress the dimensions of all sequence spaces, and the input-output dimensions of all operators.

Two input-output operators H_1, H_2 are equal if if they have the same action, i.e. $H_1(u) = H_2(u)$ for all $u \in \ell_2$. We shall largely deal with *linear time-invariant (LTI)* operators $\mathcal{L}$, and *static nonlinear* operators $\mathcal{N}$.

A linear time-invariant operator $\mathcal{L}$ is stable if it is induced ℓ_2-norm bounded.

3 Problem Setup

We are concerned with identification problems of interconnected structured nonlinear systems. As is well known [18], we may rearrange a general class of interconnected systems to the linear fractional transformation (LFT) form shown in Figure 1. Here, u is the $\mathbf{R}^p$-valued applied input signal, y is the $\mathbf{R}^q$-valued measured output signal, and $\mathcal{L}$ is a linear time-invariant system. As we will not treat noise models in this paper, we have suppressed exogenous noise inputs.

This is a general situation. Indeed, *every* finite dimensional nonlinear system (and associated identification problem) can be rearranged to this LFT framework. This framework allows us to explore specific classes of structured nonlinear system identification problems in a common framework.

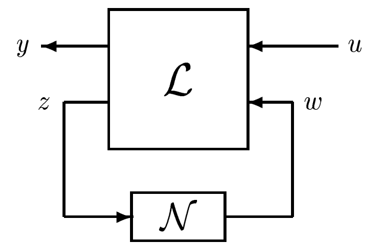

Figure 1: Nonlinear Model Structure

All the nonlinearities in the interconnection are gathered into the m-input, n-output *static* block $\mathcal{N}$. This is the block that requires identification, i.e. $\mathcal{L}$ is known. Note also that the nonlinear block $\mathcal{N}$ has a block-diagonal structure, owing to the fact that nonlinearities appear in specific components in our interconnection. That is, in actuality a particular output of $\mathcal{N}$ may depend on only a subset of the inputs z. Define $[\mathcal{N}_1, .., \mathcal{N}_n]$ as the component maps of $\mathcal{N}$, each possibly multi-input, but single-output. We will also require that the inputs to each nonlinearity are distinct, in that no two nonlinearities share a common input. This can be achieved without loss of generality by duplicating outputs of $\mathcal{L}$.

This situation is depicted conceptually as:

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}_1 & & \\ & \ddots & \\ & & \mathcal{N}_n \end{bmatrix} \tag{1}$$

The structure of the nonlinear block $\mathcal{N}$ is known and fixed. We will make the assumption that the component nonlinearities $\mathcal{N}_k(\cdot)$ are $\mathcal{C}^1$, i.e. they admit continuous first partial derivatives everywhere. Nothing else is known about $\mathcal{N}$, and it is precisely in this sense that our identification problem is non-parametric.

Definition 2 *Let $\mathbb{N}$ be the set of $\mathcal{C}^1$ mappings from $\mathbf{R}^m$ to $\mathbf{R}^n$ with structure described above. We will refer to the set of models given by the interconnection shown in Figure 1, subject to $\mathcal{N} \in \mathbb{N}$, as the model structure $\mathcal{M}$. For a fixed nonlinear block $\mathcal{N}^\circ \in \mathbb{N}$, we write $\mathcal{M}(\mathcal{N}^\circ)$ to mean the input-output operator from u to y in Figure 1 with the nonlinear block being fixed at $\mathcal{N}^\circ$.*

4 Identifiability

We are now in a position to introduce our various notions of identifiability. Roughly these capture the idea that for an identifiable model, there is only one nonlinearity from the set of nonlinearities under consideration that will reproduce the exact same input-output behavior.

Definition 3 *Suppose $\mathcal{N}^\circ \in \mathbb{N}$. The model structure $\mathcal{M}$ is* identifiable at $\mathcal{N}^\circ$ *if for any $\mathcal{N}^1 \in \mathbb{N}$,*

$$\mathcal{M}(\mathcal{N}^1) = \mathcal{M}(\mathcal{N}^\circ) \text{ implies that } \quad \mathcal{N}^1 = \mathcal{N}^\circ.$$

The model structure $\mathcal{M}$ is locally identifiable at $\mathcal{N}^\circ$ *if there exists $\epsilon > 0$ such that*

$$\mathcal{M}(\mathcal{N}^1) \neq \mathcal{M}(\mathcal{N}^\circ).$$

$$\forall \mathcal{N}^1 \in \mathbb{N} \text{ with } \mathcal{N}^1 \neq \mathcal{N}^\circ \text{ and } \|\mathcal{N}^1 - \mathcal{N}^\circ\| < \epsilon$$

The model structure $\mathcal{M}$ is identifiable everywhere *if it is identifiable at all $\mathcal{N}^\circ \in \mathbb{N}$ for which the interconnection of Figure 1 is well-posed.*

The model structure $\mathcal{M}$ is locally identifiable everywhere *if it is locally identifiable at all $\mathcal{N}^\circ \in \mathbb{N}$ for which the interconnection of Figure 1 is well-posed.*

5 Main Results

Consider again the interconnected nonlinear model structure shown in Figure 1. Recall that $\mathcal{L}$ is known and $\mathcal{N}$ is a nonlinear block to be identified.

Partition $\mathcal{L}$ comformably with its inputs and outputs as

$$\mathcal{L} = \begin{bmatrix} L_{yu} & L_{yw} \\ L_{zu} & L_{zw} \end{bmatrix}.$$

We will exclusively consider the case where z is measured, i.e. z can be inferred from knowledge of u and y. This is implied by (but not equivalent to) the condition $L_{zw} = 0$. While we do not need this stronger hypothesis, we will assume $L_{zw} = 0$ to simplify our arguments.

Now suppose we have two nonlinear maps $\mathcal{N}^\circ \in \mathbb{N}$ and $\mathcal{N}^1 \in \mathbb{N}$ that cannot be distinguished using input-output experimentation, i.e. for all u

$$\begin{aligned} 0 &= \mathcal{M}(\mathcal{N}^1)u - \mathcal{M}(\mathcal{N}^\circ)u \\ &= L_{yw}(\mathcal{N}^1 - \mathcal{N}^\circ)(L_{zu}u) \\ &= L_{yw}\mathcal{N}(L_{zu}u) \end{aligned}$$

where $\mathcal{N}(z) := (\mathcal{N}^1 - \mathcal{N}^\circ)(z) := \mathcal{N}^1(z) - \mathcal{N}^\circ(z)$. Note that $\mathcal{N} \in \mathbb{N}$ also. Therefore the model structure $\mathcal{M}$ is not identifiable at $\mathcal{N}^\circ$ if there exists a nonzero static non-linear map $\mathcal{N} \in \mathbb{N}$ such that

$$L_{yw}\mathcal{N}L_{zu} = 0 \tag{4}$$

If such a map exists, it is clear that it can be made arbitrarily small because scaling does not affect equation (4). Also note that equation (4) does not depend on $\mathcal{N}^\circ$. In summary we have

Proposition 5 *Consider the model structure $\mathcal{M}$. Assume that $L_{zw} = 0$. Then, $\mathcal{M}$ is identifiable at $\mathcal{N}^\circ$*

$\iff$ *$\mathcal{M}$ is identifiable everywhere*

$\iff$ *$\mathcal{M}$ is locally identifiable at $\mathcal{N}^\circ$*

$\iff$ *$\mathcal{M}$ is locally identifiable everywhere.*

Given that all our notions of identifiability coalesce in the z-measured case, we will simply use the term "identifiable".

Let us define the structure set

$$\mathcal{X} = \{X \in \mathbf{R}^{n \times m} : X_{ij} = 0 \text{ if the i-th output of } \mathcal{N} \text{ does not depend on the j-th input}\}.$$

Remember that the model structure $\mathcal{M}$ is identifiable if and only if $\nexists\ \mathcal{N} \in \mathbb{N},\ \mathcal{N} \neq 0$ such that $L_{yw}\mathcal{N}L_{zu} = 0$. We will show in the Appendix that the existence of such a nonlinearity is equivalent to the existence of a constant matrix in the above define set $\mathcal{X}$.

We can now state our main result:

Theorem 6 *The model structure $\mathcal{M}$ of Figure 1 is identifiable if and only if*

$$\nexists\ X \in \mathcal{X}, X \neq 0 : \quad L_{yw}XL_{zu} = 0 \tag{7}$$

Proof: See Appendix ∎

We now turn our attention to developing a readily computable test for the identifiability condition of Theorem 6, that is, does there exist such an X? The essential idea here is as follows. The constraint $X \in \mathcal{X}$ is linear, and so the condition (7) is equivalent to checking if a collection of (multivariable) LTI systems is linearly independent over the field of reals. This in turn can be checked efficiently using $\mathcal{H}_2$ norm based computations.

The identifiability condition (7) reduces to a matrix positivity test. The matrix involved is of size $r \times r$, where r is the dimension of the subspace $\mathcal{X}$. This matrix is computed by solving one Lyapunov equation of size $(2rn_x)$ where n_x is the number of states of the original system L.

We will now describe explicitly how to perform this test. Consider the model structure $\mathcal{M}$ of Figure 1. Let

$$L \sim \left[\begin{array}{c|cc} A & B_u & B_w \\ \hline C_y & D_{yu} & D_{yw} \\ C_z & D_{zu} & D_{zw} \end{array}\right]$$

be a state space realization of L. Let $K_i: \; i = 1, \cdots, r$ be basis matrices for $\mathcal{X}$, i.e. any $X \in \mathcal{X}$ can be written as

$$X_\theta = \sum_{i=1}^{r} K_i \theta_i.$$

Define Q_θ to be the system

$$Q_\theta := L_{yw} X_\theta L_{zu} = \sum_{i=1}^{r} L_{yw} K_i L_{zu} \theta_i = \sum_{i=1}^{r} T_i \theta_i,$$

where T_i is a system with state space realization

$$T_i \sim \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B_u \\ B_w K_i C_z & A & B_w K_i D_{zu} \\ \hline D_{yw} K_i C_z & C_y & D_{yw} K_i D_{zu} \end{array}\right].$$

Now we wish to check if there is a nontrivial choice for $\theta = [\theta_1, \cdots, \theta_r]$ such that $Q_\theta = 0$. Observe that Q_θ has realization

$$Q_\theta \sim \left[\begin{array}{c|c} F & G\Theta \\ \hline H & J\Theta \end{array}\right]$$

where F, G, H, J are defined as

$$F = \text{diag}(A_i), \qquad G = \text{diag}(B_i)$$

$$H = \left[\begin{array}{ccc} C_1 & \cdots & C_r \end{array}\right], \quad J = \left[\begin{array}{ccc} D_1 & \cdots & D_r \end{array}\right]$$

and $\Theta := \left[\begin{array}{ccc} \theta_1 I & \cdots & \theta_r I \end{array}\right]^* \in \mathbf{R}^{rp \times p}$.

Let W be the observability grammian of Q_θ which can be computed as the solution of the Lyapunov equation $W = F^* W F + H^* H$. We now use the fact that $Q_\theta = 0$ if and only if $\|Q_\theta\|_{\mathcal{H}_2} = 0$. Using results from e.g. [27],

$$\begin{aligned} \|Q_\theta\|_{\mathcal{H}_2} &= \text{trace } (\Theta^* J^* J \Theta + \Theta^* G^* W G \Theta) \\ &= \text{trace } (\Theta^* [J^* J + G^* W G] \Theta) \\ &= \text{trace } (\Theta^* \begin{bmatrix} \Lambda_{11} & \cdots & \Lambda_{1r} \\ \vdots & \ddots & \vdots \\ \Lambda_{r1} & \cdots & \Lambda_{rr} \end{bmatrix} \Theta) \\ &= \text{trace } (\sum_{i,j=1}^{r} \theta_i \theta_j \Lambda_{ij}) \\ &= \theta^* \Omega \theta, \end{aligned}$$

where $J^* J + G^* W G$ was partitioned to yield the $p \times p$ matrices Λ_{ij}, and p is the dimension of the signal u. Further, Ω is defined to have ij'th entry [trace (Λ_{ij})]. Notice that Ω depends only on the known quantities L and the nonlinearity structure $\mathcal{X}$, and can be readily computed.

To summarize, we have that $\nexists \; \theta \neq 0$ such that $Q_\theta = 0$ if and only if Ω is positive definite or negative definite. We have shown the following result:

Theorem 8 *The model structure $\mathcal{M}$ is identifiable if and only if*

$$\Omega > 0 \text{ or } \Omega < 0,$$

with Ω defined as above.

6 Appendix

In the following section let L and R be LTI systems of appropriate dimensions and let $\mathcal{N} \in \mathbb{N}$. We will prove the main step of our main result (Theorem 6) namely that

$$(\exists \mathcal{N} \neq 0 : L\mathcal{N}R = 0) \Longleftrightarrow (\exists X \in \mathcal{X}, \; X \neq 0 : LXR = 0)$$

Define $G_t = \nabla \mathcal{N}|_{(Ru)(t)}$ for $t \geq 0$ and $G_t = 0$ for $t < 0$, where $\nabla \mathcal{N}$ is the Jacobian of $\mathcal{N}$ and u is the input signal to the system R.

Also define $L^{(N)}$ to be the truncated Toeplitz matrix

$$L^{(N)} = \begin{bmatrix} L_0 & 0 & \cdots & 0 \\ L_1 & L_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ L_N & L_{N-1} & \cdots & L_0 \end{bmatrix}$$

where L_k are the Markov parameters of L and define $R^{(N)}$ likewise. Also define the set of consecutive gradients $\mathcal{G} = \{(G_k, \ldots, G_{k+N}) : k \in \mathbb{Z}\}$ and the vectors $\mathbf{X} = (X_0, \ldots, X_N)$ where $X_i \in \mathcal{X}$. Finally define the linear map

$$\Phi^{(N)}(\mathbf{X}) = (L^{(N)} \mathbf{X} R^{(N)}, \; \Delta \mathbf{X})$$

where $\Delta \mathbf{X} = \left[\begin{array}{ccc} X_0 - X_1 & \cdots & X_{N-1} - X_N \end{array}\right]^T$.

The following lemmas are needed to prove the main result.

Lemma 9 *$\nabla \mathcal{N}$ has the same structure as $\mathcal{N}$ as imposed by the model structure $\mathcal{M}$, i.e $\mathcal{N} \in \mathbb{N} \implies \nabla \mathcal{N}|_v \in \mathcal{X}$.*

Lemma 10 *From the linearity of L and R and the chain rule it is easily verified that $L\mathcal{N}R = 0 \implies L\{G_t\}R = 0$*

Lemma 11 *If $L\mathcal{N}R = 0$ and $\nabla \mathcal{N}|_0 = 0$ then evaluating the gradient $\nabla \mathcal{N}$ at $v = Rz^{-k}u$ and simple manipulation yields that $Lz^{-k}\{G_t\}z^k R = 0$ for all $k \geq 0$. Observe that $z^{-k}\{G_t\}z^k = \{0, \ldots, 0, G_0, G_1, \ldots\}$*

Lemma 12 *$L\{G_t\}R = 0 \implies Lz^k\{G_t\}z^{-k}R = 0$ for all $k \geq 0$. Note that $z^k\{G_t\}z^{-k} = \{G_k, G_{k+1}, \ldots\}$*

Lemma 13 *Let $M \in \mathbf{R}^{p \times p}$ then it is easy to show that $\forall j \geq p: \;\; M^j x = 0 \implies M^p x = 0$.*

The following four lemmas cover different cases which combined prove the needed Theorem 19.

Lemma 14 *If* $\nabla\mathcal{N}|_0 \neq 0$ *and* $L\mathcal{N}R = 0$ *then* $\nabla\mathcal{N}|_0 \in \mathcal{X}$ *and* $L\,\nabla\mathcal{N}|_0\,R = 0$.

Lemma 15 *Define the range space of* R *to be* $\mathcal{R}$. *If* $L\mathcal{N}R = 0$ *and* $\mathcal{R} \neq \mathbf{R}^m$ *then* $\exists x \neq 0$ *so that* $\begin{bmatrix} x^* & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{X}$ *and* $L \begin{bmatrix} x^* & 0 \\ 0 & 0 \end{bmatrix} R = 0$.

Proof: A representation of the range space of R at any time t is $\mathcal{R} = Span\{R_0, R_1, \ldots\}$ where R_i are the Markov parameters of R. $\mathcal{R}$ is a subspace of $\mathbf{R}^m$ where m is the dimension of the signal z.

Partition $\mathcal{N}$ as in equation (1) where $\mathcal{N}_i$ has m_i inputs. Recall that the inputs to each non-linearity are separated so that $\sum_{i=1}^n m_i = m$. Define the subspaces $S_i = \begin{bmatrix} 0 & \cdots & 0 & I_{m_i} & 0 & \cdots & 0 \end{bmatrix} \mathcal{R} \subseteq \mathbf{R}^{m_i}$ Since $\mathcal{R} \neq \mathbf{R}^m$ at least one of the above subspaces is strictly contained in $\mathbf{R}^{m_i}$, without loss of generality let $S_1 \subset \mathbf{R}^{m_1}$ which implies $\exists x \in S_1^\perp, x \neq 0$. Define $X = \begin{bmatrix} x^* & 0 \\ 0 & 0 \end{bmatrix}$. Note that $0 \neq X \in \mathcal{X}$. Now observe that with $r = Ru$

$$XRu = \begin{bmatrix} x^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} x^* r_1 \\ 0 \end{bmatrix} = 0$$

since x is perpendicular to $r_1 \in S_1$. In conclusion $LXRu = 0\ \forall u$. ■

Lemma 16 *If* $\mathcal{R} = \mathbf{R}^m$ *and* $Null(\Phi^{(N)}) = \{0\}$ *then* $L\mathcal{N}R = 0$ *implies that* $\mathcal{N} = 0$, *i.e. these conditions cannot occur if* $\mathcal{N} \neq 0$.

Proof: Since $Null(\Phi^{(N)}) = \{0\}$ there exists a left inverse $\Psi^{(N)}$ such that $\Psi^{(N)}\Phi^{(N)} = I$. Which implies $\Psi^{(N)}(L^{(N)}\mathbf{X}R^{(N)}, \Delta\mathbf{X}) = \mathbf{X},\ \forall\mathbf{X}$. Since the gradients G_k are in $\mathcal{X}$, $\Psi^{(N)}(L^{(N)}GR^{(N)}, \Delta G) = G$ also holds for all $G \in \mathcal{G}$. But $L^{(N)}GR^{(N)} = 0\ \forall G \in \mathcal{G}$ which implies there exists matrices A, B such that $G = \begin{bmatrix} A \\ B \end{bmatrix} \Delta G$.

Define $g_k = \begin{bmatrix} G_k & \cdots & G_{k+N-1} \end{bmatrix}^T$ and $\delta_k = (g_k - g_{k+1})$. Then

$$g_{k+1} = B\,\delta_k \quad \forall k \tag{17}$$

or alternatively $(I + B)g_{k+1} = B\,g_k,\ \forall k$. Multiplying by $(I+B)$ from the left and using the previous equation recursively yields

$$(I + B)^2 g_{k+1} = (I + B)Bg_k = B(I + B)g_k = B^2 g_{k-1}$$

this procedure can be repeated and yields

$$(I + B)^l g_k = B^l g_{k-l},\ \forall k,\ \forall l \geq 0.$$

Note that $g_{k-l} = 0,\ \forall l \geq k + N$ since $G_j = 0,\ \forall j < 0$ by definition. Therefore

$$(I + B)^l g_k = 0,\ \forall k,\ \forall l \geq \max\{0, k + N\}.$$

Let p be the size of B and consider three cases:

1. If $k \leq -N$ then $(I + B)^p g_k = 0$ since $l = p \geq \max\{0, k + N\}$
2. If $-N \leq k \leq p - N$ then $(I + B)^p g_k = 0$ since $l = p \geq k + N \geq 0$
3. If $k \geq p - N$ then $(I + B)^{k+N} g_k = 0$ and using Lemma 13 $(I+B)^p g_k = 0$ since $k+N \geq p = size(I+B)$

Combined this proves that $(I + B)^p g_k = 0,\ \forall k$. Using Equation 17 again this implies

$$\begin{aligned} 0 &= (I + B)^p (g_k - g_{k+1}) \\ &= (I + B)^{p-1}[I(g_k - g_{k+1}) + B(\delta_k)] \\ &= (I + B)^{p-1}(g_k - g_{k+1} + g_{k+1}) \\ &= (I + B)^{p-1} g_k. \end{aligned}$$

Repeating this process finally yields $g_k = 0,\ \forall k$ Note that $g_k = 0$ also implies $G_t = \nabla\mathcal{N}|_{(Ru)(t)} = 0,\ \forall t$, i.e. the gradient of $\mathcal{N}$ is zero everywhere in $\mathbf{R}^m$ which implies that $\mathcal{N} \equiv 0$. ■

Lemma 18 *If* $\mathcal{R} = \mathbf{R}^m$ *and* $Null(\Phi^{(N)}) \neq \{0\}$ *then* $\exists\mathcal{N} \neq 0 : L\mathcal{N}R = 0$ *implies* $\exists X \in \mathcal{X} : X \neq 0$ *and* $LXR = 0$.

Proof: We will show that there is a converging sequence of matrices $X^{(N)}$ and that the limit of this sequence fulfills $LXR = 0$. Let $\mathbf{X}^{(N)} \in Null(\Phi^{(N)})$. Note that $\Delta\mathbf{X}^{(N)} = 0$ implies $X_i = X_j\ \forall i, j$. Because the elements X_i are identical for vectors in the nullspace it can be equivalently represented by the subspace $S^{(N)}$ defined to be the projection of $Null(\Phi^{(N)})$ onto its first element X_1. Note that $S^{(N)} \subseteq \mathcal{X}$ and for all $X^{(N)} \in S^{(N)}$ holds $L^{(N)}X^{(N)}R^{(N)} = 0$. Also note that $S^{(N+1)} \subseteq S^{(N)}$ and that $S^{(N)} \neq \{0\},\ \forall N$ due to Lemma 16. Let $s_N = dim(S^{(N)})$ then the sequence of integers $\{s_N\}$ is monotone non-increasing and bounded below by 1, therefor it converges after a finite number of steps, say κ, and $\exists X \in S^{(\kappa)} \subseteq \mathcal{X} : X \neq 0$ such that $LXR = 0$. ■

Theorem 19 *Let* L, R *and* $\mathcal{N}$ *be defined as above then*

$$(\exists\mathcal{N} \neq 0 : L\mathcal{N}R = 0) \Longleftrightarrow (\exists X \in \mathcal{X},\ X \neq 0 : LXR = 0)$$

Proof: ($\Longleftarrow$) If $\exists X \neq 0$ then define $\mathcal{N} = X$

($\Longrightarrow$) Assume $\exists\mathcal{N} \neq 0 : L\mathcal{N}R = 0$ which implies $L\{G_t\}R = 0$ with G_t defined as above.

First note that the case where $\nabla\mathcal{N}|_0 \neq 0$ is covered by Lemma 14 and that the case where $\mathcal{R} \neq \mathbf{R}^m$ is treated in Lemma 15.

If $\nabla\mathcal{N}|_0 = 0$ and $\mathcal{R} = \mathbf{R}^m$ then $Null(\Phi)$ is not trivial because this would contradict $\mathcal{N} \neq 0$ (Lemma 16) and $\exists X \in \mathcal{X},\ X \neq 0 : LXR = 0$ according to Lemma 18.

In conclusion if $\exists\mathcal{N} : L\mathcal{N}R = 0$ then either $X = \nabla\mathcal{N}|_0$ or $X = \begin{bmatrix} x^* & 0 \\ 0 & 0 \end{bmatrix}$ or $X \in S^{(\kappa)}$ (all in the set $\mathcal{X}$) fulfill $LXR = 0$. ■

References

[1] R. Bellman and K.J. Åström, "On Structural Identifiability," *Math. Biosci.*, v.7, pp. 329–339, (1970).

[2] S.A. Billings, S.Y. Fakhouri, "Identification of a Class of Non-linear Systems Using Correlation Analysis," *Proc. IEE,* v.125, pp.691-7 (1978).

[3] M. Boutayeb, M. Darouach, H. Rafaralahy, G. Krzakala, "A New Technique for Identification of MISO Hammerstein Model," *Proc. 1993 ACC,* San Francisco, CA, v.2, pp.1991-2 (Jun. 1993).

[4] S.P. Boyd and L.O. Chua, "Uniqueness of a Basic Nonlinear Structure," *IEEE Trans. Circuits Syst.*, v.CAS-30, n.9, pp. 649–651, (Sept. 1983).

[5] S.P. Boyd and L.O. Chua, "Uniqueness of Circuits and Systems Containing One Nonlinearity," *Transactions On Automatic Control*, v.AC-30, n.7, pp. 674–681, (July 1985).

[6] M. Chapell, K. Godfrey and S. Vajda, "Global Identifiability of Non-Linear Systems with Specific Inputs: a Comparison of Methods," *Math. Biosci.*, v.102, pp. 41–73, (1990).

[7] C. Chen, S. Fassois, "Maximum likelihood identification of stochastic Wiener-Hammerstein-type nonlinear systems," *Mech. Syst. Sig. Proc.*, v.6, n.2, pp.135-53 (Mar. 1992).

[8] M. Claassen, E. Wemhoff, A. Packard and K. Poolla, "Identifibility Issues in General Interconnected Systems," *American Control Conference*, pp. 230–234, (June 2000).

[9] S. Diop and M. Fliess , "Nonlinear Observability, Identifiability and Persistent Trajectories," *30th CDC*, Brighton U.K., pp. 714–719, (1991).

[10] H.G.M. Dötsch and P.M.J. Van den Hof, "Test for Local Structural Identifiability of High-Order Non-Linear Parametrized State Space Models,"*Automatica*, v.32, n.6, pp.875–883, (1996).

[11] W. Greblicki, "Non-parametric orthogonal series identification of Hammerstein systems," *Int. J. Syst. Sci.*, v.20, n.12, pp.2355-67 (Dec. 1989).

[12] M.S. Grewal and K. Glover, "Identifiability of Linear and Nonlinear Dynamical Systems," *IEEE Trans. Automatic Control Short Papers*, pp. 833–837, (1976).

[13] T.A. Johansen, "Identification of Non-linear Systems using Empirical Data and Prior Knowledge–An Optimization Approach," *Automatica*, v.32, n.3, pp.337–56, (Mar. 1997).

[14] A. Krzyzak, "Identification of discrete Hammerstein systems by the Fourier series regression estimate," *Int. J. Syst. Sci.*, v.20, n.9, pp.1729-44 (Sept. 1989).

[15] L. Ljung, *System Identification, Theory for the User*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, (1987).

[16] L. Ljung and T. Glad, "On Global Identifiability for Arbitrary Model Parameterizations," *Automatica*, v.30, n.2, pp. 265–276, (1994).

[17] K.S. Narendra, P.G. Gallman, "An Iterative Method for the Identification of Nonlinear Systems Using the Hammerstein Model," *IEEE Trans. Auto. Contr.*, v.11, n.7, pp.546-50, (Jul. 1966).

[18] A. Packard and J.C. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 9, pp. 71–110, 1993.

[19] M. Pawlak, "On the series expansion approach to the identification of Hammerstein systems," *IEEE Trans. Auto. Contr.*, v.36, n.6, pp.763-7 (Jun. 1991).

[20] J. Sjöberg, Q. Zhang, L. Ljung, A. Benveniste, B. Delyon, P.Y. Glorennec, H. Hjalmarsson, and A. Juditsdy, "Nonlinear Black-box Modeling in System Identification: a Unified Overview," *Automatica*, v.31, n.12, pp.1691–1724, (1995).

[21] T. Söderström and P. Stoica, *System Identification*, Prentice-Hall, London, (1989).

[22] P. Stoica, "On the Convergence of an Iterative Algorithm Used for Hammerstein System Identification," *IEEE Trans. Auto. Contr.*, v.26, n.4, pp.967-69 (Aug. 1981).

[23] G. Vandersteen and J. Schoukens, "Measurement and Identification of Nonlinear Systems consisting out of Linear Dynamic Blocks and One Static Nonlinearity," *IEEE Instrumentation and Measurement Technology Conference*, v.2, pp.853–8, (1997).

[24] E. Walter, "Identifiability of State Space Models," *Lecture Notes in Biomathematics*, v.46, Springer Berlin, (1981).

[25] E. Walter, "Identifiability of Parametric Models," *Pergamon Press*, Oxford.

[26] E. Wemhoff, A. Packard and K. Poolla, "On the Identification of Nonlinear maps in a General Interconnected System," *American Control Conference*, v.5, pp. 3456–3461, (June 1999).

[27] K. Zhou, J. Doyle and K. Glover, "Robust and Optimal Control", *Prentice-Hall*, Upper Saddle River, New Jersey, (1996).