Overview of Some Theoretical & Experimental Results on Modeling and Control of Shear Flows

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Abstract— **This paper reviews the state-of-the-art in the modeling and control of transitional and turbulent shear flows, with particular emphasis on mixing applications. The review is divided into two parts. The first part provides a literature survey of analytical and experimental approaches to the modeling and control of mixing in shear flows. The second part presents some recent results in the development of reduced-order models and model-based control of mixing in some prototypical shear flow problems.**

I. INTRODUCTION

The problem of active flow control has gathered significant research attention in recent times owing to the growing number of potential industrial applications. Combustion mixing control, form drag reduction in bluff body flows, skin friction drag reduction on large aerodynamic bodies, jet exhaust signature and noise reduction, are just a few of the applications that the aerospace industry stands to benefit in from the use of intelligent flow control concepts. Significant advances have been made in developing and understanding the flow physics underlying various practically relevant turbulence phenomena, using state-of-the-art experimental diagnostic techniques to probe complicated flow phenomena, and developing high fidelity numerical simulations for accurate predictions. Intuitive methods (largely experimental) to manipulate the turbulent shear flows have also been proposed and tested successfully in several situations. However, there remains a gap in the systematic development of flow control concepts and technology. This includes the lack of: (i) computationally inexpensive models containing the essential flow physics and means to evaluate the effectiveness of control solutions; (ii) models of control solutions with which to study trade-off's between what is desirable and achievable; and (iii) models of actuation concepts and their scaling. The combination of phenomenological modeling (grounded on flow physics), dynamical systems theory and control theory appears to be an attractive avenue to bring about an analytical and design framework for evaluating and recommending innovative and effective flow control concepts. It is believed that physics-based, reduced-order models, which are relatively inexpensive to compute and yet capture sufficient physics to be meaningful, are required for such a combination to be viable and bridge the aforementioned gap.

The modeling of shear flow phenomena encompasses the following approaches:

- Fine-grain discretization of the governing, Navier-Stokes equations, namely Direct Numerical simulations (DNS) [20].
- Larger grain Large Eddy Simulation (LES) [18] or Reynolds Averaged Navier-Stokes (RANS) [29] simulations where averaging is performed over small scales whose effect on larger scales that are kept in the computation is then modeled as additional terms in the equations of motion.
- Galerkin-type approaches such as Proper Orthogonal Decomposition (POD) [3] where the equations of motion are projected on the flow-specific basis and a system of ordinary differential equations are obtained by discretization.
- Numerical simulation using vortex elements [16], and vortex filaments [17].
- Coarse-grained phenomenological modeling such as the use of vortex models or stability-analysis based methods such the use of Parabolized Stability Equations (PSE) that capture the behavior of large-scale, coherent structure motion.

The last three aproaches can be amenable to analysis using control theory methods. For numerical approach to controlling turbulent channel flows using direct Navier-Stokes simulation, see [5]. For an approach using numerical vortex methods see [15]. For development of POD methods for control see for example [12].

An example of transitional and turbulent fluid dynamic phenomena of significant practical interest is flow separation. This problem represents a good example of a flow control application where substantial experimental evidence for the successful application of active control exists, with some numerical simulation verification, but model-based control is lacking. Applications for separation control include external flows such as flow past high-angle-of-attack airfoils, and internal flows such as aggressively expanding diffusers. Various means for delaying the onset of separation have been proposed, including passive methods such as vortex generators, and active methods such as steady blowing and suction techniques, and synthetic jets. Some studies have also explored the use of unsteady actuation for mitigating/delaying separation. The success of such control has been attributed to to the excitation of "favorable" large-scale organized motion (viz., coherent structures) in the flow [28]. Although some organization of the unsteady flow structures was evidenced and concommitant improvement in the performance (e.g. diffuser pressure recovery, airfoil lift to drag ratio) was seen, the benefits of using unsteady control were not explored in detail or explained satisfactorily. In particular, the dependence of the performance on unsteady control parameters such as forcing frequencies and their amplitudes were mostly not studied. Some numerical simulations of the use of unsteady control were also performed in high M, but low Re (with laminar boundary layers) separated flows demonstrating the effectiveness of unsteady control in organizing the separation dynamics ([13]). Such simulations tend to be quite expensive and do not lend themselves to parametric control analyses. The use of periodic oscillations to delay/reduce the extent of separation was also investigated by Wygnanski et al. (e.g. see [30], [24]), who have demonstrated the effectiveness of unsteady blowing in controlling flow separation. Flow physics-based explanations of the controlled flow and its dependence on the unsteady forcing parameters were proposed, but a dynamical model of the observed flow phenomena has not emerged. Consequently, a phenomenological/empirical/analytical model of the effects of unsteady forcing on the separation dynamics is yet unavailable. Thus, optimization of the control parameters and recommendation of new unsteady separation control strategies continue to rely heavily on experimentation in laboratory scale models, with the hope that favorable results will be obtained in full-scale devices. The lack of reduced order models that capture the essential separated flow physics, permiting parametric analysis to explore new control strategies, represents a challenging gap.

One of the major challenges in utilizing unsteadiness in the control of separation is therefore to deliver physics-based models of controlled flows to describe the modification of separation through actuation, and the effect of control on the formation and evolution of coherent structures and hence the impact on the performance functional. The "quality" of physics-based models of controlled flows should be judged, particularly for practical purposes, by their ability to guide control design and predicting the anticipated levels of actuation to achieve prescribed performance benefits. Scaleability of such models with flow conditions (e.g. change in the geometry, flow speed) is essential.

Our own approach, influenced by studies of Cortelezzi et al. (e.g. see [7], [8] and the references therein), is based primarily on modeling and reduced-order numerical simulations using vortex elements. The numerical approaches have also been coupled to control theory utilizing mostly linear control theory concepts for the purpose of stabilization: by now there is a large amount of literature on this topic [9], [4]. However, as pointed out before, in many shear flow control problems the issue is that of *destabilization* of existing structures to create new ones i.e. using the internal dynamics of the system to create favorable flow conditions with small control input. This task often requires understanding of transport and mixing of fluid particles in the flow. Dynamical systems theory of mixing has been developed over the last 15 years, mostly in the context of Hamiltonian systems. As two-dimensional vortex models are given in the form of a set of ordinary differential equations that represent a Hamiltonian system, these methods are directly applicable. Control theory of Hamiltonian systems has seen substantial development (see e.g. [23]). These two theories can be coupled fruitfully with vortex models to provide a control-theoretical framework for control of mixing. Thus the vortex-model based approaches, while on the coarsest scale as far as modeling is concerned, are amenable to theoretical treatment.

In the rest of the paper we describe some studies based on reduced-order modeling, mostly through vortex element modelling, and discuss the achievements and shortcomings.

II. CONTROL OF MIXING

A. *Optimize-then-control strategy*

One of the problems in shear flow and mixing control is that often the required procedure is open loop. In addition, even if feedback is possible, the models tend to be complex and control design prohibitively complicated. One of the possible approaches in this context is to design simplified flow models, optimize mixing within the context of the model (thus finishing the task in the case of open-loop design) or design a stabilizing controller for the optimal flow. However, see the recent attempt to use feedback for mixing in the Poiseuille flow [1].

The theoretical study of control of mixing was initiated in [10] where Kolmogorov-Sinai entropy was proposed as a measure of mixing for closed flows. There are two basic requirements a measure of mixing must satisfy: it should increase with the mixed area, and with the rate of mixing. The Kolmogorov-Sinai entropy satisfies both of those properties. A basic mechanism of mixing is that of composition of shear flows in different directions (the stretch-and-fold mechanism [26]). Different *mixing protocols* of two-dimensional shear flows to determine maximal entropy at constant energy protocol were investigated. This result was motivated by a question posed by Ottino [26] in the context of mixing in commercial mixers. The question is easily posed in the standard control theory context: consider a control vector field $\dot{x}$ on a manifold M satisfying

$$\dot{x} = u(t)v_1 + (1 - u(t))v_2,$$

where $u(t)$ is the control function admitting values 0 or 1 on time intervals of length 1 and v_1 and v_2 are vector fields on M. In Fig. 1 we show schematically shear velocity fields $v_1 = (ay, 0), v_2 = (0, bx)$ with $a, b > 0$, the problem studied in [10]. The problem of maximal mixing is finding $u(t)$ that maximizes the Kolmogorov-Sinai entropy h that for smooth flows can be expressed as an integral over M of the sum of positive Lyapunov exponents. For example, for two-dimensional, area-preserving v_i there is one positive Lyapunov exponent λ^+ depending on $x \in M$ and

$$h = \int_M \lambda^+ d\mu,$$

where $d\mu$ is the two form that v_i preserve.

However, recall that λ^+ is an average of the norm of the derivative of the map along the trajectories of the system and thus the Kolmogorov-Sinai entropy is a complicated quantity depending on particle paths in fluid flows - it might not be amenable to simple optimal control design. Thus the philosophy of optimize-then stabilize is the most

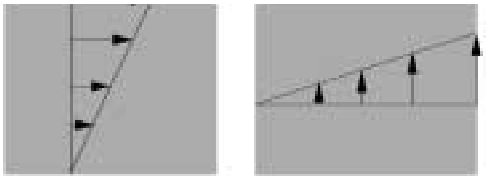

Fig. 1. Shear flows for entropy maximization.

useful one when Kolmogorov-Sinai Entropy is used in the cost function.

There are other measures of mixing that one can use, one of them being the mixing variance coefficient (MVC) described later, that was introduced in [21], useful in open flows. Optimization of mixing with respect to yet another measure, useful when transport between different regions of the flow domain needs to be maximized, is described next.

A.1 Controlling vortex motion and chaotic advection

In the study [25] the question of *optimizing and controlling* transport and mixing in two-dimensional flows was pursued. Akin to first chaotic advection studies [2], [27] a simple vortex model - a single vortex in a corner subject to a potential field was studied. In the absence of time-dependent forcing (introduced by modulating the potential field) the vortex is either at a stable equilibrium position or is moving on a periodic trajectory in the plane.

The first question that is asked is if the vortex motion, described by a Hamiltonian system of equations, is controllable. The answer was shown to be positive using the transformation to *flat coordinates* [23], [11]. Having this result, one can search over all the trajectories of the vortex in a bounded domain of the plane and determine the optimal one with respect to a suitable measure. In this work the measure was chosen to be the flux through the separatrix, shown in green in Fig. 2 as a function of the position of the vortex on the optimal vortex curve (blue) thus linking control theory and chaotic advection theory. Once the optimal trajectory was found, it was stabilized using a feedback law. The control input for this is shown in the figure in red, again as a function of the vortex position. The search for the optimal trajectory - which was restricted to lie within the black circle to preserve stability of the recirculation zone - was pursued using the numerical simplex method. Various characterizations of mixing utilizing dynamical systems concepts were pursued.

III. Model-based control of mixing in a jet in cross-flow

When a jet emanates from an orifice in a wall into a uniform crossflow along the wall (and perpendicular to the jet), it bends in the direction of the cross-flow. The jet trajectory depends on the momentum flux ratio (jet relative to cross-flow). A study was conducted in an attempt to model some of the large-scale phenomenological features in this flow to develop a reduced-order model and a control theory-based framework for mixing enhancement between the two streams. Details will be reported elsewhere ([14]). Here we briefly discuss the model and some key results.

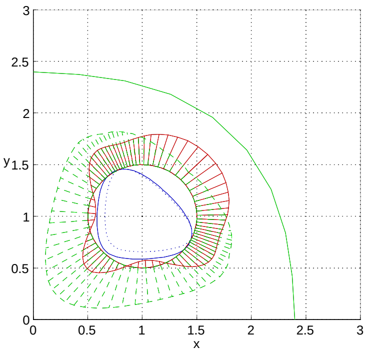

Fig. 2. Optimal mixing in the vortex-in-a-corner flow.

The dynamically significant flow structures appear to be the shear layer structures generated by the Kelvin-Helmholtz (KH) instability, the counter-rotating vortex pair (CVP), a typically weak horseshoe vortex (appearing for low jet Reynolds numbers near the jet and cross-flow bottom juncture), and an upright/wake vortex system that develops very close behind the jet (resembling a wake of vortices). Of these, we only discuss and model the KH and CVP dynamics, arguing that these shear flow structures dominate the flow mixing characteristics. In doing so, we use inviscid descriptions of the flow features and structures, ignoring fine-scale turbulence driven viscous effects. We do not attempt to model or explain the appearance of the KH or CVP structures, but, given their presence, we provide a modeling framework for studying the KH-CVP system dynamics to analyze/derive control solutions for mixing enhancement.

The array of vortex rings forming on the edge of the jet were represented simply by "horizontal" (transverse to the jet direction) rods of vorticity, representing the upstream KH ring and two growing (stretching)"legs" to represent the jet portion that has been bent along the jet direction (see Fig. 3 for an illustration of this conceptual model). The transverse rod is moving at a speed of U/2 (typical convection velocity of structures in a shear layer) and the legs are growing with respect to the rod at a rate of U/2 (since they are in a portion of the jet which is moving at a higher velocity, say U)

The parabolized vortex model itself was given as follows. If we fix the momentum ratio (jet to cross-stream), the control parameters are λ_{KH} which is the spacing of the KH vortices, ϕ_{KH} which indicates the phase difference between the KH crossings and the pulsation of the CVP at a fixed (control) frequency, Γ_0 which denotes the initial CVP circulation and $\sigma_{0_{KH}}$ which is the initial core radius for the KH roll-up (a function of the initial boundary layer thickness). The expressions for the velocity fields can be written as

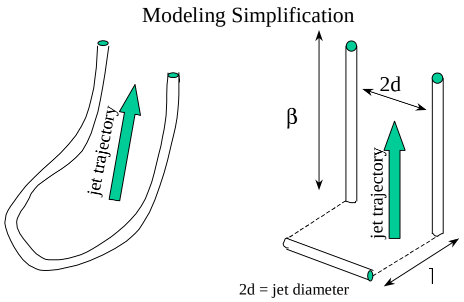

Fig. 3. A simplified model for unsteady flow structures of a jet in cross flow.

$$v_x(x,y,t) = \frac{-\Gamma_{CVP}}{2\pi}\frac{y}{r_{pos}^2}(1-e^{\frac{-r_{pos}^2}{\sigma_{CVP}^2}}) + \frac{\Gamma_{CVP}}{2\pi}\frac{y}{r_{neg}^2}(1-e^{\frac{-r_{neg}^2}{\sigma_{CVP}^2}})$$

where

$$\begin{aligned} r_{pos}^2 &= (x+d)^2 + y^2 \\ r_{neg}^2 &= (x-d)^2 + y^2 \\ \sigma_{CVP}^2 &= \sigma_0^2 + 4\nu t \\ \Gamma_{CVP} &= \Gamma_0 + (\frac{\Gamma_{KH}U}{2\lambda_{KH}})t \end{aligned}$$

$$\begin{aligned} v_y(x,y,t) = {} & \frac{\Gamma_{CVP}}{2\pi}\frac{x+d}{r_{pos}^2}(1-e^{\frac{-r_{pos}^2}{\sigma_{CVP}^2}}) \\ & + \frac{-\Gamma_{CVP}}{2\pi}\frac{x-d}{r_{neg}^2}(1-e^{\frac{-r_{neg}^2}{\sigma_{CVP}^2}}) \\ & - \frac{U}{6}\sin\theta + \frac{\Gamma_{KH}}{2\pi}\sum_{i=1}^{Ntot}\frac{-z_i}{r_i^2}(1-e^{\frac{-r_i^2}{\sigma_i^2}}) \end{aligned}$$

where

$$\begin{aligned} r_i^2 &= (y-l)^2 + z_i^2 \\ z_i &= (i-1)\lambda_{KH} - \frac{U}{2}t - \delta \\ \sigma_i^2 &= \sigma_{KH_0}^2 + 4\nu t \\ \delta &= \frac{\phi_{KH}}{2\pi}\lambda_{KH}; \phi_{KH} \in [0, 2\pi) \end{aligned}$$

$$\theta = 0.0025s^3 - 0.0453s^2 + 0.1114s + 1.4518; s = Ut$$

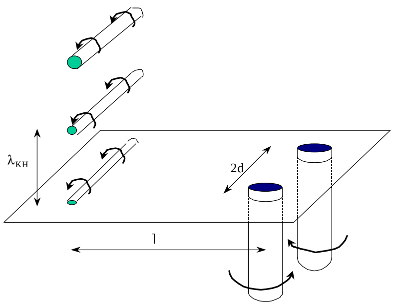

Fig. 4. Phenomenological model of jet in cross flow.

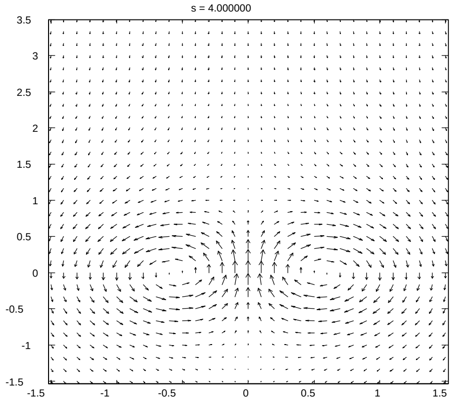

Fig. 5. The velocity field at the cross-plane $s = 4.0$.

All vortex cores were assumed to be distributed as Gaussian functions, taking into account that their core is growing in time (thereby mimicking viscous diffusion and avoiding singularities). A series of these KH rods were first arranged parallel to each other in a plane perpendicular to the model plane and allowed to cross the plane with a velocity of U/2. The CVP in the plane is growing and pulsing as described above (see Fig. 4). The corresponding velocity field at a downstream cross plane (with respect to the jet trajectory) is given in Fig. 5.

Open loop control studies were performed in an attempt to evaluate the effects of unsteady forcing on the jet mixing characteristics. Particles of one kind were introduced in a circular region representing the jet and particles of another kind were introduced in a circle around it to study mixing. The particles were then allowed to travel with the velocity field from the model and their final positions were tracked. The mixing variance coefficient (MVC, mentioned earlier and defined below) was used as a mixing metric. The MVC values can be compared to see what kind of actuation leads to better mixing (lower values indicate better mixing).

A region of size $6D_j \times 6D_j$ that covers the mixing region of interest is divided into $2^s \times 2^s$ equally sized bins. N particles are advected from $t = 0$ to $t = T_{final}$ with an adaptive step-size, 4^{th} order Runge-Kutta. N is kept constant by

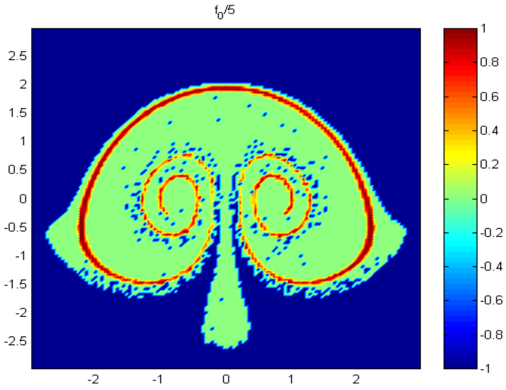

Fig. 6. Low frequency forcing case mixing results.

randomly reinserting particles that exit the mixing region.

The concentration in bin(k,l) at scale s is represented by

$$C^s_{kl} = \frac{n_{A_{kl}}}{n_{A_{kl}} + n_{B_{kl}}}$$

and the MVC at scale s is defined as

$$MVC^s = \frac{4}{2^s \times 2^s} \sum_k \sum_l (C^s_{kl} - 0.5)^2$$

Note that the MVC lies between zero and unity. The following figure (Fig. 6) displays a contour map of MVC (the extent of mixing) for a flow state forced with a "low" frequency, displaying large, spread regions of low MVC (near 0), indicating good entrainment and mixing. This result of effective mixing with low frequency forcing is in direct contrast to most prior experimental findings that propose the use of relatively high frequency excitation aimed at manipulating the shear layer structures alone. However, it agrees with the conclusions of the previously described vortex-in-a-corner study. These results are now being validated with Direct Numerical Simulations as well as experiments, where several unsteady forcing concepts will be explored. Being computationally inexpensive (by orders of magnitude compared to direct numerical simulations), such low order models are very valuable for parametric and control analyses.

IV. REDUCED-ORDER MODELING AND ACTIVE CONTROL OF FLOW SEPARATION

A reduced-basis model was developed ([6]) to predict pressure recovery enhancement in a planar diffuser for active separation control. A point vortex simulation was used to model the quasi-two dimensional flow. The effects of the expansion angle and the actuation parameters on the evolution of the coherent structure dynamics, averaged flow characteristics and pressure recovery were explored. These are reported in detail in [6]. Laboratory experiments conducted in a planar diffuser with a turbulent boundary layer at low Mach number and at high Reynolds number were used to validate the model.

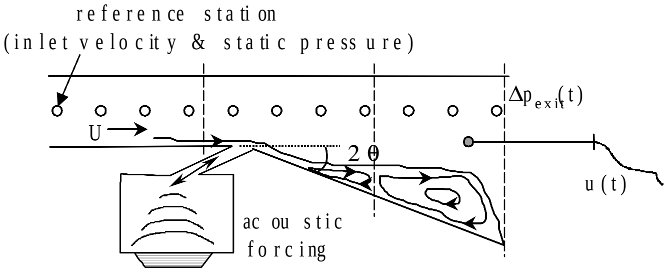

Fig. 7. Experimental facility schematic.

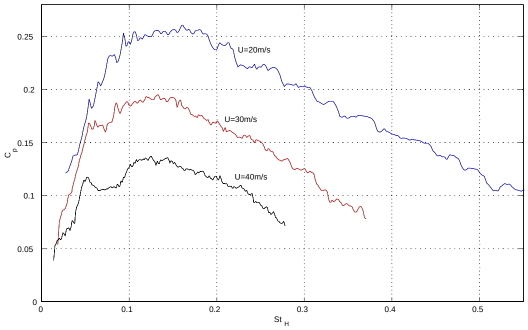

Fig. 8. *Pressure recovery coefficient (C_p) at the diffuser exit plane for the forced separated flow, showing sensitivity of performance to frequency.*

The experiments were performed in a two-dimensional air facility at UTRC. A facility schematic, displaying the velocity probe and static pressure tap locations, is shown in Fig. 7. The flow was operated in the range $30000 < Re_w < 140000$, with incoming fully developed turbulent boundary layers.

Wall mounted static pressure taps were used for time-averaged and unsteady pressure reocovery measurements. The pressure recovery coefficient C_p was computed as $\Delta p/(0.5\rho U^2)$, where Δp is the static pressure difference between the reference static pressure tap and that at any given downstream tap, and U is the inlet core velocity. Acoustic forcing was used for excitation, comprising a box with a wall-mounted speaker and a thin slot, providing effective forcing.

Open-loop control experiments were used to determine the response of the pressure recovery to various forcing frequency bands. Evidence for the formation of vortical structures and their interactions was found using smoke-visualization and unsteady velocity measurements. The C_p improvements are found to depend sensitively on the forcing frequency (Fig. 8); the diffuser included angle was approximately 23°.

A two-dimensional vortex method was used to simulate the vortex dynamics of the separated flow. The diffuser was modeled as a channel with an asymmetric diverging section. A representative flow field from a forced vortex simulation is shown in Fig. 9 displaying coherent vortex forma-

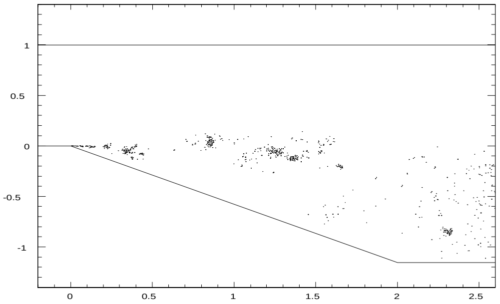

Fig. 9. *Vortex formation of a diffuser flow forced with a Strouhal frequency of $St = 0.3$.*

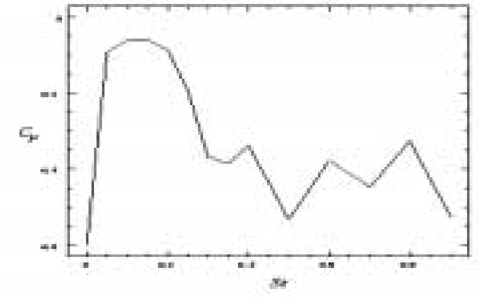

Fig. 10. *Pressure recovery coefficient as a function of forcing frequency in the range $St < 1$.*

tion. Forcing was imposed slightly upstream of the diffuser expansion corner at a modest excitation amplitude near $St = 0.3$. The circulation of these vortices is determined in agreement with the Kutta condition at the expansion corner. The vortex evolution can be computed in the physical domain with the Biot-Savart law and a panel method for the no-penetration condition at the diffuser wall. The diffuser domain is mapped on a half-plane using a conformal transformation, derived from the Schwarz-Christoffel formula. The evolution of the vortices is simulated in the half-plane; these details are discussed further in [6].

The two-dimensional vortex model successfully reproduced the dominant features of the forced separated flow. Experimentally observed flow response to actuation involving the formation of coherent vortices is qualitatively well represented. This reduced-order model seems to capture the shear layer dynamics, the organizing effect of external (unsteady) forcing on the large-scale vortex dynamics and the wall effects surprisingly well. Consequently, performance enhancement due to unsteady forcing (namely, sensitivity of the diffuser pressure recovery to the forcing frequency) is also captured well by the model. Figure 10 displays the pressure recovery enhancement as a result of forcing in various frequency bands, displaying the preferential enhancement near $St = 0.15$, as also noted in the experiments.

V. Conclusions

We have reviewed several approaches to model-based control of shear flows. Unique applications of low order modeling for control was demonstrated in three case studies encompassing mixing and high-speed flow control applications. While numerical and experimental efforts have produced interesting results, such as excitation of coherent structures under unsteady forcing, the description of these from a control theory or low order modeling perspective represents a big gap and challenge. Particularly, the lack of reduced-order models prevents the application of several analytical techniques developed in control theory of linear and nonlinear systems. Since the use of high-fidelity numerical simulations for control analyses continues to be a prohibitively expensive tool, a potentially fruitful approach is the use of low order models (e.g. "coarse" vortex methods). These dynamical systems are typically of Hamiltonian nature with control inputs and are thus amenable to control-theoretic analysis. The desired performance often requires good mixing. Measures of mixing are problem-dependent. Kolmogorov-Sinai entropy is a useful measure for closed flows, the mixing variance coefficient for open flows, and flux through distinguished surfaces for flows in which transport between flow regions is important. We should note however that cost functions involving mixing measures such as Kolmogorov-Sinai entropy can be quite complicated. If optimal control is the goal, open loop optimization with stabilization of the desired trajectory of the system might be the best approach. Flow physics-based conceptual, phenomenological models will be essential for formulating the reduced-order models (e.g. the jet in cross-flow and flow separation problems presented here).

In flow control problems, 'small input control' is often desired where expenditure of energy on control is much smaller than the total energy available in the flow. In this context, use of internal dynamics is critical. Chaotic internal dynamics might be of great use in such situations. The use of chaos control in open shear flows is a relatively recent development [22], where the use of low order unsteady flow models can be beneficial. Characterization of the internal dynamics in terms appropriate for control can be done in the context of ergodic theory [19].

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