l_∞ Problem for Finite-Word-length Controller Design

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Abstract: A l_∞ control problem is formulated for digital finite-word-length (FWL) controllers with synchronous sampling and fixed-point arithmetic. The l_∞ FWL controllers design problem is reduced to the problem of solving a Quadratic Matrix Inequality (QMI) such that the closed-loop system is asymptotically stable. The approach can be generalized to deal with other problems such as H_∞, H_2 and LQR FWL control problem.

Key words: Finite-word-length controller, l_∞ control, Quadratic Matrix Inequality, Stabilization, Digital control Theory

1. Introduction

Finite-word-length (FWL) controller design has been an important issue in modern control theory and engineering. The fixed-point arithmetic offers the advantages of speed, memory space, cost and simplicity over floating-point arithmetic [1]. However, due to FWL effects, a performance degradation of the closed-loop system are usually caused since the infinite-precision controller is implemented using a fixed-point processor. Recently many results have been reported in the literature dealing with FWL controller implementation. There exists an optimal realization of a given controller, so that the synthesis in these optimal coordinates will minimizes a proposed coefficient sensitivity measure or the noise gain from round-off effects[2] [3]. A controller designed without regard to controller synthesis can then be implemented in an optimal realization for this given controller. It is known that such controllers are not optimal overall. The design and synthesis problems are not independent problems. Some improvements are made in [4], where the finite-word-length covariance control has been studied. All dynamic controllers which assign state covariances to closed-loop system are characterized in the presence of quantization error in the control computer and in the A/D and D/A devices. It also presents a general design idea, that is, control problems reduce to a problem in linear algebra.

In this paper, we consider the l_∞ control design problem when the controller synthesized in a digital computer with synchronous sampling and fixed-point arithmetic. It is shown that the problem reduces to a problem of solving a Quadratic Matrix Inequality (QMI).

2. Problem Formulation

Consider the linear time-invariant system and the controller (with the assumption of infinite precision implementation)

$$\begin{bmatrix} x_p(k+1) \\ y_p(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_p & D_p & B_p \\ C_p & D_y & 0 \\ M_p & D_z & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ w_p(k) \\ u(k) \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k) \\ z(k) \end{bmatrix} \quad (2)$$

where x_p and x_c are the plant and the controller states, respectively, y_p is output of interest, w_p is the finite energy disturbance, z and u are the measured output and the control input, respectively.

However, in most application cases, the controller is synthesized in a digital computer with finite wordlength and fixed-point arithmetic, and we must take the quantization errors into consideration. It is well known [2] [3] that the effects of the quantization error in the control computer depend on the realization of the controller. To this end, we shall study the control design problem in a transformed set of controller parameters $A_c = T_c^{-1}\tilde{A}_c T_c$, $B_c = T_c^{-1}\tilde{B}_c$, $C_c = \tilde{C}_c T_c$, $D_c = \tilde{D}_c$ and write the controller dynamics

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k)+e_x(k) \\ z(k)+e_z(k) \end{bmatrix} \quad (3)$$

where $e_x(k)$ is the quantization error introduced by the controller state computation $x_c(k)$ in the control computer (with wordlength β_x),and $e_z(k)$ is the quantization error introduced by the A/D converter (with wordlength β_z).

The plant is described by

$$\begin{bmatrix} x_p(k+1) \\ y_p(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_p & D_p & B_p \\ C_p & D_y & 0 \\ M_p & D_z & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ w_p(k) \\ u(k)+e_u(k) \end{bmatrix} \quad (4)$$

where $e_u(k)$ is the quantization error introduced by the D/A converter (with wordlength β_u). Under sufficient excitation conditions we can approximate the quantization errors $e_x(k)$, $e_z(k)$ and $e_u(k)$ are zero-mean white noise processes $W_x=diag[\ldots q_i \ldots]$, $W_z=q_z I$, $W_u=q_u I$, respectively, where $q_i = (1/12)2^{-2\beta_i}$, $q_z = (1/12)2^{-2\beta_z}$, $q_u = (1/12)2^{-2\beta_u}$, and β_i is the length of the fractional part of the word storing the i-th controller state variable [4].

Define the matrix

$$W=\begin{bmatrix} q_u I & 0 & 0 & 0 \\ 0 & W_p & 0 & 0 \\ 0 & 0 & W_z+V & 0 \\ 0 & 0 & 0 & T_c W_x T_c^T \end{bmatrix}$$

where W_p and V are the covariance of the plant noise $w_p(k)$ and the measurement noise $D_z w_p(k)$, respectively. The closes-loop system is described by

$$x(k+1)=(A+BGM)x(k)+(D+BGE)w(k) \quad (5)$$

$$y(k)=Cx(k)+Fw(k), \quad (6)$$

with $A=\begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} B_p & 0 \\ 0 & I \end{bmatrix}$, $C=\begin{bmatrix} C_p & 0 \end{bmatrix}$, $D=\begin{bmatrix} B_p & D_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $F=\begin{bmatrix} 0 & D_y & 0 & 0 \end{bmatrix}$, $M=\begin{bmatrix} M_p & 0 \\ 0 & I \end{bmatrix}$, $E=\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$, $T=\begin{bmatrix} I & 0 \\ 0 & T_c \end{bmatrix}$, $\tilde{G}=\begin{bmatrix} D_c & \tilde{C}_c \\ \tilde{B}_c & \tilde{A}_c \end{bmatrix}$, $G=T^{-1}\tilde{G}T=\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$

and

$$x=\begin{bmatrix} x_p^T & x_c^T \end{bmatrix}^T,\ w=\begin{bmatrix} e_u^T & w_p^T & e_z^T+D_z^t w_p^T & e_x^T \end{bmatrix}^T,\ y=y_p$$

The FWL l_∞ control problem can be states as follows.

Let a performance bound $\gamma>0$ be given. Determine whether or not there exists a controller in (4) which asymptotically stabilizes the system (3) and yields the output *y* such that $\|y\|_{l_\infty}<\gamma$ for any disturbances *w* with $\|w\|_{l_2}<1$. Parametrize all such controllers when one exists.

3. Main Results

This problem reduces the following

Theorem 1: Let a controller G and a performance bound $\gamma>0$ be given. Then the following statements are equivalent: (1)The controller G solves the FWL l_∞ control problem; (2) There exists a matrix X>0 such that $\|CXC^T+FF^T\|<\gamma^2$ and the following Quadratic Matrix Inequality holds

$$(\Theta+\Gamma G\Lambda)R(\Theta+\Gamma G\Lambda)^T<Q$$

where $\Theta=[A\ D]$, $\Gamma=B$, $Q=X$, $\Lambda=[M\ E]$, $R=\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}$

Proof: The result follows from Lemma 1.

Lemma 1: Consider system $\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$ and suppose the system is asymptotically stable. Then the energy-to-peak gain $\gamma_{ep}\overset{\Delta}{=}\sup_{\|w\|_{l_2}\le 1}\|y\|_{l_\infty}$ is given by

$$\gamma_{ep}=\|CXC^T+DD^T\|^{1/2},\ X=AXA^T+BB^T$$

or alternatively

$$\gamma_{ep}=\inf_Q\left\{\|CQC^T+DD^T\|^{1/2}:Q>AQA^T+BB^T\right\}$$

Proof: see [5] [6].

Now the FWL l_∞ control problem have been reduced to a single problem of solving a matrix inequality $(\Theta+\Gamma G\Lambda)R(\Theta+\Gamma G\Lambda)^T<Q$ for the controller parameter *G*, where the other matrices are appropriately defined in terms of the plant data and a Lyapunov matrix X. The QMI can be directly solved if the matrix X satisfying some certain existence conditions. Once such an X is found, then the controller can be computed by the formula given in the literatures including [4], [7], [8] etc. The literatures also present unified methods to search the Lyapunov matrix.

4. Conclusion Remarks

In this paper, we have shown that the FWL l_∞ control problem can be reduced to a QMI problem. The matrix inequality approach to the l_∞ control problem is essentially from [9], the l_∞ control problem with finite wordlength and fixed-point arithmetic considerations, however, is first discussed in this paper. For discrete-time systems, there are lots of similar problems to be solved, such as LQG control problem, H_∞ control problem, H_2 control problem , subject to synchronous or skewes sampling between measurement and control, and subject to finite pricision computing in the A/D, D/A devices and in the controller state noise with variance related with the wordlength.

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