AN ADAPTIVE FILTER FOR CALIBRATION AND ESTIMATION OF REDUNDANT SIGNALS

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ABSTRACT

This paper presents formulation of an adaptive filter for real-time calibration of redundant signals consisting of sensor data and/or analytically derived measurements. The measurement noise covariance matrix is adjusted as a function of the *a posteriori* probabilities of failure of the individual signals. An estimate of the measured variable is obtained as a weighted average of the calibrated signals. The weighting matrix is recursively updated in real time instead of being fixed *a priori*. The filter software is presently hosted in a Pentium platform and is portable to other commercial platforms. The filter can be used to enhance the Instrumentation & Control System Software in tactical and transport aircraft, and nuclear and fossil power plants

INTRODUCTION

Redundant sensors are often installed to generate spatially averaged time-dependent estimates of critical variables for reliable monitoring and control of complex dynamical processes such as aircraft [Daly et al., 1979] and power plants [Deckert *et al*., 1983]. Sensor redundancy is often augmented with analytical measurements that are obtained from physical characteristics and/or model of the plant dynamics in combination with other available sensor data [Desai *et al*., 1979; Ray *et al*., 1983; Ray and Desai, 1986]. Both sensor and analytical redundancies are referred to as redundant signals in the sequel.

Individual signals in a redundant set may often exhibit deviations from each other after a length of time. These differences could be caused by slowly time-varying sensor parameters (e.g., amplifier gain), plant parameters (e.g., structural stiffness, and heat transfer coefficient), transport delays, etc. Consequently, some of the redundant signals could be deleted by a fault detection and isolation (FDI) algorithm if these signals are not periodically calibrated. On the other hand, failure to isolate a degraded signal (for example, due to increased threshold bound in the FDI algorithm) could cause an inaccurate estimate of the measured variable. In this case, the plant performance and stability may be adversely affected if the inaccurate estimate is used as an input to the decision and control system.

This paper presents a calibration and estimation filter for redundancy management of sensor data and analytical measurements. The salient features of the filter are delineated below.

- All signals are simultaneously calibrated on-line to compensate for their relative errors.
- The weights of individual signals for computation of a least-square estimate of the measured variable are adaptively updated as functions of the respective *a posteriori* probabilities of failure.

In the event of an abrupt disruption of a redundant signal in excess of its allowable bound, the respective signal is isolated by the FDI algorithm, and only the remaining signals are calibrated to provide an unbiased estimate of the measured variable. For a gradual degradation (e.g., a sensor drift), the faulty signal is not immediately isolated but its influence on the estimate and calibration of the remaining signals is diminished as a function of its deviation from the remaining signals. This is achieved by decreasing the relative weight of the degraded signal as a monotonic function of the magnitude of its residual (i.e., deviation from the estimate) that is a measure of its relative degradation. Thus, if the error bounds of the FDI algorithm are appropriately increased to reduce the probability of false alarms, an undetected fault would have smaller bearing on the accuracy of signal calibration and estimation as a result of the adaptively reduced weight. Therefore, the resulting delay in detecting a gradual degradation could be tolerated as the weighted estimate is practically unaffected. Furthermore, since the weight of a gradually degrading signal is smoothly reduced, the eventual isolation of the fault would not cause any abrupt change in the estimate.

The calibration and estimation filter is validated based on redundant sensor data of throttle steam temperature collected from an operating power plant. Development and validation of the filter algorithm are presented in the main body of the paper along with concluding remarks. The

Appendix presents the theory of multiple hypotheses based on *a posteriori* probability of failure of a single signal.

SIGNAL CALIBRATION AND ESTIMATION

A redundant set of ℓ sensors and/or analytical measurements of a n-dimensional plant variable are modeled at the k^{th} sample as:

$$m_k = (H + \Delta H_k)\, x_k + b_k + e_k \tag{1}$$

where

m_k is the $(\ell \times 1)$ (uncalibrated) redundant signals;

H is the $(\ell \times n)$ *a priori* determined matrix of scale factor having rank n, with $\ell > n \geq 1$;

DH_k is the $(\ell \times n)$ matrix of scale factor errors;

x_k is the $(n \times 1)$ true value of the measured variable;

b_k is the $(\ell \times 1)$ vector of bias errors; and

e_k is the $(\ell \times 1)$ measurement noise, such that

$$E[e_k] = 0 \text{ and } E[e_k\, e_l^T] = R_k d_{kl}\,.$$

Remark 1: The noise covariance matrix R_k of uncalibrated signals plays an important role in the adaptive filter for both signal calibration and estimation. Later it is shown how R_k is recursively tuned based on the history of calibrated signals. ■

Equation (1) is rewritten in a more compact form as:

$$m_k = H\, x_k + c_k + e_k \tag{2}$$

where the correction c_k due to the combined effect of bias and scale factor errors is defined as:

$$c_k \equiv \Delta H_k\, x_k + b_k \tag{3}$$

A recursive relation of the correction c_k is modeled similar to a random walk process as:

$$\begin{aligned} &c_{k+1} = c_k + v_k \\ &E[v_k] = 0;\; E[v_k v_j^T] = Q\, d_{kj} \text{ and } E[v_k\, e_j^T] = 0 \;\forall k, j \end{aligned} \tag{4}$$

where the stationary noise v_k represents modeling uncertainties. The objective is to obtain an unbiased predictor estimate $\hat{c}_k$ of the correction c_k so that the signal m_k can be calibrated at each sample.

Let us construct a filter that calibrates each signal with respect to the remaining redundant signals. The filter input is the parity vector p_k of the uncalibrated signal m_k, which is defined [Potter and Suman, 1977; Ray and Desai, 1986; Ray and Luck, 1991] as:

$$p_k = V\, m_k \tag{5}$$

where the rows of the projection matrix $V \in \Re^{(\ell - n)\times \ell}$ form an orthonormal basis of the left null space of the scale factor matrix $H \in \Re^{\ell \times n}$ in Eq. (1)., i.e.,

$$VH = 0_{(\ell-n)\times n} \text{ and } VV^T = I_{(\ell-n)\times(\ell-n)} \tag{6}$$

and the columns of V span the parity space that contains the parity vector. A combination of Eqs. (2), (4), (5) and (6) yields:

$$p_k = V\, c_k + e_k \tag{7}$$

where the noise $e_k \equiv V\, e_k$ having $E[e_k] = 0$ and $E\left[e_k\, e_j^T\right] \equiv V R_k V^T d_{kj}$.

Remark 2: If the scale factor error matrix DH_k belongs to the column space of H, then the parity vector p_k is independent of the true value x_k of the measured variable. Therefore, for $\|VDH_k\, x_k\| << \|V b_k\|$ that includes small scale factor errors, the calibration filter operates approximately independent of x_k. ■

Now we proceed to construct a recursive algorithm to predict the estimated correction $\hat{c}_k$ based on the principle of a linear least square estimator that has the structure of an optimal minimum-variance filter [Jazwinski, 1970; Gelb, 1974] and uses Eqs. (4) and (7):

$$\left.\begin{aligned} &\hat{c}_{k+1} = \hat{c}_k + K_k g_k && \text{given } \hat{c}_0 \\ &P_{k+1} = (I - K_k V) P_k + Q && \text{given } P_0 \text{ and } Q \\ &K_k = P_k V^T \left(V\,[R_k + P_k]\, V^T\right)^{-1} && \text{given } R_k \\ &g_k \equiv p_k - V\,\hat{c}_k && \text{innovation} \end{aligned}\right\} \tag{8}$$

Upon evaluation of the unbiased estimated correction $\hat{c}_k$, the uncalibrated signal m_k is compensated to yield the calibrated signal y_k as:

$$y_k = m_k - \hat{c}_k \tag{9}$$

Using Eqs. (5) and (9), the innovation g_k in Eq. (8) can be expressed as the projection of the calibrated signal y_k onto the parity space, i.e.,

$$\gamma_k = V y_k \tag{10}$$

By setting $\Gamma_k \equiv K_k V$, we obtain an alternative form of the recursive relations in Eq. (8) as:

$$\left.\begin{aligned} \hat{c}_{k+1} &= \hat{c}_k + G_k\, y_k && \text{given } \hat{c}_0 \\ P_{k+1} &= (I - G_k)P_k + Q && \text{given } P_0 \text{ and } Q \\ G_k &= P_k V^T \left(V\left[R_k + P_k\right]V^T\right)^{-1} V && \text{given } R_k \end{aligned}\right\} \quad (11)$$

Remark 3: The matrix $\left(V\left[R_k + P_k\right]V^T\right)^{-1}$ in Eqs. (8) and (11) exists because the rows of V are linearly independent, $R_k > 0$, and $P_k \geq 0$. ■

Next we obtain an unbiased weighted least squares estimate $\hat{x}_k$ of the measured variable x_k based on the calibrated signal y_k as:

$$\hat{x}_k = \left(H^T R_k^{-1} H\right)^{-1} H^T R_k^{-1} y_k \qquad (12)$$

Remark 4: The inverse R_k^{-1} of the measurement covariance matrix R_k serves as the weighting matrix for generating the estimate $\hat{x}_k$ and is used as a filter matrix. ■

Remark 5: Compensation of a (slowly varying) undetected error in the j^{th} signal out of ℓ redundant signals causes the magnitude $\left|{}_j\hat{c}_k\right|$ in the correction vector $\hat{c}_k$ to be the largest. Therefore, a limit check on each element of $\hat{c}_k$ allows detection and isolation of the degraded signal(s). The bounds of limit check, which could be different for individual elements of $\hat{c}_k$, are selected by trade-off between the probability of false alarms and the allowable error in the estimate $\hat{x}_k$ of the measured variable. ■

Degradation Monitoring

Following Eq. (12), we define the residual h_k of the calibrated signal y_k as:

$$\eta_k = y_k - H\hat{x}_k \qquad (13)$$

The residuals represent a measure of relative degradation of individual signals. For example, under normal conditions, all calibrated signals are clustered together, i.e., $\|h_k\| \approx 0$, although this may not be true for the residuals $\left(m_k - H\,\hat{x}_k\right)$ of uncalibrated signals.

While large abrupt changes in excess of the error threshold are easily detected and isolated by a standard diagnostics procedure (e.g., Ray and Desai (1986)), small errors (e.g., slow drift) can be identified from the *a posteriori* probability of failure of the calibrated signals. The *a posteriori* probability of failure is recursively computed from the history of residuals based on the following ternary hypotheses:

$$\begin{aligned} &H^0: \textit{ Normal behavior pdf } \; {}_jf^{\,0}(\bullet) \equiv {}_jf\left(\bullet \,\middle|\, H^0\right) \\ &H^1: \textit{ High (positive) failure pdf } \; {}_jf^{\,1}(\bullet) \equiv {}_jf\left(\bullet \,\middle|\, H^1\right) \quad (14) \\ &H^2: \textit{ Low (negative) failure pdf } \; {}_jf^{\,2}(\bullet) \equiv {}_jf\left(\bullet \,\middle|\, H^2\right) \end{aligned}$$

where the left subscript refers to of the j^{th} signal for $j = 1, 2, \cdots, \ell$, and the right superscript indicates the normal or failure mode. The density function for each residual is determined *a priori* from experimental data and/or instrument manufacturers' specifications. Only one test is needed here to accommodate both positive and negative failures in contrast to the binary hypotheses that require two tests.

Let us apply the recursive relations for multi-level hypotheses testing of single variables, derived in the Appendix, to each signal residual. Then, for the j^{th} signal at the k^{th} sampling instant, *a posteriori* probability of failure ${}_j\Pi_k$ is obtained following Eq. (A-17) as:

$$\left.\begin{aligned} {}_j\Psi_k &= \left(\frac{{}_jp + {}_j\Psi_{k-1}}{2\,(1 - {}_jp)}\right)\left(\frac{{}_jf^{\,1}({}_jh_k) + {}_jf^{\,2}({}_jh_k)}{{}_jf^{\,0}({}_jh_k)}\right) \\ {}_j\Pi_k &= \frac{{}_j\Psi_k}{1 + {}_j\Psi_k} \end{aligned}\right\} \quad (15)$$

where ${}_jp$ is the *a priori* probability of failure of the j^{th} sensor during one sampling period, and the initial condition of each state, ${}_j\Psi_0$, $j = 1, 2, \cdots, \ell$, needs to be specified.

Based on the *a posteriori* probability of failure, we now proceed to formulate a recursive relation for the measurement noise covariance matrix R_k that influences both calibration and estimation as seen in Eqs. (8) to (12). Its initial value R_0, which is determined from experimental data and/or instrument manufacturers' specifications, provides the *a priori* information on individual signal channels and conforms to the normal operating mode when all calibrated signals are clustered together, i.e., $\|h_k\| \approx 0$. In the absence of any signal degradation, R_k remains close to its initial value R_0. Significant changes in R_k may take place if one or more signals start degrading. The following model captures this phenomenon:

$$R_k = \sqrt{R_k^{rel}}\; R_0 \;\sqrt{R_k^{rel}} \quad \text{with} \quad R_0^{rel} = I \qquad (16)$$

where R_k^{rel} is a positive-definite diagonal matrix representing only *a posteriori* relative information of the individual signal channels and is independent of the specific structure of the sensor system; R_k^{rel} is recursively generated as follows:

$$R_{k+1}^{rel} = diag\left[h({}_j\Pi_k)\right],\ i.e.,\ {}_j r_{k+1}^{rel} = h({}_j\Pi_k) \qquad (17)$$

where ${}_j r_k^{rel}$ and ${}_j\Pi_k$ are respectively the relative variance and *a posteriori* probability of failure of the j^{th} signal at the k^{th} instant; and $h:[0,1)\to[1,\infty)$ is a continuous monotonically increasing function with boundary conditions $h(0)=1$ and $h(\varphi)\to\infty$ as $j \to 1$.

Remark 6: The implication of Eq. (17) is that credibility of a signal monotonically decreases with increase in its variance that tends to infinity as its *a posteriori* probability of failure approaches 1. The magnitude of the relative variance ${}_j r_k^{rel}$ is set to the minimum value of 1 for zero *a posteriori* probability of failure. In other words, the j^{th} diagonal element ${}_j w_k^{rel} \equiv 1/{}_j r_k^{rel}$ of the weighting matrix $W_k^{rel} \equiv \left(R_k^{rel}\right)^{-1}$ tends to zero as ${}_j\Pi_k$ approaches 1. Similarly, the relative weight ${}_j w_k^{rel}$ is set to the maximum value of 1 for ${}_j\Pi_k$=0. Consequently, a gradually degrading sensor carries monotonically decreasing weight in the computation of the estimate $\hat{x}_k$ in Eq. (12). ■

Next we set the bounds on the states ${}_j\Psi_k$ of the recursive relation in Eq. (15). The lower limit of ${}_j\Pi_k$ (which is an algebraic function of ${}_j\Psi_k$) is set to the probability ${}_j p$ of intra-sample failure. On the other extreme, if ${}_j\Pi_k$ approaches 1, the weight ${}_j w_k^{rel}$ (that approaches zero) may prevent fast restoration of a degraded sensor following its recovery. Therefore, the upper limit of ${}_j\Pi_k$ is set to $(1-{}_j a)$ where ${}_j a$ is the allowable probability of false alarms of the j^{th} signal. Consequently, the function $h(\bullet)$ in Eq. (17) is restricted to the domain $[\,{}_j p, (1-{}_j a)]$ to account for probabilities of intra-sampling failures and false alarms. Following Eq. (15), the lower and upper limits of the states ${}_j\Psi_k$ thus become $\frac{{}_j p}{1-{}_j p}$ and $\frac{1-{}_j a}{{}_j a}$, respectively. Consequently, the initial state in Eq. (15) is set as: ${}_j\Psi_0 = \frac{{}_j p}{1-{}_j p}$ for $j = 1, 2, \cdots, \ell$.

SUMMARY AND CONCLUSIONS

This paper develops and validates an adaptive filter for on-line calibration of redundant signals and estimation of the measured plant variable. The redundancy may consist of both sensor signals and/or analytical measurements that are derived from other sensor signals and physical characteristics or a model of the plant. All redundant signals are simultaneously calibrated by additive corrections that are recursively estimated. A weighted least square estimate of the measured variable is generated in real time where the weighting matrix is adaptively adjusted as a function of the *a posteriori* probability of failure of the calibrated signals. The effects of intra-sample failure and probability of false alarms are taken into account in the recursive filter that has been tested for on-line calibration of four redundant sensors of the throttle steam temperature in a commercial-scale fossil power plant. The software is presently hosted in a Pentium platform and is portable to other commercial platforms. The important features of this real-time adaptive filter are summarized below:

- A model of the physical process is not necessary for calibration and estimation if sufficient redundancy of sensor data and/or analytical measurements is available.
- All signals are simultaneously calibrated on-line to compensate for their relative errors.
- The weights of individual signals for computation of a least-square estimate are adaptively updated as functions of the respective *a posteriori* probabilities of failure.

The proposed calibration and estimation filter can enhance the Instrumentation & Control System Software in tactical and transport aircraft, and nuclear and fossil power plants. However, a limitation of this filter is its inability to handle common mode faults (i.e., similar faults, possibly due to a common source, in all or a majority) of redundant sensors because the filter algorithm relies on the relative error among the individual signals. Additional information (e.g., analytic redundancy) is needed to deal with common-mode faults.

APPENDIX: Multiple Hypotheses Testing

Let $\{\eta_k,\ k = 1, 2, 3, \cdots\}$ be independent observations of a single variable (e.g., residual of a signal) at consecutive sampling instants. We assume M distinct possible modes of failure in addition to the normal mode of operation that is designated as the mode 0. Thus, at each sampling instant, there are $(M+1)$ mutually exclusive and exhaustive modes. Each of these $(M+1)$ modes is treated as a Markov state. The hypotheses of failure of $(M+1)$ modes at the k^{th} sample are defined for $, i=1,2,\cdots,M$ as follows:

$$H_k^0 : \textit{Normal a priori pdf } f^0(\bullet) \equiv f\left(\bullet \middle| H^0\right)$$
$$H_k^i : \textit{Abnormal a priori pdf } f^i(\bullet) \equiv f\left(\bullet \middle| H^i\right) \qquad \text{(A-1)}$$

We assume a one-one correspondence between the set of $(M+1)$ modes and the set of their failure hypotheses $H_k^j,\ j=0,1,2,\cdots,M$. In the sequel, the terms, mode and hypothesis, are synonymously used.

We define the *a posteriori* probability p_k^j of the j^{th} mode at the k^{th} sample as:

$$p_k^j \equiv P[H_k^j \mid Z_k],\ j=0,1,2,\cdots,M \qquad \text{(A-2)}$$

based on the cumulative information $Z_k \equiv \bigcap_{i=1}^{k} z_i$ over k samples where the events $z_i \equiv \{h_i \in B_i\}$ are mutually independent and B_i is the region of interest at the i^{th} sample. The problem is to derive a recursive relation for *a posteriori* probability Π_k of any one of the M failure modes at the k^{th} sample:

$$\Pi_k \equiv P\left[\bigcup_{j=1}^{M} H_k^j \mid Z_k\right]$$
$$= \sum_{j=1}^{M} P\left[H_k^j \mid Z_k\right] \Rightarrow \Pi_k = \sum_{j=1}^{M} \pi_k^j \qquad \text{(A-3)}$$

Equation (A-3) holds because of the exhaustive and mutually exclusive properties of the Markov states, $H_k^j,\ j=1,2,\cdots,M$. To construct a recursive relation for Π_k, we introduce the following definitions:

Joint probability: $x_k^j \equiv P[H_k^j,\ Z_k]$ (A-4)

A priori probability: $l_k^j \equiv P\left[z_k \middle| H_k^j\right]$ (A-5)

Transition probability: $a_k^{i,j} \equiv P\left[H_k^j \middle| H_{k-1}^i\right]$ (A-6)

Then, because of independence of the events z_k and Z_{k-1}, Eq. (A-4) takes the following form:

$$x_k^j = P[H_k^j,\ z_k, Z_{k-1}]$$
$$= P[z_k | H_k^j] P[H_k^j,\ Z_{k-1}] \qquad \text{(A-7)}$$

Furthermore, the exhaustive and mutually exclusive properties of the Markov states $H_k^j,\ j=0,1,2,\cdots,M$, and independence of Z_{k-1} and H_k^j lead to:

$$P\left[H_k^j, Z_{k-1}\right] = \sum_{i=0}^{M} P\left[H_k^j, H_{k-1}^i, Z_{k-1}\right]$$
$$= \sum_{i=0}^{M} P\left[Z_{k-1} \middle| H_{k-1}^i\right] P\left[H_k^j \middle| H_{k-1}^i\right] P\left[H_{k-1}^i\right] \qquad \text{(A-8)}$$
$$= \sum_{i=0}^{M} P\left[H_k^j \middle| H_{k-1}^i\right] P\left[H_{k-1}^i,\ Z_{k-1}\right]$$

The following recursive relation is obtained from a combination of Eqs. (A-4) to (A-8) as:

$$\xi_k^j = \lambda_k^j \sum_{i=0}^{M} a_k^{i,j}\ \xi_{k-1}^i \qquad \text{(A-9)}$$

We introduce a new term

$$y_k^j \equiv \frac{x_k^j}{x_k^0} \qquad \text{(A-10)}$$

that reduces to the following form by use of Eq. (A-9):

$$\psi_k^j = \left(\frac{\lambda_k^j}{\lambda_k^0}\right)\left(\frac{a_k^{0,j} + \sum_{i=1}^{M} a_k^{i,j}\,\psi_{k-1}^i}{a_k^{0,0} + \sum_{i=1}^{M} a_k^{i,0}\,\psi_{k-1}^i}\right) \qquad \text{(A-11)}$$

to obtain the *a posteriori* probability p_k^j in Eq. (A-2) in terms of x_k^j and y_k^j as:

$$p_k^j = \frac{P[H_k^j, Z_k]}{P[Z_k]} = \frac{P[H_k^j, Z_k]}{\sum_{i=0}^{M} P\left[H_k^i, Z_k\right]}$$
$$= \frac{x_k^j}{x_k^0 + \sum_{i=1}^{M} x_k^i} = \frac{y_k^j}{1 + \sum_{i=1}^{M} y_k^i} \qquad \text{(A-12)}$$

A combination of Eqs. (A-3) and (A-12), leads to the *a posteriori* probability Π_k of failure as:

$$\Pi_k = \frac{\Psi_k}{1+\Psi_k} \text{ with } \Psi_k \equiv \sum_{j=1}^{M} y_k^j \qquad \text{(A-13)}$$

The above expressions can be realized by a simple recurrence relation under the following four assumptions:

- ***Assumption A-1***: At the starting point (i.e., k=0), all signals operate in the normal mode, i.e., $P[H_0^0]=1$

and $P[H_0^j]=0$ for $j=1,2,\cdots,M$. Therefore, $x_0^0=1$ and $x_0^j=0$ for $j=1,2,\cdots,M$.

- ***Assumption A-2***: Transition from the normal mode to any abnormal mode is equally likely. That is, if p is the *a priori* probability of failure during one sampling interval, then $a_k^{0,0}=1-p$ and $a_k^{0,i}=\frac{p}{M}$ for $i=1,2,\cdots,M$, and all k.
- ***Assumption A-3***: No transition takes place from an abnormal mode to the normal mode implying that $a_k^{i,0}=0$ for $i=1,2,\cdots,M$, and all k. The implication is that a failed sensor does not return to the normal mode (unless replaced or repaired).
- ***Assumption A-4***: Transition from an abnormal mode to any abnormal mode including itself is equally likely. That is, $a_k^{i,j}=\frac{1}{M}$ for $i,j=1,2,\cdots,M$, and all k.

A recursive relation for Ψ_k is generated based on the above assumptions and using Eq. (A-11) for $j=1,2,\cdots,M$:

$$y_k^j=\frac{p+\sum_{i=1}^{M} y_{k-1}^i}{(1-p)M}\left(\frac{l_k^j}{l_k^0}\right) \text{given } y_0^j=0 \text{ for} \quad \text{(A-14)}$$

which is simplified by use of the relation $\Psi_k \equiv \sum_{i=1}^{M} y_k^i$ in Eq. (A-13) as:

$$\Psi_k=\left(\frac{p+\Psi_{k-1}}{(1-p)M}\right)\sum_{j=1}^{M}\frac{l_k^j}{l_k^0} \quad \text{given } \Psi_0=0 \quad \text{(A-15)}$$

If the probability measure associated with each abnormal mode is absolutely continuous relative to that associated with the normal mode, then the ratio λ_k^j/λ_k^0 of *a priori* probabilities converges to a Radon-Nikodym derivative as the region B_k in the expression $z_k \equiv \{h_k \in B_k\}$ approaches zero measure [Wong and Hajek, 1985]. This Radon-Nikodym derivative is simply the likelihood ratio $f^j(\eta_k)/f^0(\eta_k)$, $j=1,2,\cdots,M$, where $f^j(\bullet)$ is the *a priori* density function conditioned on the hypothesis H^j, $j=0,1,2,\cdots,M$. Accordingly, Eq. (A-15) becomes:

$$\Psi_k=\left(\frac{p+\Psi_{k-1}}{(1-p)M}\right)\sum_{j=1}^{M}\frac{f^j(h_k)}{f^0(h_k)} \quad \text{given } \Psi_0=0 \quad \text{(A-16)}$$

For the specific case of two abnormal hypotheses (i.e., $M=2$) representing positive and negative failures, the recursive relations for Ψ_k and Π_k in Eqs. (A-16) and (A-13) become:

$$\left.\begin{aligned}\Psi_k&=\left(\frac{p+\Psi_{k-1}}{2(1-p)}\right)\left(\frac{f^1(h_k)+f^2(h_k)}{f^0(h_k)}\right)\\ \Pi_k&=\frac{\Psi_k}{1+\Psi_k}\end{aligned}\right\}\text{given } \Psi_0=0 \quad \text{(A-17)}$$

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