On the Development of Generalized Hamiltonian Realizations

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Abstract

The generalized Hamiltonian realization problem is discussed in this paper. Three kinds of realizations are investigated. The first is the generalized Hamiltonian realization of a dynamic system. As an example, the excitation control system [4] is investigated. The feedback dissipative realization of controlled Hamiltonian systems is then considered. A necessary and sufficient condition for existence of this realization is obtained. Finally, the approximate realization is considered. A normal form result is implemented to provide certain computable conditions.

1 Introduction

In recent years, the problem of energy-based Lyapunov functions has been investigated by several authors [1-3],[5],[13]. The Lyapunov candidates are chosen from the Hamiltonian functions of generalized Hamiltonian systems. Here a new approach provides a method for solving stabilzation problems for controlled Hamiltonian systems.

A generalized Hamiltonian dynamic system is proposed in [4], [6] as

$$\dot{x} = M(x) \bigtriangledown H, \quad x \in R^n \tag{1}$$

where $M(x)$ is an $n \times n$ matrix, called the *structure matrix.* H is the Hamiltonian function of the system. We also denote by $X_H = M(x) \bigtriangledown H$ the Hamiltonian vector field defined by H. A generalized controlled Hamiltonian system is defined as

$$\dot{x} = M(x) \bigtriangledown H + \sum_{i=1}^{m} M(x) \bigtriangledown G_i u_i, \quad x \in R^n \tag{2}$$

where G_i are Hamiltonian functions for input vector fields.

As proposed in [4], [6], we allow $M(x)$ to be an arbitrary matrix. Decompose $M(x) = J(x)+P(x)$, where $J(x)$ is skew-symmetric and $P(x)$ is symmetric. Furthermore, we may decompose $P(x) = -R(x) + S(x)$, where both $R(x)$ and $S(x)$ are semi-positive definite and the ranks of $S(x)$ and $R(x)$ are the numbers of positive eigenvalues and the numbers of negative eigenvalues of $M(x)$ respectively. Then the decomposition is unique. Hence, we have

$$M(x) = J(x) - R(x) + S(x) \tag{3}$$

We call system (1) (or controlled system (2)) a dissipative system (controlled dissipative system respectively) if $S(x) = 0$. From [1-3] we can see that such systems are particularly important in stabilization analysis.

A key point in applying this new approach to a general control system is the need to express the system as a controlled Hamiltonian system. To the authors' knowledge, there is no systematic method of handling this problem. The purpose of this paper is to explore possible solutions to this problem.

2 Generalized Hamiltonian Realization

Definition 2.1. *A dynamic system*

$$\dot{x} = f(x), \quad x \in R^n \tag{4}$$

is said to have a Hamiltonian realization if there exists a suitable coordinate chart and a Hamiltonian function H such that equation (4) can be expressed as (1). A controlled dynamic system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x \in R^n \tag{5}$$

is said to have a Hamiltonian realization if there exists a suitable coordinate chart and Hamiltonian functions H, $G_1, \cdots, C_m$ such that equation (5) can be expressed as (2). If in a realization, the decomposition of $M(x)$ has $S(x) = 0$, it is called a dissipative realization (or controlled dissipative realization respectively).

Consider system (4). The most useful case is to convert it into (1) where M is constant. Denote $A_i = (\frac{\partial}{\partial x_i} f)^T, \quad i = 1, \cdots, n$ We have the following:

Proposition 2.2. *System (4) has a constant and non-singular M realization if the following equation (6)*

$$\begin{pmatrix}
A_2 & -A_1 & & & \\
A_3 & & -A_1 & & \\
 & & \ddots & & \\
A_n & & & & -A_1 \\
 & A_3 & -A_2 & & \\
 & & \ddots & & \\
 & A_n & & & -A_2 \\
 & & \vdots & & \\
 & & & A_n & -A_{n-1}
\end{pmatrix}
\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = 0 \tag{6}$$

has constant solution $X_i \in R^n$, $i = 1, \cdots, n$, which makes the following matrix N non- singular:

$$N = col(X_1^T,\ X_2^T\ \cdots\ X_n^T)$$

Proof. Denote the i-th component of Nf by H_i. Then a straightforward computation shows that

$$\frac{\partial H_i}{\partial x_j} = A_j X_i, \quad i, j = 1, \cdots, n$$

The solution of (6) implies that

$$\frac{\partial H_i}{\partial x_j} = \frac{\partial H_j}{\partial x_i}, \quad i \neq j$$

According to Poincare's Lemma [7], Nf is closed. That is, there exists H, such that $Nf = \bigtriangledown H$. i.e., $f = M \bigtriangledown H$. Note that R^n is simply connected and the result is global. □

Remark. *1. For a fixed coordinate frame, the conditions in Proposition 2.2 are also necessary. 2. The above result can also be used for finding a common M for a set of vector fields by find a common solution for them. 3. For a constant vector field, any non-singular M can be its structure matrix. Because in this case (6) holds automatically.*

3 Excitation Control

As an example, we consider the excitation system [4]

$$\begin{cases}
\dot{\delta} = \omega - \omega_0 \\
\dot{\omega} = \frac{\omega_0}{M} P_m - \frac{D}{M}(\omega - \omega_0) - \frac{\omega_0 E_q' V_s}{M x_{d\Sigma}'} sin\delta \\
\dot{E}_q' = -\frac{1}{T_d'} E_q' + \frac{1}{T_{d0}} \frac{x_d - x_d'}{x_{d\Sigma}'} V_s \cos\delta + \frac{1}{T_{d0}} V_f
\end{cases} \tag{7}$$

Consider $u = V_f$ as the control, and set $x_1 = \delta$, $x_2 = \omega - \omega_0$, $x_3 = E_q'$ and denote $a = \frac{\omega_0}{M} P_m$, $b = \frac{D}{M}$, $c = \frac{\omega_0 V_s}{M x_{d\Sigma}'}$, $d = \frac{1}{T_d'}$, $e = \frac{1}{T_{d0}} \frac{x_d - x_d'}{x_{d\Sigma}'} V_s$, and $h = \frac{1}{T_{d0}}$, then system (7) becomes

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} x_2 \\ a - bx_2 - cx_3 \sin x_1 \\ -dx_3 + e\cos x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} u \tag{8}$$

Now we consider all the possible Hamiltonian realizations. Since g is constant, we can ignore it. For f equation (6) becomes

$$\left(\begin{array}{ccccc}
1 & -b & 0 & 0 & cx_3 \cos x_1 \\
0 & -c\sin x_1 & -d & 0 & 0 \\
0 & 0 & 0 & 0 & -c \sin x_1
\end{array}\right.$$
$$\left.\begin{array}{cccc}
e \sin x_1 & 0 & 0 & 0 \\
0 & 0 & cx_3 \cos x_1 & e \sin x_1 \\
-d & -1 & b & 0
\end{array}\right) X = 0 \tag{9}$$

If we consider the constant terms and terms linear in x_1 and x_3 only, we get the following linear system:

$$\begin{pmatrix}
1 & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -d & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -d & -1 & b & 0 \\
0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\
0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} X = 0 \tag{10}$$

It is easy to check that the nonlinear terms in x do not provide any new equations. Let $X = (n_{11},\ n_{12},\ \cdots,\ n_{33})$. Since the rank of the coefficient matrix of (10) is 7, we may simply choose $n_{33} = \frac{c}{e}$; $n_{21} = \alpha$ Then, up to a constant coefficient, the solution of (10) is given by

$$N = \begin{pmatrix} b & 1 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & \frac{c}{e} \end{pmatrix};\ M = \mu \begin{pmatrix} 0 & 1 & 0 \\ \alpha & -b & 0 \\ 0 & 0 & \frac{e\alpha}{c} \end{pmatrix}$$

where μ is an arbitrary coefficient. We may simply set $\mu = 1$. Now for any non-zero α M provides a Hamiltonian realization. If we consider the dissipative realization, it is easy to verify that the only possible solution is $\alpha = -1$. Using this M, system (8) can be rewritten as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{e}{c} \end{pmatrix} \right)$$
$$\times \begin{pmatrix} -a + cx_3 \sin x_1 \\ x_2 \\ -c\cos x_1 + \frac{cd}{e} x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} u \tag{11}$$

Choosing a suitable output, the dissipative realization of (7) is expressed as

$$\begin{cases}
\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = (J(x) - R(x)) \frac{\partial H}{\partial x} + g(x) u \\
y = g^T \frac{\partial H}{\partial x}
\end{cases} \tag{12}$$

where

$$J(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{e}{c} \end{pmatrix}$$

$$g = (0 \;\; 0 \;\; h)^T, \quad y = -\frac{c}{T_{d0}} \cos x_1 + \frac{cd}{T_{d0}e} x_3,$$

$$H(x) = -cx_3 \cos x_1 - ax_1 + \frac{cd}{2e} x_3^2 + \frac{1}{2} x_2^2$$

This is exactly the form used in [4]. According to Remark 3, the dissipative realization of the excitation control system (7) is unique (up to a constant coefficient).

4 Feedback Dissipative Realization

The dissipative form of Hamiltonian systems is important for using port-controlled stability analysis. We consider a general Hamiltonian system as

$$\begin{cases} \dot{x} = M \bigtriangledown H + g(x)u = (J - R + S) \bigtriangledown H + g(x)u \\ y = g^T \bigtriangledown H \end{cases} \tag{13}$$

where J is skew symmetric and R and S are symmetric and positive semi- definite.

The following is an example of a general Hamiltonian system.

Example 4.1. [8] The Rotational/Translational Actuator (RTAC) system can be expressed, via the coordinate transformation introduced in [9], as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \epsilon \sin x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \end{cases} \tag{14}$$

where $0 < \epsilon < 1$. Choosing a nominal output $y = x_4$ and the Hamiltonian function as

$$H = \frac{1}{2}(x_1^2 + x_2^2 + \sin^2(x_3) + x_4^2) + \epsilon x_1 \sin x_3$$

then the system (14) can be expressed as the following generalized controlled Hamiltonian system

$$\begin{cases} \dot{x} = M \bigtriangledown H + g(x)u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \times \begin{pmatrix} x_1 - \epsilon \sin x_3 \\ x_2 \\ (\epsilon x_1 + \sin x_3) \cos x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y = g^T \bigtriangledown H = x_4 \end{cases} \tag{15}$$

□ To use the stabilization method proposed in [3], we have to convert such generalized controlled Hamiltonian systems into dissipative type systems. We consider the following state feedback control

$$u = K(x) \bigtriangledown H + v \tag{16}$$

We say that system (13) has a *feedback dissipative realization* if there exists a control as given in (16) such that the feedback system (15) is a dissipative Hamiltonian system. Then we have

Proposition 4.2. *System (13) has a dissipative type realization (around an equilibrium point x_0), iff there exists an $m \times n$ matrix $K(x)$, such that the following matrix is negative semi- definite(locally).*

$$g(x)K(x) + K^T(x)g^T(x) + (M + M^T) \leq 0 \tag{17}$$

Proof. Because the symmetric part of $M + gK$ is negative semi-definite. □

We are particularly interested in the case when both M and $g = (g_1 \;\cdots\; g_m)$ are constant. In this case we seek a particular output feedback control of the form

$$u = K \bigtriangledown H + v$$

where K is a constant matrix.

Let $P = \frac{1}{2}(M + M^T)$, then we have the following corollary.

Corollary 4.3. *System (13) with constant M and g has a dissipative type realization if there exists an $m \times n$ matrix K such that the following matrix is negative semi-definite :*

$$gK + K^T g^T + P \leq 0 \tag{18}$$

Note that we can decompose $P = -R + S$, with positive semi-definite R and S. Assuming $span\{col(S)\} \subset span\{col(g)\}$, it is easy to find K which satisfies (18). In fact, if $S = g\alpha$ we can simply choose $K = -\alpha$.

Example 4.4. Recall Example 4.1. It is easy to see that a solution of (18) is $K = (0 \; 0 \; 1 \; 0)$. Hence the RTAC system (15) has a feedback dissipative realization. □

5 Approximate Hamiltonian Realizations

When a precise Hamiltonian realization does not exist for a dynamic system but it may be approximated by a Hamiltonian system up to certain degree. Such an approximation may be enough for stability analysis. This approach allows a coordinate change and so provides more freedom to manipulate systems.

Definition 5.1. *System (4) with $f(0) = 0$ is said to have a k-th order approximate realization if there exists a coordinate $z = z(x)$ with $z(0) = 0$ such that under z system (4) becomes*

$$\dot{z} = M \bigtriangledown H + r(z) \tag{19}$$

where M is constant, $deg(H) \leq k + 1$ and $r(z) = 0(\|z\|^{k+1})$. $deg(H)$ is the degree of the lowest degree *non-vanishing terms of the Taylor expanssion of H. If $M = J - R$ with skew symmetric J and symmetric $R \geq 0$, (19) is called a k-th order approximate dissipative realization.*

We have the following stability result:

Proposition 5.2. *Assume system (4) has an approximate dissipative realization (19), $r(z) \in Span\{R\}$, and 0 is a local minimum point of $H(x)$. Then system (4) is stable at 0.* **Proof.** First we claim that the condition $r(z) \in Span\{R\}$ is a coordinate-independent condition. It is not difficult to show that under a coordinate change $z = z(x)$, with the Jacobian matrix J_z we have

$$r(z) = J_z r(x(z)); \quad R(z) = J_z R J_z^T(x(z))$$

The claim follows.

Next, choosing H as the Lyapunov function and denoting $r(z) = R\xi(z)$, then we have

$$\frac{d}{dt}H|_{(19)} = -dHR(\bigtriangledown H + \xi)$$

According to the order of ξ, it is easy to see that $\frac{d}{dt}H|_{(19)} \leq 0$ locally. □

Remark. *It is obvious that the condition $r(z) \in Span\{R\}$ can be relaxed by $r(z) = r_1(z) + r_2(z)$, where*

$$r_1(z) \in Span\{R\}, \quad L_{r_2}H(z) \leq 0 \tag{20}$$

Consider system (4) again and assume $f(0) = 0$ and denote the Jacobian of f at zero by L. Let the vector field $X = Lx$ and for any vector field Y denote $ad_L Y = ad_X Y$. Denote by P^k the set of vector fields in R^n with k-th degree homogeneous components. Then 1. P^k is a linear vector space over R. 2. Let $L \in P^1$ be a given vector field. Then the Lie derivative $ad_L : P^k \to P^k$ is a linear mapping.

Now fix $Lx \in P^1$. According to the above, the range of the mapping $ad_L : P^k \to P^k$ is a subspace of P^k. Thus we can decompose P^k as

$$P^k = ad_L(P^k) \oplus S^k$$

where S^k is a complement of $ad_L(P^k)$. Note that S^k is not unique. The following theorem provides a normal form expression for the system (4).

Theorem 5.3.[10] *Consider system (4) with $f(0) = 0$. Let $L = J_f(0)x$, where $J_f(0)$ is the Jacobian matrix of f at zero. Then there exists a local diffeomorphism $x = x(z)$ around zero, such that (4) can be locally expressed as*

$$\dot{z} = s^{(1)}(z) + s^{(2)}(z) + \cdots + s^{(r)}(z) + R_r(z), \tag{21}$$

where

$$s^{(1)}(z) = J_f(0)z; \; s^{(i)}(z) \in S^i; \; i = 2, \cdots, r;$$
$$R_r(z) = 0(\|z\|^{r+1})$$

(21) is called a normal form of (4). An algorithm was proposed in [11].

Denote the coefficient matrix in (6) by A and its approximation up to the k-th degree by A^k. That is, $A = A^k + 0(\|x\|^{k+1})$, with $deg(A^k) = k$. Then consider the following equation:

$$A^k \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = 0, \quad , X_i \in R^n, \; i = 1, \cdots, n \tag{22}$$

The following proposition is staightforward.

Proposition 5.4. *Assume (22) has a constant solution X, which defines the matrix N given by*

$$N = col(X_1^T, \; X_2^T, \; \cdots, \; X_n^T)$$

non-singular. Then system (4) can be expressed as

$$\dot{x} = M \bigtriangledown H + 0(\|x\|^{k+2}), \quad deg(H) = k + 2 \tag{23}$$

where $M = N^{-1}$. If (22) doesn't have a non-zero constant solution for $k = 0$, then a Hamiltonian (or an approximate Hamiltonian) realization doesn't exist. Thus we assume there exists a largest k such that the solution exists.

Assume system (4) has a (k-1)-th order approximate Hamiltonian realization. Correspondingly, (23) can be written as

$$\dot{x} = M \bigtriangledown H + 0(\|x\|^k) = M \bigtriangledown H + q_k + 0(\|x\|^{k+1}) \tag{24}$$

where $deg(H) \leq k$, $deg(q_k) = 0(\|x\|^k)$. Denote $dim(P^k) - dim(ad_L(P^k)) = t$, where L is the Jacobian of $f(x)$ at zero. We want to find a k-th order approximation from (24). Since further discussion involves a coordinate change, we use $\bigtriangledown_x H$ for $\bigtriangledown H$ to emphasize the gradient is taken with respect to x etc. We have the following:

Proposition 5.5. *Assume there exist t homogeneous polynomials $H_1, \cdots, H_t$ of degree $k+1$ such that*

$$ad_L(P^k) + Span\{M \bigtriangledown_x H_1, \cdots, M \bigtriangledown_x H_t\} = P^k$$

Then system (24) has a k-th approximate Hamiltonian realization. **Proof.** We can find a vector field $T(x) \in P^k$ such that

$$q(x) = ad_L(T(x)) + c_1 M \bigtriangledown H_1 + \cdots + c_t M \bigtriangledown H_t$$

Then we make a coordinate transformation

$$x = z + T(z), \quad J_z = (I + \frac{\partial T}{\partial x})^{-1} = I - J_T + 0(\|z\|^{2k})$$

One sees easily that the coordinate transformation does not affect the terms of degree less than k. That is, the term $M \bigtriangledown H$ in (24) remains unchanged. To be precise, since

$$\begin{aligned}\bigtriangledown_x H(x) = J_z \bigtriangledown_z H(x(z)) = (I - J_T + 0(\|z\|^{2k}))\times \\ \bigtriangledown_z H(z)(I - J_T^T + 0(\|z\|^{2k}) = \bigtriangledown_z H(z) + 0(\|z\|^{2k})\end{aligned}$$

then

$$M \bigtriangledown_x H(x) = M \bigtriangledown_z H(z) + 0(\|z\|^{2k}) \qquad (25)$$

Also the new k-th degree term

$$\tilde{q}(z) \in Span\{M \bigtriangledown_x H_1, \cdots, M \bigtriangledown_x H_t\}$$

and it can be expressed as

$$\tilde{q}(z) = c_1 M \bigtriangledown_z H_1(z) + \bigtriangledown_z H_t(z) + 0(\|z\|^{2k}) \qquad (26)$$

Substituting (25) and (26) into (24), the conclusion follows. $\square$

The following corollary is from the proof of Proposition 5.5.

Corollary 5.6. *Assume there exist t homogeneous polynomials $H_1, \cdots, H_t$ of degree $k+1$ and a homogeneous vector field $T(x) \in P^k$ such that*

$$q(x) = ad_L T(x) + c_1 M \bigtriangledown_x H_1 + \cdots + c_t M \bigtriangledown_x H_t$$

Then system (24) has a k-th approximate Hamiltonian realization as

$$\dot{z} = M \bigtriangledown_z (H(z) + c_1 H_1(z) + \cdots + c_t H_t(z)) + 0(\|z\|^{k+1}) \qquad (27)$$

where $x = z + T(z)$. In fact, this result can be expressed in a more elegent form. Let $d_k = dim(P^k)$. Then [11]

$$d_k = dim(P^k) = \frac{n(n+k-1)!}{k!(n-1)!}$$

The following lemma is a consequence of the results. **Lemma 5.7.** *Let $X \in P^k$, $N \in GL(n, R)$. Then $\Phi_N : P^k \to P^k$, defined as $X \mapsto NX$, is a group representation [12] of $GL(n, R)$ to $GL(d_k, R)$. Moreover,*

$$\Phi_n = N \otimes I_{s_k} \qquad (28)$$

where

$$s_k = \frac{d_k}{n} = \frac{(n+k-1)!}{k!(n-1)!}$$

Denote by Q_k the linear space of k-th degree homogenious polynomials. It is easy to see that $dim(Q_k) = s_k$. Then the gradient $\bigtriangledown : Q_{k+1} \to P^k$ is a linear mapping. So $\bigtriangledown Q_{k+1} \subset P^k$ is a subspace. Using Lemma 5.7, it is easy to verify the following result. **Proposition 5.8.** *System (24) has a k-th order approximate Hamiltonian realization, with the same M, iff*

$$q(x) \in ad_L(P_k) + \Phi_N(\bigtriangledown Q_{k+1}) \qquad (29)$$

Example 5.9. Consider the following system

$$\begin{cases} \dot{x}_1 = sin(x_2) + 2(1 - cos(x_1)) \\ \dot{x}_2 = -x_1 + x_2 log(1 + x_1 + x_2) \end{cases} \qquad (30)$$

Express it in approximate form as

$$\begin{aligned}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} + \begin{pmatrix} x_1^2 \\ x_1 x_2 + x_2^2 \end{pmatrix} \\ - \begin{pmatrix} \frac{1}{6}x_2^3 \\ \frac{1}{2}x_1^3 + x_1^2 x_2 + \frac{1}{2}x_1 x_2^2 \end{pmatrix} + 0(\|x\|^4)\end{aligned} \qquad (31)$$

It is easy to determine that the largest k for the k-th approximated equation of (6) to have a solution is $k = 2$, when the equation can be obtained by comparing the coefficients of different terms as

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} n_{11} \\ n_{12} \\ n_{21} \\ n_{22} \end{pmatrix} = 0 \qquad (32)$$

The only non-zero solution (up to a constant coefficient) is

$$N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 1 & 4 \end{pmatrix}$$

Then we have

$$M = N^{-1} = -\frac{1}{18} \begin{pmatrix} 4 & -2 \\ -1 & -4 \end{pmatrix} \qquad (33)$$

Then the system (30) can be expressed as

$$\begin{aligned}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} := M \bigtriangledown H + q_3 + R = -\tfrac{1}{18} \begin{pmatrix} 4 & -2 \\ -1 & -4 \end{pmatrix} \bigtriangledown H \\ - \begin{pmatrix} \frac{1}{6}x_2^3 \\ \frac{1}{2}x_1^3 + x_1^2 x_2 + \frac{1}{2}x_1 x_2^2 \end{pmatrix} + 0(\|x\|^4)\end{aligned} \qquad (34)$$

where

$$H = -x_1^2 - 4x_1 x_2 + \frac{1}{2}x_2^2 - \frac{4}{3}x_1^3 + x_1^2 x_2 + 2x_1 x_2^2 + \frac{4}{3}x_2^3$$

Next, we consider $ad_L P^3$, with $Lx = J_f(0)x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$. Set a basis $e_1, \cdots, e_8$ of P^3 as

$$\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 x_2^2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_2^3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} \quad \begin{pmatrix} 0 \\ x_1^2 x_2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ x_2^3 \end{pmatrix}$$

A straightforward computation shows that

$$\begin{aligned} V_1 &= ad_L e_1 = 3e_2 + e_5 \\ V_2 &= ad_L e_2 = -e_1 + 2e_3 + e_6 \\ V_3 &= ad_L e_3 = -2e_2 + e_4 + e_7 \\ V_4 &= ad_L e_4 = -3e_3 + e_8 \\ V_5 &= ad_L e_5 = -e_1 + 3e_6 \\ V_6 &= ad_L e_6 = -e_2 - e_5 + 2e_7 \\ V_7 &= ad_L e_7 = -e_3 - 2e_6 + e_8 \\ V_8 &= ad_L e_8 = -e_4 - 3e_7 \end{aligned}$$

Morever, the $rank(V_1\ V_2\ V_3\ V_4\ V_5\ V_6\ V_7\ V_8) = 6$ and the first 6 vectors are linearly independent.

Simply choose $H_1 = \frac{1}{4}x_1^4$ and $H_2 = \frac{1}{4}x_2^4$. (In general, we have to find a solution of (28), which produces a set of linear equations.) From (33), we have

$$M \bigtriangledown H_1 = -\tfrac{1}{18}\begin{pmatrix} 4x_1^3 \\ -x_1^3 \end{pmatrix} = -\tfrac{2}{9}e_1 + \tfrac{1}{18}e_5$$
$$M \bigtriangledown H_2 = -\tfrac{1}{18}\begin{pmatrix} -2x_2^3 \\ -4x_2^3 \end{pmatrix} = \tfrac{1}{9}e_4 + \tfrac{2}{9}e_8$$

It is easy to verify that V_1, V_2, V_3, V_4, V_5, V_6, and $M \bigtriangledown H_1$, $M \bigtriangledown H_2$ are linearly independent. Then we can express q_3 as a linear combination by solving a linear equation. The numerical solution is $b_1 = -0.75$; $b_2 = -3.5$; $b_3 = -1.333333$; $b_4 = -2.333333$; $b_5 = 0.833333$; $b_6 = 0.416667$; and $c_1 = 12$; $c_2 = 10.5$. Now we can choose new coordinates z by

$$x = z + T(z) = z + \sum_{i=1}^{6} b_i e_i = z+$$
$$\begin{pmatrix} -0.75z_1^3 - 3.5z_1^2 z^2 - 1.333333z_1 z_2^2 - 2.333333z_2^3 \\ 0.833333z_1^3 + 0.416667z_1^2 z_2 \end{pmatrix}$$

Using Corollary 5.6, system (34) can be further expressed as

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = -\frac{1}{18}\begin{pmatrix} 4 & -2 \\ -1 & -4 \end{pmatrix} \bigtriangledown H + 0(\|z\|^4) \quad (35)$$

where $H = -z_1^2 - 4z_1 z_2 + \frac{1}{2}z_2^2 - \frac{4}{3}z_1^3 + z_1^2 z_2 + 2z_1 z_2^2 + \frac{4}{3}z_2^3 + 3z_1^4 + \frac{10.5}{4}z_2^4$. □

6 Conclusion

Three different realization problems were considered in this paper. The first involved transforming an affine (control) system to a generalized Hamiltonian (control) system. A set of linear algebraic equations were provided. A sufficient condition to obtain the realization is that these equations have a constant solution and the matrix N constructed by the solution elements is nonsingular. The excitation control system in power systems was investigated. It was proved that the dissipative realization obtained in [4] is the unique solution. The second problem considered when a generalized controlled Hamiltonian system has a feedback dissipative realization. A necessary and sufficient condition was presented. It was also shown that the RTAC system [8] has a feedback dissipative realization. Finally, an approximate realization was considered. The normal form algorithm was used to develop the realization. The constructive proof provides an algorithm to construct an approximate realization.

Acknowledgments: This work was supported by a Royal Society Royal Fellowship, by G69774008, G59837270, G1998020308 and the National Key Project of China.

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