Guaranteed Multi-Loop Stability Margins and the Gap Metric

Keith Glover* Glenn Vinnicombe George Papageorgiou

Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, England

tel: +44-1223-339706 fax: +44-1223-332662

Abstract

The gap metric and nu-gap metric have many appealing properties for assessing the uncertainty in a plant in a feedback configuration. This short paper addresses the question of relating these generalised stability margins to more traditional single-loop and multi-loop robustness measures. In particular we show that when a system is controlled by a controller which provides a stability margin of ϵ in the gap metric for the weighted plant then if the weighting matrices are diagonal the plant will have robust stability to easily interpreted simultaneous gain and phase variations at all the inputs and outputs.

Keywords: Flight control, $\mathcal{H}_\infty$ loop shaping, stability margins, gap metric.

1 Introduction

The gap metric [1] and nu-gap metric [6] have many appealing theoretical properties and are convincingly appropriate measures for uncertainty in feedback systems. The $\mathcal{H}_\infty$ loop shaping approach to robust design [3] maximises robustness to gap-metric uncertainty for a weighted plant. The weighting matrices are essential to determine appropriate levels of bandwidth and disturbance rejection in the closed loop system. For the feedback connection of the weighted plant $W_2(s)G(s)W_1(s)$ in feedback with a controller $K_\infty(s)$ the stability margin is defined as,

$$b(W_2GW_1, K_\infty) = \left\| \begin{bmatrix} K_\infty \\ I \end{bmatrix} (I - W_2GW_1)^{-1} \begin{bmatrix} I & W_2GW_1 \end{bmatrix} \right\|_\infty^{-1}$$

A main result [6] is then that the closed loop will be stable for all perturbed plants, G_Δ, such that $\delta_\nu(W_2G_\Delta W_1, W_2GW_1) < b(W_2GW_1, K_\infty)$.

2 The Single-Input/Single-Output Case

Note that if $G = M^{-1}N$ is a normalised coprime factorization, and a perturbed system G_Δ is defined as $G_\Delta = G(1+\delta_1)/(1-\delta_2)$ with $|\delta_1| \le \epsilon < 1$ and $|\delta_2| \le \epsilon$ then $\delta_\nu(G, G_\Delta) < \epsilon$, since we can write $G_\Delta = (M - \delta_2 M)^{-1}(N + \delta_1 N)$. The following lemma, which is closely related to results in [5] and [4], gives a frequency domain characterization of such systems.

Lemma 2.1 *$G_\Delta = G(1+\delta_1)/(1-\delta_2)$ with $|\delta_1| \le \epsilon < 1$ and $|\delta_2| \le \epsilon$, if and only if*

$$\frac{|1 - G_\Delta/G|}{1 + |G_\Delta|/|G|} \le \epsilon$$

In addition this characterization gives that if $G_\Delta = Gre^{j\phi}$ then,

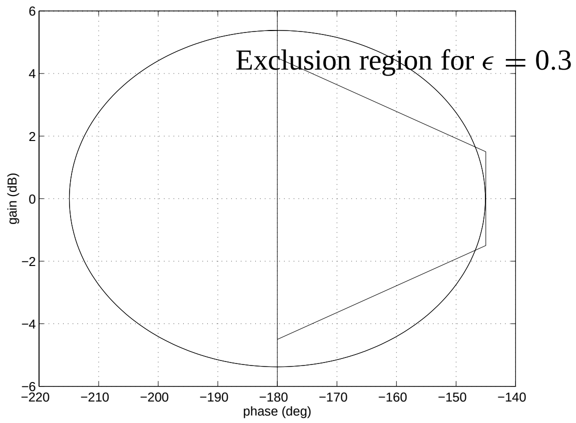
$$\sin^2(\phi/2) \le \frac{(1+\epsilon - r(1-\epsilon))(-1+\epsilon+r(1+\epsilon))}{4r}$$

which gives a clear picture of the allowable gain and phase variations in a Nichols chart exclusion region. Such a region is found to be particularly helpful in flight control stability analysis [2]. This formula also implies the guaranteed gain and phase margins in [7], namely a gain margin of $\pm 20\log_{10}(1+\epsilon)/(1-\epsilon)$ dB and a phase margin of $\pm 2\arcsin\epsilon$. An alternative way of deriving this region is to note that we can write $G_\Delta = G\frac{(1+\delta_1)}{\sqrt{1-\epsilon^2}}\frac{\sqrt{1-\epsilon^2}}{(1-\delta_2)}$ and also that the sets $\frac{1+\delta_1}{\sqrt{1-\epsilon^2}}$ and $\frac{\sqrt{1-\epsilon^2}}{1-\delta_2}$ with $|\delta_1| < \epsilon$ and $|\delta_2| < \epsilon$ give identical disks in the complex plane. Hence if $b(W_2GW_1, K_\infty) = \epsilon$, then G can be perturbed by a term $\frac{(1+\delta)^2}{1-\epsilon^2}$ with $|\delta| < \epsilon$ and the closed-loop system will remain stable. The exclusion region for $\epsilon = 0.3$, which is a typical value, is depicted in the figure together with a standard Nichols chart exclusion region used in flight control stability analysis.

3 The Multi-loop case

In the multivariable case the weighting matrices can substantially distort gap metric balls of uncertainty. However if we restrict ourselves to diagonal weighting

*correspondence e-mail: kg@eng.cam.ac.uk



matrices and diagonal perturbations then we have,

$$\begin{aligned} W_2 G_\Delta W_1 &= W_2(I+\Delta_1)G(I-\Delta_2)^{-1}W_1 \\ &= (I+\Delta_1)W_2GW_1(I-\Delta_2)^{-1} \\ &= (I+\Delta_1)NM^{-1}(I-\Delta_2)^{-1} \\ &= (N+\Delta_1 N)(M-\Delta_2 M)^{-1} \\ &= (N+\Delta_N)(M-\Delta_M)^{-1} \end{aligned}$$

where NM^{-1} is a normalised coprime factorization of W_2GW_1, and

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} \right\|_\infty < \epsilon$$

if $\|\Delta_i\|_\infty < \epsilon = b(W_2GW_1, K_\infty)$ for $i = 1, 2$, then

$$\delta_\nu(W_2G_\Delta W_1, W_2GW_1) < \epsilon$$

[6] and stability is guaranteed.

Now $G_\Delta = (I+\Delta_1)G(I-\Delta_2)^{-1}$ looks asymmetric with respect to inputs and outputs. However, as in the single-loop case we can rewrite this as

$$G_\Delta = \frac{1}{\sqrt{1-\epsilon^2}}(I+\Delta_1)G(I-\Delta_2)^{-1}\sqrt{1-\epsilon^2}$$

and note that the sets $\frac{1+\delta_1}{\sqrt{1-\epsilon^2}}$ and $\frac{\sqrt{1-\epsilon^2}}{1-\delta_2}$ with $|\delta_1| < \epsilon$ and $|\delta_2| < \epsilon$ are identical. Hence each input and output channel can be simultaneously perturbed by a term $\frac{1+\delta}{\sqrt{1-\epsilon^2}}$ with $|\delta| < \epsilon$ and the system will remain stable.

For example if $\epsilon = 0.3$, which is a typical value, then a gain change of up to $\sqrt{\frac{1+\epsilon}{1-\epsilon}} = 1.37(2.69\text{dB})$ or a phase change of up to $\arcsin(\epsilon) = 17.5\,\text{deg}$ independently on each input and output can be allowed without losing stability. That is, half of the allowable gain/phase offset in the single-loop case can be tolerated simultaneously at each input and output.

The above argument allows independent perturbations to all channels simultaneously. To obtain a single channel result we can set,

$$\begin{aligned} I+\Delta_1 &= \text{diag}(1+\delta_1, 1-\delta_2, \cdots, 1-\delta_2), \\ I-\Delta_2 &= (1-\delta_2)I \end{aligned}$$

which implies that

$$(I+\Delta_1)G(I-\Delta_2)^{-1} = \text{diag}\left(\frac{1+\delta_1}{1-\delta_2}, 1, 1, \cdots 1\right) \text{G}$$

So, the allowable single channel offsets are the same as in the single-loop case.

In the above, the perturbations have been restricted to being complex scalars. It can be easily shown that the results also apply for dynamic uncertainty satisfying the same gain/phase bounds provided the continuity condition $\delta_\nu(G, G_\Delta) < 1$ is satisfied.

4 Conclusions

We have demonstrated that the stability margin implied by the nu-gap metric when applied in $\mathcal{H}_\infty$ loop shaping with diagonal pre- and post-compensator weights, gives robustness to easily computed perturbations at the plant inputs and outputs.

References

[1] T. T. Georgiou and M. C. Smith. Optimal robustness in the gap metric. *IEEE Transactions on Automatic Control*, 35:673–686, 1990.

[2] J-F. Magni, S. Bennani, and J. Terlouw, editors. *Robust Flight Control: A Design Challenge*. Number 224 in Lecture Notes in Control and Information Sciences. Springer-Verlag, first edition, 1997.

[3] D.C. McFarlane and K. Glover. *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*. Lecture Notes in Control and Information Sciences. Springer-Verlag, first edition, 1990.

[4] H. Panagopoulos and K. J. Astrom. PID control and $\mathcal{H}_\infty$ loop shaping. *International Journal of Robust and Nonlinear Control*, 2000. to appear.

[5] J.A. Sefton. *A Geometrical Approach to Feedback stability*. PhD thesis, University of Cambridge, Dept. of Engineering, 1991.

[6] G. Vinnicombe. Frequency-domain uncertainty and the graph topology. *IEEE Trans. Automatic Control*, 38(9):1371–1383, 1993.

[7] G. Vinnicombe. *Measuring the Robustness of Feedback Systems*. PhD thesis, University of Cambridge, Dept. of Engineering, 1993.