A REMARK ON THE DESIGN OF TIME–VARYING STABILIZERS FOR STOCHASTIC DIFFERENTIAL SYSTEMS WITHOUT UNFORCED DYNAMICS

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Abstract

The purpose of this paper is to extend to stochastic differential systems without unforced dynamics the stabilization techniques for controllable driftless systems developped by Pomet in [3].

1 Introduction

In this paper, we are concerned with the stabilization problem of stochastic differential systems with unforced dynamics.

Using the techniques developped in [1] to establish a stochastic version of Jurdjevic–Quinn's theorem combined with Pomet's approach [3] to design time–varying stabilizers for driftless deterministic systems we show how to stabilize affine stochastic differential systems without unforced dynamics by means of time–varying state feedback laws.

2 Problem Setting

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and denote by $x_t \in \mathbb{R}^n$ the stochastic process solution of the control stochastic differential system

$$dx_t = \sum_{j=0}^{p} u^j X_j(x_t)dt + \sum_{j=1}^{p} \sum_{k=1}^{m} u^j Y_{jk}(x_t)dw_t^k \qquad (1)$$

where :

1. $(w_t)_{t\geq 0}$ is a standard Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^m$.

2. X_j, $0 \leq j \leq p$, and Y_{jk}, $1 \leq j \leq p$, $1 \leq k \leq m$, are smooth vector fields defined on $\mathbb{R}^n$.

3. u^j, $0 \leq j \leq p$, are real–valued control laws.

4. rank span $\left\{ \mathrm{ad}_{X_0}^k X_j \ , \ 0 \leq j \leq p, \ k \in \mathbb{N} \right\} = n$.

In the following, we define a periodic time–varying function $V(t,x)$ used in the design of the stabilizing control law and meant to be a Lyapunov function for the overall closed–loop system.

Let α be a time–varying function mapping $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}$ such that α is 2π–periodic and odd with respect to time, vanishes for $x = 0$ and such that

$$|\alpha(t,x)| \; ||X_0(x)|| \leq K\,(1 + ||x||) \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

For any $s \geq 0$ and $x \in \mathbb{R}^n$, denote by $\psi(s,t,x)$ $(s \leq t)$ the solution at time t of the differential equation :

$$\dot{x} = \alpha(t,x) X_0(x) \qquad (2)$$

starting from the state x at time s.

Furthermore, be means of well–known results of the theory of P.D.E., one can prove the following result.

Theorem 2.1 *The function V defined on $\mathbb{R} \times \mathbb{R}^n$ by*

$$V(t,x) = \frac{1}{2} ||\psi(0,t,x)||^2 \qquad (3)$$

satisfy the following properties :

(1) V is 2π–periodic with respect to time, twice differentiable with respect to x and admits an infinitesimal upper limit.

(2) $V(t,x) = 0 \Leftrightarrow x = 0$.

(3) V is the solution of the P.D.E. with initial condition at $t = 0$:

$$\begin{aligned} &\nabla_t V(t,x) + \alpha(t,x) \nabla_x V(t,x) X_0(x) = 0 \\ &V(0,x) = \frac{1}{2} ||x||^2. \end{aligned} \qquad (4)$$

3 The Main Result

The purpose of this section is to prove that the 2π–periodic time–varying control law :

$$\begin{aligned} u_0(t,x) &= \alpha(t,x) - X_0V(t,x) \\ u^i(t,x) &= -\frac{X_iV(t,x)}{1+\left(\mathrm{Tr}\left(\sum_{k=1}^{m} Y_{ik}(x)Y_{ik}(x)^{\star}\nabla_x^2 V(t,x)\right)\right)^2} \end{aligned} \tag{5}$$

renders the stochastic differential system (1) asymptotically stable in probability provided further requirement on α are fullfiled.

Theorem 3.1 *Assume that*

$$\left.\begin{array}{l} X_iV(t,x)=0 \quad , \quad \forall i\in\{0,...,p\} \\ \\ \dfrac{\partial^j \alpha}{\partial t^j}(t,x)=0 \quad , \quad \forall j\in \mathbb{N}^{\star} \end{array}\right\} \Rightarrow x=0. \tag{6}$$

Then, the equilibrium solution of the closed–loop system deduced from (1) by applying the time–varying state feedback law (5) is asymptotically stable in probability.

Proof Since the functions α, V and u_i, $0\le i\le p$, are 2π–periodic with respect to time they induce functions defined on $S^1\times \mathbb{R}^n$ where $S^1=\mathbb{R}/2\pi$.

Hence, the 2π–periodic closed–loop system deduced from (1) can be considered as the time–invariant stochastic differential system on $S^1\times\mathbb{R}^n$ given by :

$$dX_t = F(X_t)dt + \sum_{k=1}^{m} G_k(X_t)dw_t^k \tag{7}$$

where

$$X_t = \begin{pmatrix} t \\ x_t \end{pmatrix} \quad , \quad F(X)=\begin{pmatrix} 1 \\ \sum_{j=0}^{p} u^j(t,x)X_j(x) \end{pmatrix}$$

and

$$G_k(X)=\begin{pmatrix} 0 \\ \sum_{j=1}^{p} u^j(t,x)Y_{jk}(x) \end{pmatrix} \qquad 1\le k\le m.$$

Then, denoting by $\mathcal{L}$ the infinitesimal generator of the closed–loop system (7) one gets :

$$\begin{aligned} \mathcal{L}V(t,x) &= -\left((X_0V)(t,x)\right)^2 + \sum_{j=1}^{p}\Big(u^j(t,x)(X_jV)(t,x) \\ &\quad + \frac{1}{2}u^j(t,x)^2\,\mathrm{Tr}\left(Y_{jk}(x)Y_{jk}(x)\nabla_x^2 V(t,x)\right)\Big) \end{aligned}$$

and, with the choice of the control laws u^j, $1\le j\le p$, one has $\mathcal{L}V(t,x)\le 0$ for every $(t,x)\in S^1\times\mathbb{R}^n$ which implies that the equilibrium solution of the closed–loop system (7) is stable in probability.

Furthermore, by application of the stochastic version of La Salle's theorem proved in [2], the stochastic process X_t solution of (7) tends in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}V(t,x_t)=0$ for all $t\ge 0$.

But, if $\mathcal{L}V(t,x_t)=0$, the choice of feedback laws we have made in (5) implies that $X_iV(t,x_t)=0$ for every $i\in\{0,...,p\}$ and hence, $u^0(t,x_t)=\alpha(t,x_t)$ and $u^j(t,x_t)=0$ for every $j\in\{1,...,p\}$.

Therefore, by successive applications of Itô's formula to the stochastic process $\mathcal{L}V(t,x_t)$ one has :

$$\mathrm{ad}_{L_0}^k X_jV(t,x_t)=0$$

for every $t\ge 0$, $j\in\{0,...,p\}$ and $k\in\mathbb{N}$ where L_0 is the vector field defined by

$$L_0=\frac{\partial}{\partial t}+\alpha X_0.$$

Moreover, straightforward inductive computations imply that, for every (t,x) such that $\alpha(t,x)\ne 0$, one has

$$\text{rank span}\left\{\mathrm{ad}_{L_0}^k X_j \ ,\ 0\le j\le p,\ k\in\mathbb{N}\right\}=n.$$

Then, arguying as in [3], one can prove, under the above rank condition, that $\mathcal{L}V(t,x_t)=0$ implies

$$\frac{\partial^j\alpha}{\partial t^j}(t,x_t)=0\ ,\ \forall j\in\mathbb{N}^{\star}$$

and

$$X_iV(t,x_t)=0\ ,\ \forall i\in\{0,...,p\}$$

which is equivalent, according to (6), to $x_t=0$.

Therefore, according to the stochastic version of La Salle's theorem, the stochastic process x_t tends in probability to 0 and hence, the equilibrium solution of the closed–loop system (7) is asymptotically stable in probability.

References

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