A STOCHASTIC JURDJEVIC–QUINN THEOREM

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Abstract

The purpose of this paper is to extend to affine in the control stochastic differential systems the well–known result of Jurdjevic–Quinn [2]. This result incorporates a previous result in the stochastic context proved in [1].

1 Introduction

The stabilization of deterministic nonlinear control systems has been intensively studied by many authors in the last past years .

Among all the works related to this topic, we wish to outline the result proved by Jurdjevic–Quinn [2] giving stabilizing state feedback laws for deterministic systems affine in the control provided the control Lie algebra of the system has full rank. The result exposed in [2] has been revisited in [5] where a more easily computable rank condition for stabilizability is stated.

A stochastic version of Jurdjevic–Quinn's theorem has been established in [1] for stochastic differential systems the drift of which is affine in the control.

In fact, it is proved in [1] that the stabilizer given in [2] for the deterministic part of the system remains valid in the stochastic case provided the system coefficients satisfy a rank condition which can be easily deduced from that stated in [5].

The technique used in [1] is based on the stochastic Lyapunov analysis developped by Khasminskii [3] and the stochastic version of La Salle's invariance principle proved by Kushner [4]. However, in the proof of Theorem 3.2 in [1], all the information given by Itô's formula, when using the stochastic La Salle theorem, has not been used. The aim of this paper is to take this fact into account in order to improve the stabilizability condition given in [1] allowing to stabilize some stochastic differential systems which do not satisfy the hypothesis of Theorem 3.2 in [1].

2 Problem setting

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and denote by $(w_t)_{t\geq 0}$ a standard $\mathbb{R}^m$–valued Wiener process defined on this space.

Consider the stochastic process solution $x_t \in \mathbb{R}^n$ of the stochastic differential system written in the sense of Stratonovitch,

$$\begin{aligned} x_t &= x_0 + \int_0^t \left(X_0(x_s) + \sum_{j=1}^{p} u^j Y_j(x_s) \right) ds \\ &\quad + \sum_{i=1}^{m} \int_0^t X_i(x_s) \circ dw_s^i \qquad (1) \end{aligned}$$

where

1. x_0 is given in $\mathbb{R}^n$.
2. u^j, $1 \leq j \leq p$, are real–valued measurable control laws.
3. X_i, $0 \leq i \leq m$, and Y_j, $1 \leq j \leq p$, are smooth Lipschitz vector fields on $\mathbb{R}^n$, vanishing in the origin, and with less than linear growth.

The aim of this paper is to compute a state feedback law u^j, $1 \leq j \leq p$ which renders the equilibrium solution $x_t \equiv 0$ of the closed–loop system deduced from (1) asymptotically stable in probability.

3 The main result

Denoting by L the infinitesimal generator of the stochastic process solution of the unforced stochastic differential system deduced from (1), one can prove the following stabilization result.

Theorem 3.1 *Assume that there exists a smooth Lyapunov function V defined on $\mathbb{R}^n$ (i.e. a C^2 function V mapping $\mathbb{R}^n$ into $\mathbb{R}$ which is proper and positive definite) such that*

1) $LV(x) \leq 0$ for every $x \in \mathbb{R}^n$.

2) The set

$$\mathcal{K} = \Big\{x \in \mathbb{R}^n \ / \ X_{i_0}^{\alpha_0} L^{\beta_0} ... X_{i_k}^{\alpha_k} L^{\beta_k} Y_j V(x) = 0 \text{ and } X_{i_0}^{\alpha_0} L^{\beta_0} ... X_{i_k}^{\alpha_k} L^{\beta_k + 1} V(x) = 0, \forall j \in \{1,..,p\}, \forall k \in \mathbb{N}, \forall i_0,..,i_k \in \{1,...,m\}, \forall \alpha_0, \beta_0,..,\alpha_k, \beta_k \in \{0,..,k\} \text{ such that } \sum_{i=0}^{k} (\alpha_i + \beta_i) = k \Big\}.$$

is reduced to $\{0\}$.

Then, the control law u defined on $\mathbb{R}^n$ by

$$u^j(x) = -Y_j V(x) \ , \ 1 \leq j \leq p, \tag{2}$$

renders the stochastic differential system (1) asymptotically stable in probability.

Proof of Theorem 2.1 : Denoting by $\mathcal{L}$ the infinitesimal generator of the closed–loop system deduced from (1) with the state feedback law u given by (2) one has

$$\mathcal{L}V(x) = LV(x) - \sum_{j=1}^{p} (Y_j V)^2(x). \tag{3}$$

Then, taking hypothesis 1) into account, it yields $\mathcal{L}V(x) \leq 0$ for every $x \in \mathbb{R}^n$, and according with the stochastic Lyapunov theorem (see [3] for example) the equilibrium solution $x_t \equiv 0$ of the closed–loop system deduced from (1) is stable in probability.

Furthermore, the stochastic La Salle theorem proved by Kushner [4] asserts that the equilibrium solution of the closed–loop system tends in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}V(x_t) \equiv 0$ for every $t \geq 0$.

But, one can deduce easily from (3) that $\mathcal{L}V(x_t) \equiv 0$ for every $t \geq 0$ if, and only if, $LV(x_t) \equiv 0$ and $Y_j V(x_t) \equiv 0$, $j \in \{1,...,p\}$, for every $t \geq 0$.

Moreover, applying Itô's formula to the stochastic processes $LV(x_t)$ and $Y_j V(x_t)$, $1 \leq j \leq p$, it yields that if $LV(x_t) \equiv 0$ and $Y_j V(x_t) \equiv 0$, $1 \leq j \leq p$, for every $t \geq 0$, one has $L^2 V(x_t) \equiv 0$, $X_i LV(x_t) \equiv 0$, $1 \leq i \leq m$, $LY_j V(x_t) \equiv 0$, $1 \leq j \leq p$, and $X_i(Y_j V)(x_t) \equiv 0$, $1 \leq i \leq m$, $1 \leq j \leq p$, for every $t \geq 0$.

Therefore, by successive applications of Itô's formula, one can prove that if $\mathcal{L}V(x_t) \equiv 0$ for every $t \geq 0$, one has $x_t \in \mathcal{K}$ for every $t \geq 0$ and consequently, according with hypothesis 2), $x_t \equiv 0$ for every $t \geq 0$.

Hence, the stochastic La Salle's invariance theorem imply that the equilibrium solution $x_t \equiv 0$ of the closed–loop system deduced from (1) tends in probability to 0 and thus, is asymptotically stable in probability.

4 Example

Let x_0 be given in $\mathbb{R}^2$, and denote by $x_t \in \mathbb{R}^2$ the solution of the stochastic differential system :

$$dx_t = \begin{pmatrix} -x_{1,t} \\ -x_{2,t} \end{pmatrix} dt + u \begin{pmatrix} x_{2,t} \\ 0 \end{pmatrix} dt + \begin{pmatrix} x_{2,t} \\ x_{1,t} \end{pmatrix} \circ dw_t \tag{4}$$

where $(w_t)_{t \geq 0}$ is a standard real–valued Wiener process and u is a real–valued measurable control law.

Then, taking the Lyapunov function V defined on $\mathbb{R}^2$ by $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ one has :

$$LV(x) = 0 \quad , \quad YV(x) = x_1 x_2 \quad \text{and} \quad LYV(x) = 0$$

for every $x \in \mathbb{R}^2$, and the hypothesis of Theorem 3.2 in [1] are not satisfied.

However, for any $x \in \mathbb{R}^2$, one has

$$X_1 YV(x) = x_1^2 + x_2^2,$$

and hence, the set $\mathcal{K}$ defined in hypothesis 2) of Theorem 2.1 is reduced to $\{0\}$.

Therefore, the state feedback law u defined on $\mathbb{R}^2$ by :

$$u(x) = -YV(x) = -x_1 x_2$$

renders the stochastic differential system (4) asymptotically stable in probability.

References

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