Observability for linear Systems with unknown Inputs

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Abstract

In this note we give a characterization of observability for a class of linear systems with unknown inputs.

1 Introduction

The problem of observing the state of a linear time-invariant multivariable system, subjected to unknown inputs, has received considerable attention in the last two decades (see e.g. [2, 6, 3, 4, 5, 1]). Different approaches to construct observers for such systems have been proposed by these authors; we point out the reference [1] where the author give a necessary and sufficient condition under which it is possible to construct a full-order observer for linear systems with unknown inputs.

In this paper, we propose a new concept : the unknown inputs-observability and we give a characterization of linear systems which are unknown inputs-observable.

We consider linear multivariable systems:

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, the matrices A, B and C having the appropriate dimensions and we suppose $\operatorname{rank}(B) = m$ and $\operatorname{rank}(C) = p$.

We assume that u is an unknown input and we propose the following definition:

Definition 1 *We will say that system* (1) *is unknown inputs-observable iff the application* $(x_0, u) \mapsto y(x_0, t, u(t))$ *which maps a pair of (init. cond., input) to the output of* (1) *(with initial condition* x_0 *and input* $t \mapsto u(t)$*) is injective*

2 Main result

We will prove the following theorem:

Theorem 1 *The linear system* (1) *is unknown-inputs observable iff the rank of the matrix*

$$\tilde{A}(s) = \begin{pmatrix} sI_n - A & B \\ C & O \end{pmatrix}$$

is equal to $n + m$ *for all* $s \in \mathbb{C}$.

The point is that the condition stated in this theorem is necessary in order to construct a full-order observer for (1) and is sufficient if $\operatorname{rank}(CB) = \operatorname{rank}(B)$ (see [1]). Thus, for linear systems with unknown inputs, the notion proposed in the above definition is the analogous of the classical notion of observability for linear systems (with known inputs).

First, the theorem is proved in the case where $\operatorname{rank}(B) \geq \operatorname{rank}(C)$. Suppose first that there exists $s \in \mathbb{C}$ such that $\operatorname{rank}(\tilde{A}(s)) < n + m$, then we can find s, α and β in $\mathbb{C}$ such that $A\alpha = s\alpha$, $C\alpha = 0$, $B\beta = 0$ and $M\beta = s\beta$. We consider then the $\mathbb{C}^n$ vector satisfying the differential equation:

$$\left\{ \begin{array}{rcl} \dot{z}(t) & = & Az(t) + B\exp(tM)\beta \\ z(0) & = & \alpha \end{array} \right. \tag{2}$$

It is easy to verify that $Cz^{(p)}(0) = 0$ for $p = 0, 1, \ldots$ from which we deduce that $Cz \equiv 0$. From z, we obtain a pair $(x, u) \not\equiv (0, 0)$ such that $Cx \equiv 0$.

We prove the converse in the case $\operatorname{rank}(C) = \operatorname{rank}(B)$, the general case following from this particular case. Clearly, the condition on the determinant of $\tilde{A}$ implies that the system (1) is observable in the classical sense; moreover if we replace A by $A + KC$ in the definition of $\tilde{A}(s)$, the rank of this last matrix remains unchanged, so we can suppose that A is nilpotent.

In this case matrix $\tilde{A}$ is squared and by multiplying by a suitable matrix, it is easily seen that

$$\det \tilde{A}(s) = (-1)^p s^n \times detC(I_n - A)^{-1}B$$

denoting by $d(s)$ this determinant, we have $\lim_{s\to\infty} d(s)/s^{n-p} = \det CB$ which proves that $d(s)$ is a polynomial of degree at most $n-p$. So we put:

$$d(s) = \lambda_0 s^{n-p} + \cdots + \lambda_{n-p}$$

and $d(s)$ being nonzero for all $s \in \mathbb{C}$, we have $\lambda_0 = \lambda_1 = \ldots = \lambda_{n-p-1} = 0; \lambda_{n-p} \neq 0$. Now, we denote by $N(s)$ the matrix $N(s) = C(I_n - sA)^{-1}B$ and by $M(s)$ the transposed of the cofactor matrix of $M(s)$ and we put:

$$M(s) \quad = \quad M_0 + sM_1 + \cdots + s^d M_d$$

Notice that $\det M(s) = (-1)^p s^{n-p} d(1/s) = (-1)^p(\lambda_0 + \lambda_1 s + \cdots + \lambda_{n-p} s^{n-p}$ and, since A is nilpotent, $M(s) = C(I_n + sA + \cdots + s^{n-1}A^{n-1}$ so from $M(s) \times N(s) = \det M(s) I_p$, we get:

$$\begin{aligned} M_0CB &= (-1)^p \lambda_0 I_p \\ M_0CAB + M_1CB &= (-1)^p \lambda_1 I_p \\ \vdots &= \vdots \\ M_0CA^{n-p}B + \cdots + M_{n-p}CB &= (-1)^p \lambda_{n-p} I_p \end{aligned}$$

Suppose now that the pair (x_0, u) is such that $x(t)$, the solution of (1) with initial condition x_0 and control law u, is such that $y(t) = 0$ for all $t \geq 0$. Starting from the equality $Cx(t) = 0$ and proceeding by derivation and elimination, we finally get the equality:

$$M_0CA^{n-p+1}x(t) + \cdots + M_{n-p}CAx(t) + \lambda_{n-p}u(t) = 0 \quad (3)$$

Since $\lambda_{n-p} \neq 0$, this proves that the function $t \mapsto u(t)$ is differentiable and, denoting by P the matrix

$$P = M_0CA^{n-p+1} + \cdots + M_{n-p-1}CA^2 + M_{n-p}CA + M_{n-p-1}C$$

we have $Px + \lambda_{n-p}u = 0$ and so $\dot{u} = -\dfrac{1}{\lambda_{n-p}}P\dot{x} = PAx$ (since $PB = 0$)then (x, u) is solution of the linear system

$$\left\{ \begin{array}{lcl} \dot{x} & = & Ax + Bu \\ \dot{u} & = & -\dfrac{1}{\lambda_{n-p}}Px \\ y & = & Cx \end{array} \right. \qquad (4)$$

Now the matrix related to this linear system verifies

$$\operatorname{rank} \begin{pmatrix} sI_n - A & B \\ O & P \\ C & O \end{pmatrix} = n + p$$

because we assume that

$$\operatorname{rank} \begin{pmatrix} sI_n - A & B \\ C & O \end{pmatrix} = n + p$$

so system (4) is observable which implies $x_0 = 0$ and $u \equiv 0$. Finally, we can conclude that system (1) is unknown inputs-observable.

In the case where $\operatorname{rank} B < \operatorname{rank} C$, it is easily seen that there exits a vector in the form $\begin{pmatrix} b \\ 0 \end{pmatrix}$ which does not belong to the vector space spanned by the columns of matrix $\tilde{A}(s)$, so we can complement matrix B with $p - m$ columns in such a way that the matrix $\bar{A}$ constructed with A, $\bar{B}$ (B complemented by $p - m$ appropriate columns) and C is of full rank for all $s \in \mathbb{C}$. Suppose now that (x_0, u) is such that $Cx(t) = 0$ for all $t \geq 0$, $x(t)$ denoting the solution of (1), clearly, $x(t)$ is also solution of

$$\dot{x}(t) = Ax(t) + \bar{B}.\begin{pmatrix} u \\ 0 \end{pmatrix} \qquad (5)$$

and it follows from the case $\operatorname{rank} C = \operatorname{rank} B$ that $x_0 = 0$ and $u \equiv 0$.

A proof by induction shows the result in the general case ($\operatorname{rank} B > \operatorname{rank} C$).

References

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