Vertex results for parametric Shifted H_∞ Performance of Weighted Interval Plants [1]

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Abstract

This note considers the parametric shifted H_∞ problem of weighted interval plants. A sufficient condition is obtained on weighting function $W(s)$ such that for any interval plant $P(s,q,r)$ the maximal shifted H_∞ norm of $W(s)P(s,q,r)$ is achieved at one of the vertex weighted plants.

1 Introduction

The idea of this paper is motivated from the problem of robust stability under structured and unstructured uncertainties and the quantitative theory of robust adaptive control. The problem can be transformed to be the computation problem of the maximal H_∞ norm or shifted H_∞ norm of a weighted transfer function with parametric uncertainties [1,2], which is also called parametric H_∞ or shifted H_∞ problems.

The standard H_∞ norm of a stable transfer function $P(s)$ is defined as $\|P(s)\|_\infty \doteq \sup_{\omega\in\mathcal{R}} |P(j\omega)|$, and the shifted H_∞ norm of a $\mathbf{H}^\delta$ stable transfer function $P(s)$ is defined as $\|P(s)\|_\infty^\delta \doteq \|P(s-\delta)\|_\infty = \sup_{\omega\in\mathcal{R}} |P(j\omega-\delta)|$, [1], where $\mathcal{R}$ denotes the set of real numbers.In the following,for brevity,we also use the following notations. $\mathbf{H}$ represents the set of all Hurwitz polynomials whose roots are all in the open left half plane and $\mathbf{H}^\delta$ the set of polynomials whose roots are all in the half plane $\{s : Res < -\delta\}$ where $\delta > 0$. Hurwitz polynomials are also called stable polynomials. For a polynomial in $\mathbf{H}^\delta$, we will call it $\mathbf{H}^\delta$ stable.

The quantitative estimates of the robustness margins in robust adaptive control involve the calculation of maximal shifted H_∞ norms over some compact convex sets in the parameter space [1]. To solve this problem, [1] extends the vertex results for parametric H_∞ problems of interval plants to the parametric shifted H_∞ problems. Motivated by this result and the fact that practical performance and robustness problems are characterized by frequency-dependent criteria, this note aims to develop vertex results for parametric shifted H_∞ problems of weighted interval plants.

[1]Supported partly by a small ARC grant and NSF of China.

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2 Main Result

Consider an uncertain stable transfer function modeled by an interval plant:

$$P(s,q,r) \doteq \frac{N(s,q)}{D(s,r)} = \frac{q_0 + q_1 s + \cdots + q_m s^m}{r_0 + r_1 s + \cdots + s^n} \quad (2.1)$$

where the coefficient vectors q and r vary in the rectangles:

$$Q \doteq \{q = [q_0, q_1, \cdots, q_m]^T : q_i^- \le q_i \le q_i^+\} \quad (2.2)$$

$$R \doteq \{r = [r_0, r_1, \cdots, r_{n-1}]^T : r_i^- \le r_i \le r_i^+\} \quad (2.3)$$

and introduce an assumption on weighting functions:

Assumption 1: *All the roots of $A(s)$ are real, less than $-\delta$, nonrepeated and for any two roots α_1, α_2 of $A(s)$ ($\alpha_2 < \alpha_1 < -\delta$), there exists at least one real root β of $B(s)$, such that $-\alpha_1 < |\beta| < -\alpha_2$.*

Then we have

Theorem 1: *Let $W(s) = \frac{B(s)}{A(s)}$ be any real transfer function which satisfies Assumption 1. Then*

$$\max_{q\in Q, r\in R} \left\| \frac{B(s)}{A(s)} P(s,q,r) \right\|_\infty^\delta = \max_{q\in V_Q, r\in V_R} \left\| \frac{B(s)}{A(s)} P(s,q,r) \right\|_\infty^\delta \quad (2.4)$$

where

$$V_Q \doteq \{q : q_i \in \{q_i^-, q_i^+\}, i = 0,1,\cdots,m\}$$

$$V_R \doteq \{r : r_i \in \{r_i^-, r_i^+\}; i = 0,1,\cdots,n-1\}.$$

3 Vertex Lemma and Proof of the Main Result

In this section, a vertex lemma for standard parametric H_∞ problems with weighting functions in [4] is reviewed and then modified for the proof of Theorem 1. At first, the following assumption is needed.

Assumption 2: *All the roots of $A(s)$ are real, negative, nonrepeated and for any two roots α_1, α_2 of $A(s)$ ($\alpha_2 < \alpha_1 < 0$), there exists at least one real root β of $B(s)$, such that $-\alpha_1 < |\beta| < -\alpha_2$.*

Lemma 1[4]: *Let $N(s)$, $D_0(s)$ and $A(s), B(s)$ be fixed polynomials, $D_0(s) \in \mathbf{H}$, $q(s)$ a polynomial of degree less than the degree of $D_0(s)$ having only even or odd coefficients, $W(s) = \frac{B(s)}{A(s)}$ satisfy Assumption 2. Then*

$$\max_{\lambda\in[0,1]} \left\| \frac{B(s)}{A(s)} \frac{N(s)}{D_0(s)+\lambda q(s)} \right\|_\infty = \max_{\lambda=0,1} \left\| \frac{B(s)}{A(s)} \frac{N(s)}{D_0(s)+\lambda q(s)} \right\|_\infty \tag{3.1}$$

Corollary 1. *Let $a(s)$ be an anti-Hurwitz polynomial with real coefficients, i.e, $a(-s) \in \mathbf{H}$, and $q(s)$ have only even or only odd coefficients, then*

$$\max_{\lambda\in[0,1]} \left\| \frac{B(s)}{A(s)} \frac{N(s)}{D_0(s)+\lambda a(s)q(s)} \right\|_\infty = \max_{\lambda=0,1} \left\| \frac{B(s)}{A(s)} \frac{N(s)}{D_0(s)+\lambda a(s)q(s)} \right\|_\infty \tag{3.2}$$

Proof: Notice that $a(-s) \in \mathbf{H}, a(s)a(-s)$ has only even coefficients, $|a(-j\omega)| \neq 0$ for any fixed $\omega \in \mathcal{R}$, and

$$\left\| \frac{B(s)}{A(s)} \frac{N(s)}{D_0(s)+\lambda a(s)q(s)} \right\|_\infty = \left\| \frac{B(s)}{A(s)} \frac{a(-s)N(s)}{a(-s)D_0(s)+\lambda a(-s)a(s)q(s)} \right\|_\infty$$

holds for any fixed $\lambda \in [0,1]$, then (3.2) can be derived directly from Lemma 1.

Now the vertex lemma for parametric shifted H_∞ problems can be given now.

Lemma 2. *Let $W(s)$ be a weighting function satisfying assumption 1, $D_0(s), D_0(s)+s^k$ be H^δ stable. Then*

$$\max_{\lambda\in[0,1]} \left\| W(s) \frac{N(s)}{D_0(s)+\lambda s^k} \right\|_\infty^\delta = \max_{\lambda=0,1} \left\| W(s) \frac{N(s)}{D_0(s)+\lambda s^k} \right\|_\infty^\delta \tag{3.3}$$

Proof: It is easy to see that if $W(s)$ satisfies assumption 1, then $W(s-\delta)$ satisfies assumption 2. Note that $D(s) \in \mathbf{H}^\delta$ is equivalent to $D(s-\delta) \in \mathbf{H}$. It can be seen from Corollary 1 that

$$\max_{\lambda\in[0,1]} \left\| W(s) \frac{N(s)}{D_0(s)+\lambda s^k} \right\|_\infty^\delta = \max_{\lambda\in[0,1]} \left\| W(s-\delta) \frac{N(s-\delta)}{D_0(s-\delta)+\lambda (s-\delta)^k} \right\|_\infty = \max_{\lambda\in\{0,1\}} \left\| W(s-\delta) \frac{N(s-\delta)}{D_0(s-\delta)+\lambda (s-\delta)^k} \right\|_\infty$$

$$= \max_{\lambda=0,1} \left\| W(s) \frac{N(s)}{D_0(s)+\lambda s^k} \right\|_\infty^\delta \tag{3.4}$$

which completes the proof of Lemma 2.

Proof of Theorem 1: Clearly, for any fixed complex numbers a and b, the following holds.

$$\max_{\lambda\in[0,1]} |a+\lambda b| = \max_{\lambda=0,1} |a+\lambda b| \tag{3.5}$$

Therefore, for any fixed $r \in R$

$$\max_{q\in Q} \left\| \frac{B(s)}{A(s)} P(s,q,r) \right\|_\infty^\delta = \max_{q\in V_Q} \left\| \frac{B(s)}{A(s)} P(s,q,r) \right\|_\infty^\delta \tag{3.6}$$

From Lemma 2, for any fixed $q \in Q$ and $r_k \in [r_k^-, r_k^+], k = 0,1,\cdots,n-1, k \neq i$, one has

$$\max_{r_i\in[r_i^-,r_i^+]} \left\| \frac{B(s)}{A(s)} P(s,q,r) \right\|_\infty^\delta = \max_{r_i=r_i^-,r_i^+} \left\| \frac{B(s)}{A(s)} P(s,q,r) \right\|_\infty^\delta \tag{3.7}$$

Thus, from (3.6-7), (2.4) holds and the proof is completed.

4 Conclusion

In this paper, the problem of the maximal shifted H_∞ norm for a weighted interval plant is investigated. A subset of weighting functions is characterized such that the maximal shifted H_∞ norm of the weighted interval plants enjoys the extremality property. The vertex result presented here has extended the previous vertex results obtained in [1,4]. But unlike the vertex result in [4], which need check only 16 vertex plants, Theorem 1 need check all the extreme point plants.

References

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